Lecture 7 - Monday, April 14

PARAMETRIC CURVES AND VECTOR FUNCTIONS (§10.1, §13.1)

Example: $x = \cos(t), y = -\sin(t)$ for $0 \leq t \leq \pi$.

Observe: Point $(x, y)$ on curve has $x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$. That means the curve will be contained in unit circle.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>

![Diagram](image)

$x(t) = \cos(t)$

$y(t) = -\sin(t)$
We can express this as a vector function of $t$:

$$\vec{r}(t) = (\cos(t), -\sin(t))$$

Its derivative is

$$\vec{r}'(t) = (-\sin(t), -\cos(t))$$

The derivative gives a tangent vector to the curve. For example at $t = \frac{\pi}{4}$ one has

$$\vec{r}'\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \vec{r}'\left(\frac{\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$
In general, if \( \vec{r}(t) = (x(t), y(t)) \), then

\[
\vec{r}'(t) = (x'(t), y'(t)) = \left( \frac{dx}{dt}, \frac{dy}{dt} \right)
\]
gives a tangent vector.

The slope of the tangent line is as usual \( \frac{dy}{dx} \).

We use the chain rule:

\( y \) is a function of \( x \)
\( x \) is a function of \( t \)

\[ \Rightarrow \ y \text{ is also a function of } t \]

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}
\]

The slope of the tangent line is

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}
\]

In the example \( \vec{r}(t) = (\cos(t), -\sin(t)) \),

\[
\vec{r}'(t) = (-\sin(t), -\cos(t))
\]

\[ = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = \frac{dy}{dt} \]

The slope of the tangent line is \( \frac{dy}{dx} = -\cos(t) = -\frac{\cos(t)}{\sin(t)} \).

For example for \( t = \frac{\pi}{4} \), the slope of the tangent is 1.
Example: For \( \vec{r}(t) = (2 \sin(3t), 2 \cos(3t)) \) \( (0 \leq t \leq 2\pi) \), describe the path and the equation of the tangent line at \( t = \frac{\pi}{12} \).

Note: \( x^2 + y^2 = 4 \sin^2(3t) + 4 \cos^2(3t) = 4 \). Hence the path is contained in the circle of radius 2.

We start at \( \vec{r}(0) = (0, 2) \) and end at \( \vec{r}(2\pi) = (0, 2) \). We pass through the point \( (0, 2) \) also at \( t = \frac{2}{3}\pi \) and at \( t = \frac{4}{3}\pi \) — that means we go around the circle 3 times. Right after \( t = 0 \), the \( x \)-coordinate increases and the \( y \)-coordinate decreases. So we must travel clockwise.
Example (cont.) Tangent line at $t = \frac{\pi}{12}$:

\[
\vec{r}'(t) = (2 \sin(3t), 2 \cos(3t)) = (2 \cdot \frac{1}{\sqrt{2}}, 2 \cdot \frac{1}{\sqrt{2}}) = (\sqrt{2}, \sqrt{2})
\]

\[
\vec{r}''(t) = (3\sqrt{2}, -3\sqrt{2})
\]

Slope of tangent line is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -1$.

Tangent line goes through $(\sqrt{2}, \sqrt{2})$ and has slope $-1$.
We set $y - \sqrt{2} = -(x - \sqrt{2})$ and see that the equation for the tangent line must be

\[
y = -x + 2\sqrt{2}
\]