Example: Describe the path of

\[ \vec{r}(t) = (x(t), y(t)) = (3t, t^2 + 4) \]  for  \(-\infty < t < \infty\)

and eliminate the parameter \(t\).

We have \(x = 3t \Leftrightarrow t = \frac{x}{3}\).

Hence \(y = t^2 + 4 = \left(\frac{x}{3}\right)^2 + 4\).

The curve is \((x, \frac{x^2}{9} + 4)\) for  \(-\infty < x < \infty\), which is a parabola.
Effects of different curve parametrizations

Case 1: Curve $\vec{r}(t) = (t, \frac{t^2}{9} + 4)$; tangent vec.: $\vec{r}'(t) = (1, \frac{2}{9}t)$

![Graph showing Case 1](image)

Case 2: Curve $\vec{r}(t) = (3t, t^2+4)$. Tangent vec: $\vec{r}'(t) = (3, 2t)$

![Graph showing Case 2](image)

Case 3: Curve $\vec{r}(t) = (-3t, t^2 + 4)$; $\vec{r}'(t) = (-3, 2t)$

![Graph showing Case 3](image)
Paths in \( \mathbb{R}^3 \)

A path in \( \mathbb{R}^3 \) can be described with a vector function

\[
\vec{r}(t) = (x(t), y(t), z(t))
\]

The derivative \( \vec{r}'(t) = (x'(t), y'(t), z'(t)) \) now gives the direction vector for the line tangent to the point \( \vec{r}(t) \).

Example: For

\[
\vec{r}(t) = (3 \sin(t), 4 \cos(t), t) \quad \text{for } t \geq 0
\]

find the parametric equations for the line tangent to the curve at \( t = \frac{\pi}{2} \).

Path starts at \((0, 4, 0)\). \( z \) always increases.

\(-3 \leq x \leq 3 \) and \(-4 \leq y \leq 4 \) for all \( t \).
Example (cont.)

\[ \vec{r}(t) = (3 \sin(t), 4 \cos(t), t) \]
\[ \vec{r}(\frac{\pi}{2}) = (3, 0, \frac{\pi}{2}) \]
\[ \vec{r}'(t) = (3 \cos(t), -4 \sin(t), 1) \]
\[ \vec{r}''(\frac{\pi}{2}) = (0, -4, 1) \]

The tangent line is \( \{ \vec{r}(\frac{\pi}{2}) + t \vec{r}''(\frac{\pi}{2}) : t \in \mathbb{R} \} \).

With parametric equations, the tangent line is

\[ x = 3, \ y = -4t, \ z = \frac{\pi}{2} + t \]
Def. The **unit tangent vector** to \( \vec{r}(t) \) is

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}
\]

In the example above \( \vec{r}'(t) = (3 \cos(t), -4 \sin(t), 1) \).

\[
|\vec{r}'(t)| = \sqrt{9 \cos^2(t) + 16 \sin^2(t) + 1}
= \sqrt{9 \cos^2(t) + 9 \sin^2(t) + 7 \sin^2(t) + 1}
= \sqrt{9 \cos^2(t) + 9 \sin^2(t) + 7 \sin^2(t) + 1}
= \sqrt{7 \sin^2(t) + 10}
\]

Hence

\[
\vec{T}(t) = \left( \frac{3 \cos(t)}{\sqrt{7 \sin^2(t) + 10}}, \frac{-4 \sin(t)}{\sqrt{7 \sin^2(t) + 10}}, \frac{1}{\sqrt{7 \sin^2(t) + 10}} \right)
\]

For example, \( \vec{T}(\pi) = (-\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}) \).
Arc length

Def + Lemma: A path described by
\( \vec{r}(t) = (x(t), y(t), z(t)), \ a \leq t \leq b \) has arc length

\[
L = \int_{a}^{b} |\vec{r}'(t)| \, dt
\]

Here we need that \( x'(t), y'(t), z'(t) \) exist and are continuous and the curve is traversed exactly once.

Intuition:

travelled distance = \( \int_{t_{\text{start}}}^{t_{\text{end}}} \text{speed}(t) \, dt \)
Proof sketch.

Take sequence $a = a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n = b$.

Compute length of the polygonal curve through $\vec{r}(a_0), \ldots, \vec{r}(a_n)$:

\[
L \approx \sum_{i=0}^{n} |\vec{r}(a_i) - \vec{r}(a_{i+1})| \\
\approx \sum_{i=0}^{n} (a_{i+1} - a_i) \cdot |\vec{r}'(a_i)| \\
\approx \int_{a}^{b} |\vec{r}'(t)| \, dt
\]

derivative gives approximation

def. integral

If $|a_i - a_{i+1}| \to 0$, the error in “$\approx$” goes to 0. □
Given the curve

$$\vec{r}(t) = (x(t), y(t), z(t)) = (\cos(2\pi t), \sin(2\pi t), 3 - t^{3/2})$$

Compute the length of the curve from the point where $t = 0$ to the point where $t = 8$.

- Recall: Arc length is $\int_0^8 |\vec{r}'(t)|\,dt$
- Find $\vec{r}''(t)$:

$$\vec{r}''(t) = (-2\pi \sin(2\pi t), 2\pi \cos(2\pi t), -\frac{3}{2}t^{1/2})$$

- Determine $|\vec{r}''(t)|$:

$$|\vec{r}''(t)| = \sqrt{(-2\pi \sin(2\pi t))^2 + (2\pi \cos(2\pi t))^2 + (-\frac{3}{2}\sqrt{t})^2}$$

$$= \sqrt{4\pi^2 + \frac{9}{4}t}$$

- Integrate

$$\int_0^8 \sqrt{4\pi^2 + \frac{9}{4}t} \, dt = \frac{4}{9} \cdot 2 \left(4\pi^2 + \frac{9}{4}t\right)^{3/2} \bigg|_0^8 = \frac{8}{27}((4\pi^2+18)^{3/2} - 8\pi^3)$$

Recall: $\int \sqrt{u} \, du = \frac{2}{3}u^{3/2}$