Midterm 1

- Date: Tuesday, Feb 3. Place: in quiz sections
- Time: 50min
- Relevant material: until (incl.) Wednesday, Jan 28
  Review: Friday, Jan 30 & Monday Feb 2
  CLUE: midterm review on Sunday, Feb 1 (time and place TBA)
- Allowed:
  - scientific calculator (no graphic calculators)
  - handwritten page of notes (1 sheet, both sides)

A good way how to prepare for the exam:

- Read the sections in the textbook
- Solve a couple of old midterms (see web page)
- If you have questions, ask Math Study Center, CLUE, TAs, me, etc..
Polar Coordinates (§10.3)

**Cartesian** coordinate system: every point in $\mathbb{R}^2$ is represented by an ordered pair $(x_0, y_0)$.

**Polar** coordinate system: every point in $\mathbb{R}^2$ is represented by an ordered pair $(r, \theta)$

$\theta =$ oriented angle with positive $x$-axis

$r =$ signed distance from origin
Examples:

<table>
<thead>
<tr>
<th>point</th>
<th>cartesian</th>
<th>polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(0, 2)</td>
<td>(2, $\pi$)</td>
</tr>
<tr>
<td>B</td>
<td>(1, 1)</td>
<td>($\sqrt{2}$, $\pi$)</td>
</tr>
<tr>
<td>C</td>
<td>(-1, -1)</td>
<td>($\sqrt{2}$, $\frac{5}{4}\pi$) or ($\sqrt{2}$, $-\frac{3}{4}\pi$) or ($-\sqrt{2}$, $\frac{\pi}{4}$)</td>
</tr>
<tr>
<td>D</td>
<td>(2, 0)</td>
<td>(2, 0) or (2, $2\pi$) or ($-2$, $\pi$)</td>
</tr>
</tbody>
</table>

Note: Polar representation is not unique.

($r$, $\theta$) is the same point as ($r$, $\theta+k\cdot2\pi$) and ($-r$, $\theta+k\cdot2\pi+\pi$) for any $k \in \mathbb{Z}$.
Conversion between cartesian and polar coordinates

Given polar coordinates \((r, \theta)\)
⇒ cartesian coordinates are 
\[ x = r \cdot \cos(\theta), \quad y = r \cdot \sin(\theta) \]

Given cartesian coordinates \((x, y)\)
⇒ polar coordinates are
- \(\theta = \tan^{-1}\left(\frac{y}{x}\right)\) (gives always \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\))
- \(r = \pm \sqrt{x^2 + y^2}\).

**Warning:** Case split needed:
- If \(x > 0\), then choose \(r = \sqrt{x^2 + y^2}\)
- If \(x < 0\), then choose \(r = -\sqrt{x^2 + y^2}\)
- If \(x = 0\), then \(\tan^{-1}\left(\frac{y}{x}\right)\) undefined. Choose \(\theta = \frac{\pi}{2}\), \(r = y\)
Example
If polar is \((3, \frac{7}{6} \pi)\), then
\[
x = 3 \cdot \cos \left(\frac{7}{6} \pi\right) = 3 \cdot \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}
\]
\[
y = 3 \cdot \sin \left(\frac{7}{6} \pi\right) = 3 \cdot \left(-\frac{1}{2}\right) = -\frac{3}{2}
\]
\[\rightarrow \text{Cartesian } \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)\]

If Cartesian is \((-1, \sqrt{3})\), then one choice for \(\theta\) is
\[\theta = \tan^{-1} \left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3}\]
Moreover \(r^2 = (-1)^2 + (\sqrt{3})^2 = 4 \Rightarrow r = \pm 2\).
Need to choose \(r = -2\).
Polar is \((-2, -\frac{\pi}{3})\) (or \((2, \frac{2}{3} \pi)\)).
Polar curves

Example 1: Same curve in
- Cartesian coordinates: \( x^2 + y^2 = 16 \)
- Polar coordinates: \( r = 4 \)

(circle of radius 4 and centered at origin)
Example 2: Consider the curve $r = 4 \cos(\theta)$ given in polar coordinates. Which equation describes the curve in terms of Cartesian coordinates?

\[
\{(r, \theta) : r = 4 \cos \theta\} \leftarrow \text{in polar}
\]
\[
= \{(x, y) : x = r \cos(\theta), y = r \sin(\theta), r = 4 \cdot \cos(\theta)\} \leftarrow \text{in cartesian}
\]

Now eliminate $\theta$ and $r$:

- $x = r \cdot \cos(\theta)$ and $r = 4 \cos(\theta) \implies x = \frac{r^2}{4}$
- $r^2 = x^2 + y^2$ and $x = \frac{r^2}{4} \implies x^2 + y^2 = 4x$

Rearranging gives $\boxed{(x - 2)^2 + y^2 = 2^2}$.

→ equation of circle of radius 2 centered at $(2, 0)$
Exercise (Lieblich, Midterm 1, Spring ’13, Ex 1c)
Find the arc length of the spiral $r = \theta^2$ for the portion of the spiral between $\theta = 0$ and $\theta = 2\pi$.

• Recall: Arc length is $\int_a^b |\vec{r}'(t)| \, dt$

• Convert to Cartesian coordinates: 
  $\vec{r}(\theta) = (\theta^2 \cos(\theta), \theta^2 \sin(\theta))$

• Compute derivatives
  $\frac{d}{d\theta} \left( \theta^2 \cdot \cos(\theta) \right) = 2\theta \cdot \cos(\theta) + \theta^2 \cdot (-\sin(\theta))$
  $\frac{d}{d\theta} \left( \theta^2 \cdot \sin(\theta) \right) = 2\theta \cdot \sin(\theta) + \theta^2 \cdot \cos(\theta)$

• Length of derivative vector is
  $|\vec{r}'(\theta)| = \sqrt{(2\theta \cdot \cos(\theta) - \theta^2 \sin(\theta))^2 + (2\theta \cdot \sin(\theta) + \theta^2 \cdot \cos(\theta))^2}$
  $= \sqrt{4\theta^2 \cos(\theta)^2 + \theta^4 \sin(\theta)^2 + 4\theta^2 \sin(\theta)^2 + \theta^4 \cos(\theta)^2}$
  $= \sqrt{4\theta^2 + \theta^4} = \theta \sqrt{\theta^2 + 4}$

• Integrate
  $\int_0^{2\pi} |\vec{r}'(\theta)| \, d\theta = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} \, d\theta = \frac{1}{3} \left( \theta^2 + 4 \right)^{3/2} \bigg|_0^{2\pi}$
  $= \frac{1}{3} \left( ((2\pi)^2 + 4)^{3/2} - 4^{3/2} \right) \approx 92.896$

Recall that $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2}$ and hence
$\int 2x \sqrt{x^2 + 4} \, dx = \frac{2}{3} (x^2 + 4)^{3/2}$.  

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The spiral from the last exercise:

\[ \vec{r}(0) = (0, 0) \]
\[ \vec{r}(2\pi) = (4\pi^2, 0) \]