Dot product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

For all vectors $\vec{a}, \vec{b}$ their **angle** $\theta$ satisfies

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$
Def. The **cross product** of \( \vec{a} = (a_1, a_2, a_3) \) and \( \vec{b} = (b_1, b_2, b_3) \) is

\[
\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)
\]

\[
= \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]

Properties of cross product:

- \( \vec{a} \times \vec{b} \) is orthogonal to both, \( \vec{a} \) and \( \vec{b} \).

- \( |\vec{a} \times \vec{b}| \) is the **area** of the parallelogram spanned by \( \vec{a} \) and \( \vec{b} \).

- Angle \( \theta \in [0, \pi] \) between \( \vec{a} \) and \( \vec{b} \) satisfies

\[
|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)
\]
Lines in space

The line through the point \( P_0 = (x_0, y_0, z_0) \) with direction vector \( \vec{v} = (a, b, c) \) has

- parametric equations
  \[
  x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct
  \]

- vector equation
  \[
  \vec{r}(t) = (x_0 + at, y_0 + bt, z_0 + ct)
  \]

- symmetric equations
  \[
  \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
  \]
**Def:** The plane with **normal vector** \( \vec{n} = (a, b, c) \), containing point \( P_0 = (x_0, y_0, z_0) \) is

\[
\{ P \in \mathbb{R}^3 : P \cdot \vec{n} = P_0 \cdot \vec{n} \}
\]

\[
= \{ (x, y, z) : ax + by + cz = ax_0 + by_0 + cz_0 \}
\]

The **angle between two planes** is the angle between their normal vectors (has to be \( \leq \frac{\pi}{2} \)).

Please see textbook for other surfaces!
Tangent lines

A vector function \( \vec{r}(t) = (x(t), y(t)) \) describes a curve in \( \mathbb{R}^2 \).

The derivative is \( \vec{r}'(t) = (x'(t), y'(t)) \).

The tangent line at \( t_0 \) is \( \{ \vec{r}(t_0) + s\vec{r}''(t_0) : s \in \mathbb{R} \} \).

The slope of the tangent line at \( t_0 \) is \( \frac{y'(t_0)}{x'(t_0)} \).

The unit tangent vector to \( \vec{r}(t) \) is

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}
\]
Arc length

Def + Lemma: A path described by
\[ \vec{r}(t) = (x(t), y(t), z(t)), \quad a \leq t \leq b \] has arc length

\[ L = \int_{a}^{b} |\vec{r}'(t)| \, dt \]

Here we need that \( x'(t), y'(t), z'(t) \) exist and are continuous and the curve is traversed exactly once.

Intuition:

\[ \text{travelled distance} = \int_{t_{\text{start}}}^{t_{\text{end}}} \text{speed}(t) \, dt \]
Conversion between cartesian and polar coordinates

Given polar coordinates \((r, \theta)\)
⇒ cartesian coordinates are \(x = r \cdot \cos(\theta),\ y = r \cdot \sin(\theta)\)

Given cartesian coordinates \((x, y)\)
⇒ polar coordinates are

- \(\theta = \tan^{-1}\left(\frac{y}{x}\right)\) (gives always \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\))
- \(r = \pm \sqrt{x^2 + y^2}\).

**Warning:** Case split needed:

- If \(x > 0\), then choose \(r = \sqrt{x^2 + y^2}\)
- If \(x < 0\), then choose \(r = -\sqrt{x^2 + y^2}\)
- If \(x = 0\), then \(\tan^{-1}\left(\frac{y}{x}\right)\) undefind. Choose \(\theta = \frac{\pi}{2}\), \(r = y\)
Curvature

For a path $\vec{r}(t)$ with smooth parametrization (i.e. $\vec{r}'$ exists and is continuous), the curvature $\kappa(t)$ can be expressed as:

- If $|\vec{r}''(t)| = 1$ for all $t$ the curvature is
  \[ \kappa(t) = |\vec{r}''(t)| \]

- For any path the curvature is
  \[ \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \]

- For any path in $\mathbb{R}^3$, the curvature is
  \[ \kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}''(t)|^3} \]

Remark: Last formula is most convenient. For a curve $(x(t), y(t))$ in $\mathbb{R}^3$, just apply to $(x(t), y(t), 0)$. 