SCALAR AND DOT PRODUCTS (§12.2, §12.3)

Recall: If \( \vec{a} = (a_1, a_2, a_3) \), then the magnitude/length is
\[
|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.
\]

If \( c \in \mathbb{R} \), then \( c\vec{a} = (ca_1, ca_2, ca_3) \).

A vector with magnitude 1 is called a unit vector.

The vector \( \frac{\vec{a}}{|\vec{a}|} \) is a unit vector.
THE DOT PRODUCT

Let \( \vec{a} = (a_1, a_2) \) and \( \vec{b} = (b_1, b_2) \) be vectors in \( \mathbb{R}^2 \), then the dot product (or inner product) is \( \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 \). Similarly, in \( \mathbb{R}^3 \) one has
\[
\vec{a} \cdot \vec{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

**Example:** \((2, 1, -1) \cdot (0, 3, -3) = 2 \cdot 0 + 1 \cdot 3 - 1 \cdot (-3) = 6.\)

**Remark:** All properties that we derive for the dot product will hold in general \( \mathbb{R}^n \) (in particular in both \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)).

**Properties:**

- For any vectors \( \vec{a} \) and \( \vec{b} \), \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \).
- For any vector \( \vec{a} \) we have \( \vec{a} \cdot \vec{a} = |\vec{a}|^2 \).
- For any vector \( \vec{a} \), \( \vec{a} \cdot \vec{0} = \vec{0} \cdot \vec{a} = 0 \).
- For vectors \( \vec{a}, \vec{b} \) and a scalar \( c \in \mathbb{R} \), \( (c \vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) \).
- For vectors \( \vec{a}, \vec{b}, \vec{c} \) one has \( (\vec{a} + \vec{b}) \cdot \vec{c} = (\vec{a} \cdot \vec{c}) + (\vec{b} \cdot \vec{c}) \).
ANGLES

Given two unit vectors $\vec{a}, \vec{b}$, the angle between $\vec{a}$ and $\vec{b}$ is the length of the circular arc defined by them.

For general vectors $\vec{a}$ and $\vec{b}$, the angle $\theta$ is the same as the angle between $\frac{\vec{a}}{|\vec{a}|}$ and $\frac{\vec{b}}{|\vec{b}|}$.
• If \( \vec{b} = -\vec{a} \) then \( \theta = \pi \)

• If \( \theta = \frac{\pi}{2} \), then \( \vec{a} \) and \( \vec{b} \) are orthogonal or perpendicular. We write \( \vec{a} \perp \vec{b} \)

The shorter arc counts \( \Rightarrow \) angles are always \( 0 \leq \theta \leq \pi \).

• If \( \vec{b} = \vec{a} \) or \( \vec{b} = c\vec{a} \) for \( c > 0 \), then \( \theta = 0 \)
SINE AND COSINE

Consider a right triangle with hypothenuse of length 1 and angle $\theta$:
Consider the unit vectors \( \vec{a} = (1, 0) \) and \( \vec{b} = (\cos(\theta), \sin(\theta)) \).

The angle between \( \vec{a} \) and \( \vec{b} \) is \( \theta \).

The dot product is \( \vec{a} \cdot \vec{b} = (1, 0) \cdot (\cos(\theta), \sin(\theta)) = \cos(\theta) \).

For all unit vectors \( \vec{a}, \vec{b} \) with angle \( \theta \) one has

\[
\vec{a} \cdot \vec{b} = \cos(\theta).
\]

For all vectors \( \vec{a}, \vec{b} \) their angle \( \theta \) satisfies

\[
\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}.
\]
Fact: Suppose \( \vec{a} \) and \( \vec{b} \) have both length > 0. Then \( \vec{a} \perp \vec{b} \) if and only if \( \vec{a} \cdot \vec{b} = 0 \).

Proof. Let \( \theta \in [0, \pi] \) be the angle between \( \vec{a} \) and \( \vec{b} \). Then

\[
\begin{align*}
\vec{a} \cdot \vec{b} &= 0 \\
\iff |\vec{a}| \cdot |\vec{b}| \cdot \cos(\theta) &= 0 \\
\iff \cos(\theta) &= 0 \\
\iff \theta &= \frac{\pi}{2}
\end{align*}
\]
Exercise (Aut 2013, Loveless, Midterm 1, Ex 1)

The forces $\vec{a}$ and $\vec{b}$ are pictured. If $|\vec{a}| = 80$ N and $|\vec{b}| = 100$ N, find the angle the resultant force makes with the positive $x$-axis. Give your answer rounded to the nearest degree.

Solution:

- The forces are $\vec{a} = (80 \cos(\frac{\pi}{3}), 80 \sin(\frac{\pi}{3})) = (40, 40\sqrt{3})$ and $\vec{b} = (-100, 0)$

- The resultant force is $\vec{u} = \vec{a} + \vec{b} = (40 + (-100), 40\sqrt{3} + 0) = (-60, 40\sqrt{3})$

- We are looking for the angle $\theta$ between $\vec{u}$ and $\vec{v} = (1, 0)$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{-60}{\sqrt{60^2 + (40\sqrt{3})^2} \cdot 1} = \frac{-60}{\sqrt{8400}}$$

- Finally

$$\theta = \cos^{-1}(\frac{-60}{\sqrt{8400}}) \approx 130.89339^\circ \approx 131^\circ$$
**Def.** Given non-zero vectors \( \vec{a}, \vec{b} \) the **projection** of \( \vec{b} \) onto \( \vec{a} \) is the unique vector \( \text{proj}_{\vec{a}}(\vec{b}) \) that is a scalar of \( \vec{a} \) and has \( \vec{a} \cdot \vec{b} = \text{proj}_{\vec{a}}(\vec{b}) \cdot \vec{a} \).

Example: \( \text{proj}_{(1,0)}(5, 3) = (5, 0) \)

**Formula:**

\[
\text{proj}_{\vec{a}}(\vec{b}) = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} \in \mathbb{R}
\]

**Proof.**

\[
\left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = (\vec{a} \cdot \vec{b}) \cdot \frac{1}{|\vec{a}|^2} (\vec{a} \cdot \vec{a}) = \vec{a} \cdot \vec{b}
\]
We define the **scalar projection** of \( \vec{b} \) onto \( \vec{a} \) as

\[
\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}
\]

**Facts:**

- \( |\text{proj}_{\vec{a}}(\vec{b})| = |\text{comp}_{\vec{a}}(\vec{b})| \)
- If \( \theta < \frac{\pi}{2} \), then \( \text{comp}_{\vec{a}}(\vec{b}) > 0 \)
- If \( \theta > \frac{\pi}{2} \), then \( \text{comp}_{\vec{a}}(\vec{b}) < 0 \)