Problem Set 4

409 - Discrete Optimization

Winter 2014

Exercise 1
For this exercise, let $G = (V, E)$ be an undirected graph with edge cost $c : E \to \mathbb{R}$. We assume here that the edges have distinct cost and are sorted so that $c(e_1) < c(e_2) < \ldots < c(e_m)$.

a) Under this assumption, show that the optimum MST is unique!

Hint: In the proof of the “swapping lemma” in the lecture we have implicitly shown the following: for two spanning trees $T, T^*$ and any edge $e \in T\setminus T^*$, there is always an edge $f \in T^*\setminus T$ so that both, $(T\setminus \{e\}) \cup \{f\}$ and $(T^*\setminus \{f\}) \cup \{e\}$ are spanning trees. You may use this fact without reproving it.

b) Consider the cost function $c'(e_i) = i$. Show that a spanning tree $T$ is optimal w.r.t. $c$ if and only if it is optimal w.r.t. cost function $c'$.

Hint: Consider how Kruskal’s algorithm behaves on both cost functions.

c) Suppose we use the following alternative algorithm to compute a minimum spanning tree:

Algorithm: Compute any spanning tree. Now perform arbitrary edge swaps until the tree is optimal.

Prove that no matter which edge swaps you choose, after at most $n^3$ edge swaps you arrive at an optimal tree.

Hint: Use b).

Exercise 2
Consider the following directed graph $G = (V, E)$ (edges are labelled with edge cost $c(e)$).

a) Use the Bellmann-Moore-Ford algorithm to compute the distances from $s$ to each other node $v$ (call those values $\ell(v)$; you can use your favourite ordering for the edges).

b) Use the labels $\ell(v)$ from a) as potentials $\pi(v)$. In the above graph, label the nodes with their potential and label the edges with their reduced costs. Are the potentials feasible? Is $c$ conservative?
c) Let $c_\pi(e)$ be the reduced cost of edge $e$ that you computed in $b$). Now run Dijkstra’s algorithm with cost function $c_\pi$ and source node $a$. Use the symbol $\ell'(v)$ to denote the computed $a$-$v$ distances.

d) How do you translate the values $\ell'(v)$ from $c$) into the actual $a$-$v$ distances w.r.t. to the original cost function $c$?