Chapter 3

Matroid Intersection

This chapter is a reproduction of a section in Lex Schrijver's lecture notes, with somewhat more details.

3.1 Introduction

In a previous chapter of this course, we learned what a *matroid* is. It is a pair $M = (X, \mathcal{I})$ where X is called the *groundset* and $\mathcal{I}$ are subsets of X that are also called the *independent sets*. Additionally, the matroid has to satisfy the following three axioms:

1. *Non-emptyness:* $\emptyset \in \mathcal{I}$
2. *Monotonicity:* If $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$
3. *Exchange property:* If $Y, Z \in \mathcal{I}$ with $|Y| < |Z|$, then there is an $x \in Z/Y$ so that $Y \cup \{x\} \in \mathcal{I}$

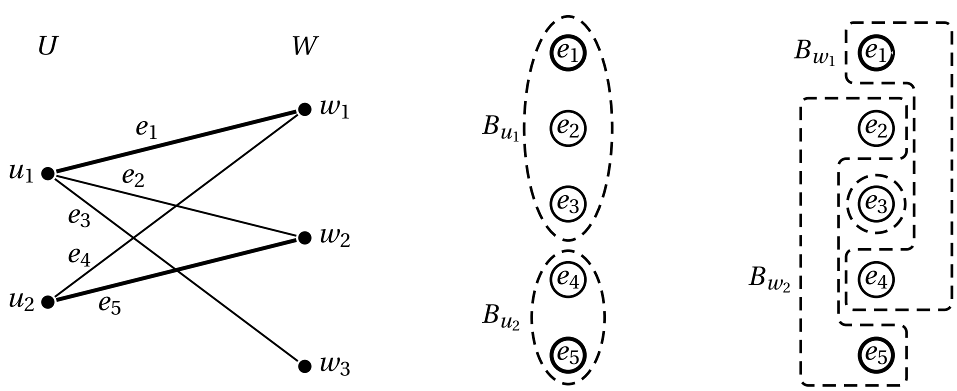
Examples for matroids are:

- The set of forests in an undirected graph form a *graphical matroid*.
- If $v_1, \ldots, v_n$ are vectors in a vector space, then $M = ([n], \mathcal{I})$ with $\mathcal{I} = \{I \subseteq [n] \mid \{v_i\}_{i \in I}$ linearly independent$\}$ is a *linear matroid*.
- A *partition matroid* with ground set X can be obtained as follows: take any partition $X = B_1 \dot\cup \ldots \dot\cup B_m$ and select numbers $d_i \in \{0, \ldots, |B_i|\}$. Then $M = (X, \mathcal{I})$ with $\mathcal{I} := \{I : |I \cap B_i| \leq d_i$ for all $i = 1, \ldots, m\}$ is a matroid.

We already learned that one can use the greedy algorithm to find a maximum weight independent set. In this chapter, we will see that a way more complex problem also can be solved in polynomial time:

MATROID INTERSECTION
Input: Matroid $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$ on the same groundset
Goal: Find $\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

To understand that this is a non-trivial problem, we want to argue that it contains *maximum bipartite matching* as a special case. To see this, take any bipartite graph $G = (V, E)$. Suppose that $V = U \cup W$ with $U = \{u_1, \ldots, u_{|U|}\}$ and $W = \{w_1, \ldots, w_{|W|}\}$ are both sides. Then we can define two matroids that both have the edge set E as ground set as follows: take $M_1 = (E, \mathcal{I}_1)$ as the partition matroid with partitions $\delta(u_1), \ldots, \delta(u_{|U|})$, all with parameter $d_i := 1$. Similarly, we introduce $M_2 = (E, \mathcal{I}_2)$ as partition matroid with partitions $\delta(w_1), \ldots, \delta(w_{|W|})$. Now the matroid intersection problem asks to select as many edges as possible, where in each neighborhood $\delta(u_i)$ and $\delta(w_j)$ we select at most one edge. This is exactly maximum bipartite matching. See the figure below for an example:



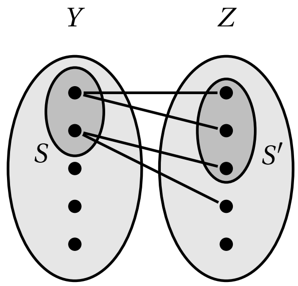
bipartite graph $G = (V, E)$ $\quad$ M_1 $\quad$ M_2

3.2 The exchange lemma

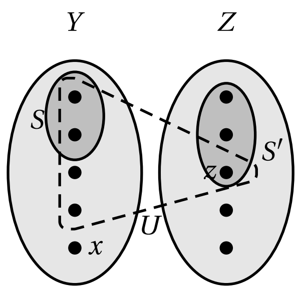
For example if we have two spanning trees T_1, T_2 in a graph, then the exchange property implies that for any $e \in T_1$, there exists *some* edge $f(e) \in T_2$ so that $(T_1 \setminus \{e\}) \cup f(e)$ is again a spanning tree. Now we will see that a stronger property is true: the map $f : T_1 \to T_2$ can be chosen to be bijective.

Lemma 3.1. *Let $M = (X, \mathcal{I})$ be a matroid and let $Y, Z \in \mathcal{I}$ be disjoint independent sets of the same size. Define a bipartite exchange graph $H = (Y \cup Z, E)$ with $E = \{(y, z) : (Y \setminus y) \cup z \in \mathcal{I}\}$. Then H contains a perfect matching.*

Proof. Suppose for the sake of contradiction that H has no perfect matching. From *Hall's condition* we know that there must be subsets $S \subseteq Y$ and $S' \subseteq Z$ so that all edges incident to S' must have their partner in S and $|S| < |S'|$.



Since $|S| < |S'|$ and S, S' are both independent sets, there is an element $z \in S'$ so that $S \cup \{z\} \in \mathcal{I}$. We can keep adding elements from Y to $S \cup \{z\}$ until we get a set $U \subseteq Y \cup \{z\}$ with $|U| = |Y|$.



There is exactly one element in $Y \setminus U$; we call it x. Then $(Y / x) \cup \{z\} = U \in \mathcal{I}$ and $(x, z) \in E$ would be an edge — a contradiction. □

We will use that *exchange graph* more intensively later. Formally, for a matroid $M = (X, \mathcal{I})$ and an independent set $Y \in \mathcal{I}$, we can define $H(M, Y)$ as the bipartite graph with partitions Y and $X \setminus Y$ where we have an edge between $y \in Y$ and $x \in X \setminus Y$ if

$$(Y \setminus y) \cup \{x\} \in \mathcal{I}.$$

3.3 The rank function

Again, let $M = (X, \mathcal{I})$ be a matroid. Recall that an inclusionwise maximal independent set is called a *basis*. Moreover, all bases have the same size which is also called the *rank* of a matroid. One can generalize this to the *rank function* $r_M : 2^X \to \mathbb{Z}_{\geq 0}$ which is defined by

$$r_M(S) := \max\{|Y| : Y \subseteq S \text{ and } Y \in \mathcal{I}\}$$

which for a subset $S \subseteq X$ of the groundset, tells how many independent elements one can select from S.

Now suppose we have two matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ over the same groundset. The rank function will be useful to decide at some point that we have found the largest joint independent set. Let us make the following observation:

Lemma 3.2. *Let $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$ with rank functions r_1 and r_2. Then for any independent set $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and any set $U \subseteq X$ one has*

$$|Y| \leq r_1(U) + r_2(X/U).$$

Proof. We have

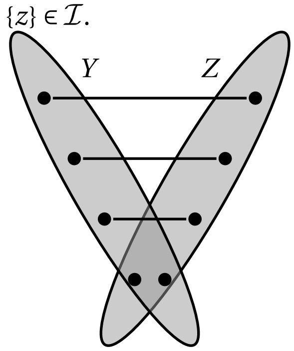
$$|Y| = \underbrace{|U \cap Y|}_{\leq r_1(U)} + \underbrace{|(X/U) \cap Y|}_{\leq r_2(X/U)} \leq r_1(U) + r_2(X/U).$$

using that Y is an independent set in both matroid. □

Later in the algorithm, we will see that this inequality is tight for some Y and U. As a side remark, for partition matroids in bipartite graphs, the lemma coincides with the fact that a vertex cover is always an upper bound to the size of any matching.

3.4 An reverse exchange lemma

We just saw that the exchange graph has a perfect matching between independent sets of the same size. We now show the converse, namely that a unique perfect matching between an independent set Y and any set Z implies that Z is also independent. In the following, we will consider perfect matchings in the graph $H(M, Y)$ between $Y \Delta Z$. What we mean is a perfect matching N, matching nodes in $Y \setminus Z$ to nodes in $Z \setminus Y$ and each edge $(y, z) \in N$ satisfies $(Y \setminus y) \cup \{z\} \in \mathcal{I}$.

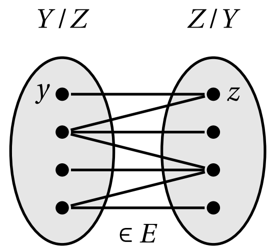


Lemma 3.3. *Let $M = (X, \mathcal{I})$ be a matroid and let $Y \in \mathcal{I}$ be an independent set and let $Z \subseteq X$ be any set with $|Z| = |Y|$. Suppose that there exists a unique perfect matching N in $H(M, Y)$ between $Y \Delta Z$. Then $Z \in \mathcal{I}$.*

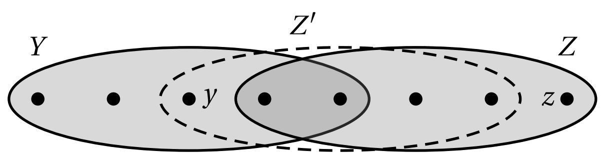
Proof. Let $E = \{(y, z) \in (Y \setminus Z) \times (Z \setminus Y) \mid (Y/y) \cup \{z\} \in \mathcal{I}\}$ be all the exchange edges between $Y \setminus Z$ and $Z \setminus Y$.
Claim: E has a leaf[1] $y \in Y/Z$.
Proof of claim: By assumption there is a perfect matching $N \subseteq E$. Start at any node $w \in Y \Delta Z$. If you are on the "right side" $Z \setminus Y$, then move along a matching edge in N; if we are on the left hand side $Y \setminus Z$, take a non-matching edge. If we every revisit a node,

[1] Recall that a *leaf* is a degree-1 node.

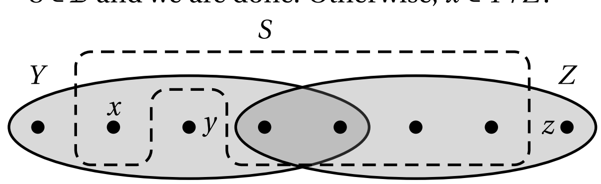
then we have found an even length path $C \subseteq E$ that alternates between matching edges and non-matching edges. Hence $N \Delta C$ is again a perfect matching, which contradicts the uniqueness. That implies that our path will not revisit a node, but that it will get stuck at some point. It cannot get stuck at a node in Z/Y because there is always a matching edge incident. Hence it can only get stuck at a node $y \in Y/Z$ that is only incident to one edge (y,z) and that edge must be in N. $\square$



Let z denote the element with $(y,z) \in N$. Note that $Z' := (Z \setminus z) \cup \{y\}$ satisfies $|Y \Delta Z'| = |Y \Delta Z| - 2$ and there is still exactly one perfect matching between $Y \Delta Z'$ (which is $N \setminus \{(y,z)\}$). Hence we can apply induction and assume that $Z' \in \mathcal{I}$.



We know that $r((Y \cup Z) \setminus y) \geq r((Y \setminus y) \cup \{z\}) = |Y|$. By the matroid exchange property, there is some element $x \in (Y \cup Z)/y$ so that $S := (Z'/y) \cup \{x\}$ is an independent set of size $|Y|$. If $x = z$ then $Z = S \in \mathcal{I}$ and we are done. Otherwise, $x \in Y/Z$.



As $|S| > |Y \setminus y|$, there must be an exchange edge between y and a node in S/Y. That contradicts the choice of y. $\square$

3.5 The algorithm

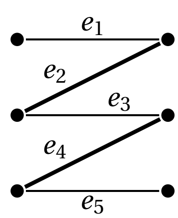
Now, suppose that we have two matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ over the same ground set. Our algorithm starts with the independent set $Y := \emptyset$ and then augments it iteratively. Suppose we already have some joint independent set $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$. We will show how to either find another set $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|Y'| = |Y| + 1$ or decide that Y is already optimal. Let us define sets

$$X_1 := \{y \in X \setminus Y \mid Y \cup \{y\} \in \mathcal{I}_1\} \quad \text{and} \quad X_2 := \{y \in X \setminus Y \mid Y \cup \{y\} \in \mathcal{I}_2\}$$

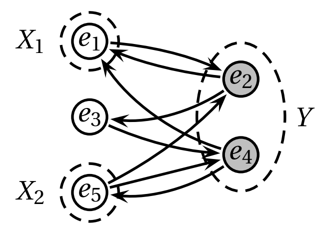
In other words, X_1 denotes the elements that could be added to the independent set Y so that we would still have an independent set in M_1. We define a directed graph $H = (X, E)$ as follows: for all $y \in Y$ and $x \in X / Y$

$$(y, x) \in E \quad \Leftrightarrow \quad (Y / y) \cup \{x\} \in \mathcal{I}_1$$
$$(x, y) \in E \quad \Leftrightarrow \quad (Y / y) \cup \{x\} \in \mathcal{I}_2$$

Let us check what this graph does for bipartite graphs (and M_1, M_2 are the partition matroids modelling both sides). In this case Y corresponds to a matching, X_1 are edges whose left-side node is unmatched by Y and X_2 are edges whose right-side node is unmatched. We also observe that a Y-augmenting path corresponds to a directed path in H.



original graph



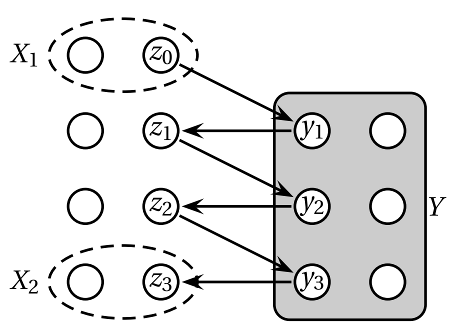
exchange graph H

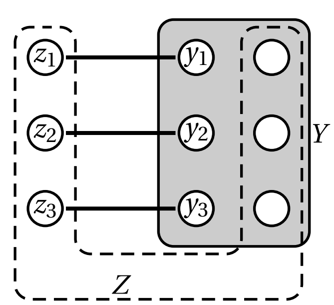
With a bit care, we can use the concept of augmenting paths also for general matroid.

Lemma 3.4. *Suppose there exists a directed path $z_0, y_1, z_1, \ldots, y_m, z_m$ starting at a vertex $z_0 \in X_1$ and ending at a node $z_m \in X_2$. If that is a* <u>*shortest*</u> *path, then*

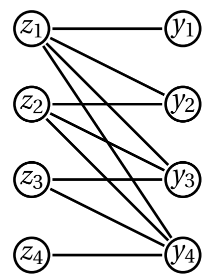
$$Y' := (Y \setminus \{y_1, \ldots, y_m\}) \cup \{z_0, \ldots, z_m\} \in \mathcal{I}_1 \cap \mathcal{I}_2$$

Proof. We will show that $Y' \in \mathcal{I}_1$, the other inclusion follows by symmetry. On the figure below, on the left hand side, we consider the directed path and on the right hand side, we consider only edges E of the exchange graph $H(M_1, Y)$ that run between $Y \setminus Z$ and $Z \setminus Y$ for $Z := (Y \setminus \{y_1, \ldots, y_m\}) \cup \{z_1, \ldots, z_m\} = Y' \setminus y_0$.





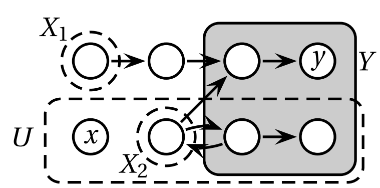
Note that the edges $\{(z_i, y_i) : i = 1, \ldots, m\}$ from the directed path form a perfect matching on $Y \Delta Z$. While E may contain more edges than that, it does not contain a *coord*, which is an edge (y_i, z_j) with $j > i$. The reason is that in this case our X_1-X_2 path would not have been the shortest possible one as we could have used the coord as shortcut. Now, consider the "complete" cordless graph $E^* := \{(y_i, z_j) : i \geq j\}$. Then this graph does have only one perfect matching. In particular, (y_1, z_1) has to be in a matching — then apply induction.



As the matching on $Y \Delta Z$ is unique, by Lemma 3.3 we have $Z = Y'/z_0 \in \mathcal{I}_1$. We know that $r_{M_1}(Y \cup Y') \geq r_{M_1}(Y \cup \{z_0\}) \geq |Y| + 1$ since $z_0 \in X_1$ is one of the "M_1-augmenting" elements. One the other hand $r_{M_1}(Y \cup Y'/\{z_0\}) \leq |Y|$ as none of the other elements of Y' is in X_1 (here we use again that we have a shortest path). Hence, the only element that could possibly augment Y'/z_0 to an independent set of size $|Y| + 1$ is z_0 itself. □

Lemma 3.5. *Suppose there is no path from a node in X_1 to a node in X_2. Then Y is optimal. In particular we can find a subset $U \subseteq X$ so that $|Y| = r_{M_1}(U) + r_{M_2}(X \setminus U)$.*

Proof. Let $U := \{i \in X : \nexists X_1 - i \text{ path in } H\}$ (or maybe more intuitively, $X \setminus U$ are the nodes that are reachable from X_1).



First, we claim that $r_{M_1}(U) = |Y \cap U|$. One direction is easy: $r_{M_1}(U) \geq r_{M_1}(U \cap Y) = |U \cap Y|$. For the other direction, suppose for the sake of contradiction that $r_{M_1}(U) > |Y \cap U|$ and hence there is some $x \in U$ so that $(Y \cap U) \cup \{x\}$ is an independent set of size $|Y \cap U| + 1$. There are two case depending on whether or not x also increases the rank of Y itself:

- *Case* $r_{M_1}(Y \cup \{x\}) = |Y| + 1$. Then $x \in X_1 \cap U$, which is a contradiction to the choice of U.

- *Case:* $r_{M_1}(Y \cup \{x\}) = |Y|$. Take a maximal independent set Z with $(Y \cap U) \cup \{x\} \subseteq Z \subseteq Y \cup \{x\}$. Then there is exactly one element $y \in Y/U$, so that $Z = (Y/y) \cup \{x\}$. This implies that we have would contain a directed edge (y,x). Then the node $x \in U$ is reachable from a element $y \notin U$, which contradicts the definition of U.

From the contradiction we obtain that indeed $r_{M_1}(U) = |Y \cap U|$. Similarly one can show that $r_{M_2}(X/U) = |Y \cap (X/U)|$ (which we skip for symmetry reasons). Overall, we have found a set U so that $|Y| = |Y \cap U| + |Y \cap (X \setminus U)| = r_{M_1}(U) + r_{M_2}(X \setminus U)$. $\square$

It follows that:

Theorem 3.6. *Matroid intersection can be solved in polynomial time.*

Proof. Start from $Y := \emptyset$ and iteratively construct the directed exchange graph; compute shortest X_1-X_2 paths and augment Y as long as possible. $\square$

The matroids that we have seen so far, all had some explicit representation. Note that the matroid intersection algorithm would work also in the *black box model*, where the only information that we have about the matroids is given by a so-called *independence oracle*. This is method that receives a set $Y \subseteq X$ and simply answers whether or not this is an independent set.

Our algorithm provides a nice min-max formula for the size of joint independent sets:

Theorem 3.7 (Edmond's matroid intersection theorem)**.** *For any matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ one has*

$$\max\{|S| : S \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{U \subseteq X}\{r_{M_1}(U) + r_{M_2}(X \setminus U)\}$$

Proof. We saw the inequality "≤" already in Lemma 3.2. When the matroid intersection algorithm terminates, then it has found a set U providing equality. $\square$