Lecturer: Thomas Rothvoss Due date: Friday, Nov 18, 2022, 11pm, on GradeScope

Problem Set 7

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise 4.15 (10pts)
Let $D = (V, A)$ be a directed graph and let $\mathcal{C}$ be the collection of directed circuits in D. For each directed circuit C in D, let χ^C be the incidence vector of C, that is $\chi^C : A \to \{0, 1\}$ with $\chi^C(a) = 1$ if C traverses a and $\chi^C(a) = 0$ otherwise. Show that f is a non-negative circulation if and only if there exists a function $\lambda : \mathcal{C} \to \mathbb{R}_{\geq 0}$ so that $f = \sum_{C \in \mathcal{C}} \lambda(C) \cdot \chi^C$. That is, the nonnegative circulations form the cone generated by $\{\chi^C \mid C \in \mathcal{C}\}$.

Exercise (Not in Schrijver – 10pts)
First answer the following:

(i) Let $D' = (V, A')$ be a directed graph with capacities c' and $s, t \in V$. We call an s-t flow $g : A' \to \mathbb{R}_{\geq 0}$ *elementary* if there is a single s-t path P in D' so that

$$g(a) = \begin{cases} \text{value}(g) & \text{if } a \in P \\ 0 & \text{if } a \notin P \end{cases}$$

Now let f^* be a maximum s-t flow in D under c'. Prove that there is an elementary s-t flow g under c' with $\text{value}(g) \geq \frac{1}{|A'|}\text{value}(f^*)$.

Now let $D = (V, A)$ be a directed graph with integral capacities $c : A \to \mathbb{Z}_{\geq 0}$, distinguished vertices $s, t \in V$ and for the sake of simplicity suppose that for each arc $a \in A$ one has $a^{-1} \notin A$. Recall that $D_f = (V, A_f)$ is the residual graph for an s-t flow f. We can also define *residual capacities* $c_f(a) := c(a) - f(a)$ for $a \in A$ with $f(a) < c(a)$ and $c_f(a^{-1}) := f(a)$ for $a \in A$ with $f(a) > 0$.

(ii) Let f be any flow under c and let f^* be a maximum value s-t flow. Prove that the residual graph D_f contains an s-t path P where $c_f(a) \geq \frac{1}{2|A|}(\text{value}(f^*) - \text{value}(f))$ for all $a \in P$.

(iii) Consider the modification of the Ford-Fulkerson algorithm where in each iteration we pick an s-t path P that maximizes $\min\{c_f(a) : a \in P\}$ where f is the current flow. Prove that this algorithm takes at most $O(|A| \ln(2\text{value}(f^*)))$ many iterations.

Exercise (Not in Schrijver – 10pts)
Let $D = (V, A)$ be a directed graph with capacities $c(a) := 1$ for all $a \in A$. Let $s, t \in V$ and assume that $\delta^{out}(t) = \emptyset$. Let $\delta^{in}(t) = \{a_1, \ldots, a_m\}$ be the arcs incoming to t. Define $M = (X, \mathcal{I})$ with $X := \{a_1, \ldots, a_m\}$ and $\mathcal{I} := \{\{a_i \in X : f(a_i) = 1\} \mid f \text{ is an } s\text{-}t \text{ flow under } c \text{ in } D\}$. Prove that M is a matroid!

Remark. The first exercise is taken from A. Schrijver's lecture notes.