MASTER FUNDS IN PORTFOLIO ANALYSIS WITH GENERAL DEVIATION MEASURES

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Abstract

Generalized measures of deviation are considered as substitutes for standard deviation in a framework like that of classical portfolio theory for coping with the uncertainty inherent in achieving rates of return beyond the risk-free rate. Such measures, derived for example from conditional value-at-risk and its variants, can reflect the different attitudes of different classes of investors. They lead nonetheless to generalized one-fund theorems in which a more customized version of portfolio optimization is the aim, rather than the idea that a single "master fund" might arise from market equilibrium and serve the interests of all investors.

The results that are obtained cover discrete distributions along with continuous distributions. They are applicable therefore to portfolios involving derivatives, which create jumps in distribution functions at specific gain or loss values, well as to financial models involving finitely many scenarios. Furthermore, they deal rigorously with issues that come up at that level of generality, but have not received adequate attention, including possible lack of differentiability of the deviation expression with respect to the portfolio weights, and the potential nonuniqueness of optimal weights.

Moreover they address in detail a phenomenon largely neglected in the past, namely that if the risk-free rate lies above a certain threshold, a master fund of the usual type will fail to exist and need to be replaced by one of an alternative type, representing a "net short position" instead of a "net long position" in the risky instruments. They show there can sometimes even be an interval for the risk-free rate in which no master fund of either type exists. A notion of basic fund, in place of master fund, is introduced get around this difficulty and serve as a single guide to optimality regardless of such circumstances.

Keywords: *deviation measures, risk measures, value-at-risk, conditional value-at-risk, portfolio optimization, one-fund theorems, master funds, basic funds, efficient frontiers, convex analysis.*

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1 Introduction

In classical portfolio theory, investors respond to the uncertainty of profits by selecting portfolios that minimize variance, or equivalently standard deviation, subject to achieving a specified level in expected gain [16, 10, 15]. The well known "one-fund theorem" [29, 27] stipulates that this can be accomplished in terms of a single "master fund" portfolio by means of a formula which balances the amount invested in that portfolio with the amount invested at the current risk-free rate. Nowadays, other approaches to uncertainty have gained in popularity. Portfolios are being selected on the basis of percentile characteristics such as *value-at-risk* (VaR), *conditional value-at-risk* (CVaR), or other properties proposed for use in risk assessment; cf. [2, 11, 13] and earlier alternatives such as in [6]. These measures have no pretension to being universal, however; VaR and CVaR depend, for instance, on the specification of a confidence level parameter, which could vary among investors. Instead, what is apparent in the alternative approaches currently being touted is a move toward a kind of partial *customization* of responses to risk, while still avoiding, as impractical, a reliance on specifying individual utility functions.

A question in this evolving environment is the extent to which parallels to the classical theory persist when the minimization of standard deviation is replaced by something else. Researchers have already looked into the possibilities in several special cases, under various limiting assumptions (recognized explicitly or imbedded implicitly). Our goal, in contrast, is to demonstrate that important parallels with classical theory exist much more broadly, despite technical hurdles, and in this way to bring out features that have not completely been analyzed, or even perceived, in the past.

We focus on the general *deviation measures* we developed axiomatically in [23]. Our idea is to substitute such a deviation measure for standard deviation in the setting of classical theory and investigate the consequences rigorously in detail. Furthermore, we aim at doing so, for the first time, in cases where the rates of return may have discrete distributions, or mixed discrete-continuous distributions (which can arise from put and call options), as well as cases where they have continuous distributions.

The deviation measures we work with are paired one-to-one, through a simple transformation involving expectations, with *risk measures* belonging a class which is close to that of Artzner, Delbaen, Eber and Heath [5], but different in partly relaxing their requirements while insisting on a property beyond theirs. A similar property was invoked by Ogryczak and Ruszczynski in [17, 18] for *safety measures*, which may be viewed as negatives of risk measures. Minimizing a deviation measure subject to a constraint on expected returns can anyway be different from minimizing the corresponding risk measure, since, as shown in [23], the first problem always has a solution but the second problem can sometimes fail to have a solution, due to a phenomenon of "acceptably free lunches."

Our axioms cover numerous choices of a deviation measure, from classical type to CVaR type, but they leave out deviation measures of VaR type, which lack convexity. Convexity is essential for answering most of the harder questions that confront us. Its importance for sound applications in finance has already been recognized as well in connection with the coherency concept in [5]. We do, however, try to indicate along the way the troubles that VaR type deviations would bring up.

The first of our main results says that a one-fund theorem holds regardless of the particular choice of the deviation measure, but with certain modifications. The optimal risky portfolio need not always be unique, and it might not always be expressible by a "master fund" as traditionally conceived, even when only standard deviation is involved. An alternative concept of "basic fund" is introduced to fill the gap. The rest of our main results pin down precisely the degree to which basic funds can, or cannot, be rescaled into master funds. This turns out to require an understanding not only of an efficient frontier for risky portfolios at price 1, associated with "master funds of positive type," but also of such a frontier for risky portfolios at price -1, associated with "master funds of negative type." The size of the risk-free rate of return plays a key role here. We prove that when it is below a certain threshold, the usual positive type prevails, but when it is above a certain threshold, the negative type has to be brought in. Moreover in special situations those thresholds can differ, leaving a gap filled by an interval of values of the risk-free rate for which neither type of master fund can replace a basic fund. We also explain how thresholds can be calculated by solving an auxiliary optimization problem.

It deserves emphasis that, in contrast to much of the previous work in this area, our results are obtained without relying on the existence of densities for the statistical distributions that arise, or even on the continuity of the distribution functions, which would preclude applications to discrete random variables or effects tied to options. We do not take for granted, or require, the differentiability of the deviation with respect to the parameters specifying the relative weights of the instruments in the portfolio. This is not merely for the sake of technical generality. An example provided in the final section of the paper illustrates how put and call options in portfolios can lead to nondifferentiability as well as to a threshold gap for the risk-free rate. Therefore no theory, unless it faces up to such troubles, can be regarded as fully applicable to portfolios involving derivative instruments.

With standard calculus being inadequate for the problems at hand, we have had to rely instead on techniques of convex analysis [19] while adhering strictly to the fundamental principles of optimization theory.[3]

The need for a "negative" efficient frontier referring to "net short positions," along with the usual "positive" one for "net long positions," is not surprising, in view of the diversity of measures that investors may be using. In line with their different opinions about risk, some investors may find the risk-free rate high enough to warrant borrowing from the market and investing that money risk-free, while others will prefer a fund in which the "longs" outweigh the "shorts." An interesting analogy can be found in [26, p. 507] in terms of a stock index futures contract which might even consist entirely of short positions.

The emergence of a variety of different master funds, optimal for different deviation measures, is an inescapable outcome of any theory which, like ours, attempts to cope with the current tendency toward customization in portfolio optimization. A master fund identified with respect to the wishes of one class of investors can no longer be proposed as obviously furnishing input for factor analysis of the market as a whole, because the financial markets react to the wishes of *all* investors. A master fund, in our general sense, can no longer be interpreted as associated with a sort of universal equilibrium. Whether some such master funds, individually or collectively, might nonetheless turn out to be valuable in factor analysis, is an issue outside the scope of this paper. CAPM-like covariance relations do indeed come out of the optimality conditions that characterize our master funds (as can be gleaned from the optimality prescriptions in [23]), but we reserve this development, requiring further elaboration of underpinnings in convex analysis, to a follow-up paper [25].

[3] Lack of familiarity with the mathematics of optimization has been a handicap in some of the finance literature in this area, going all the way back to Markowitz. In his pioneering work [16], for example, he excluded short positions by constraining the portfolio weights to be nonnegative. He neglected, however, to take into account that Lagrange multipliers for those inequality constraints could come into play, in which case a closed-form solution to the optimality conditions for a master fund would be impossible. Supposing that the multipliers can be taken to be zero is equivalent (because of convexity) to supposing that, if shorting were allowed, there would anyway be no short positions at optimality. There is no support for that conviction, however, and indeed, numerical calculations are known to produce quite different answers when shorting is allowed and when it is not. Similar looseness about whether solutions to optimization problems even exist must be the reason why the magnitude of the risk-free rate was not perceived to have an effect, and the need for master funds representing net short positions went undetected. The need for allowing at least some short positions as possibilities in a master fund was emphasized by [26, pp. 500, 505].

2 Deviation and Risk

We start by reviewing what we mean by "deviation measures" and explaining how they are related to "risk measures" in the terminology of Artzner et al. [5]. Roughly speaking, deviation measures evaluate the degree of nonconstancy in a random variable (i.e., the extent to which outcomes may deviate from expectations), whereas risk measures evaluate overall prospective loss (from the benchmark of zero loss). Deviation measures correspond, under a basic pairing, with risk measures that have an additional "expectation boundedness" property not contemplated in [5]. This pairing is valuable for tranforming known examples of risk measures into examples of deviation measures, as well as in illuminating how the valuable notion of coherency introduced in [5] translates to the context of deviations.

We consider a space Ω, the elements ω of which can represent future states or scenarios (perhaps just finitely many), and suppose it to be supplied with a probability measure P and the other technicalities that make it a legitimate probability space. We treat as random variables (r.v.'s) the (measurable) functions X on Ω for which $E[X^2] < \infty$; the space of such functions will be denoted, for short, by $\mathcal{L}^2(\Omega)$. For X in $\mathcal{L}^2(\Omega)$, the mean $\mu(X)$ and variance $\sigma^2(X)$ are well defined in particular:[4]

$$\begin{aligned} &\mu(X) = EX = \textstyle\int_\Omega X(\omega)dP(\omega), \\ &\sigma^2(X) = E[X - EX]^2 = \textstyle\int_\Omega [X(\omega) - \mu(X)]^2 dP(\omega). \end{aligned} \tag{1}$$

To assist in working with constant r.v.'s, $X(\omega) \equiv C$, the letter C will always denote a constant in the real numbers $I\!R$.

By a *deviation measure* will be meant any functional $\mathcal{D}$ that assigns to each random variable X (understood to be in $\mathcal{L}^2(\Omega)$ always) a value $\mathcal{D}(X)$ in accordance with the following axioms:

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$; equivalently, $\mathcal{D}(X) = \mathcal{D}(X - EX)$ for all X,
(D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda\mathcal{D}(X)$ for all X and $\lambda \geq 0$,
(D3) $\mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(X')$ for all X and X',
(D4) $\mathcal{D}(X) > 0$ for nonconstant X, whereas $\mathcal{D}(X) = 0$ for constant X.

A deviation measure $\mathcal{D}$ will be called *coherent* if, in addition to these four axioms, it satisfies

(D5) $\mathcal{D}(X) \leq EX - \inf X$ for all X.

These axioms come from our paper [23], where the notion of a general deviation measure was first formulated at this level.[5] The equivalence in D1 is evident from taking C to equal $-EX$, and on the other hand, noting that $[X + C] - E[X + C] = X - EX$ for any constant C. Clearly D4 puts the focus of $\mathcal{D}$ on the uncertain part of an r.v. X and insists on this part not going undetected. The combination of D2 and D3 implies the *convexity* of $\mathcal{D}$, which is a key property in all contexts of optimization and makes the tools of convex analysis available to our endeavor.[6] We speak of the additional property in D5 as the *lower-range boundedness* of $\mathcal{D}$.

Coherency will be discussed later in this section. Its chief effect mathematically is to make available, through duality, certain probabilistic interpretations of deviation that can be very helpful in analysis and even in modeling. Such considerations also lie behind our focus on properties of $\mathcal{D}$ with respect to all $\mathcal{L}^2(\Omega)$, instead of just a subspace $\mathcal{X}$ of $\mathcal{L}^2(\Omega)$ that generated by some particular collection of random variables.

[4]Our choice of $\mathcal{L}^2(\Omega)$ is dictated by the need for a framework in which the magnitude of $X - EX$ can be measured by general deviations, with standard deviation as a particular case. In [5] and [7], expectations had no role, and the focus instead was on $\mathcal{L}^\infty(\Omega)$.

[5]In [13], a class of measures was described axiomatically in terms of D1, D2 (effectively), and the version of D4 requiring only weak inequality, but no D3. Such measures lack convexity and other crucial properties.

[6]In the presence of that convexity, D1 is actually implied by the seemingly much weaker condition that $\mathcal{D}(C) = 0$ for all constants C.

The example of standard deviation, $\mathcal{D}(X) = \sigma(X)$, dominates classical portfolio theory and is *symmetric* in the sense that $\mathcal{D}(-X) = \mathcal{D}(X)$. Similar nonsymmetric examples of deviation measures satisfying the axioms include the standard *semideviations* $\mathcal{D}(X) = \sigma_+(X)$ and $\mathcal{D}(X) = \sigma_-(X)$, where

$$\sigma_-^2(X) = E[\max\{EX - X, 0\}^2], \qquad \sigma_+^2(X) = E[\max\{X - EX, 0\}^2]. \tag{2}$$

The first of these emphasizes the downside of X, while the second emphasizes the upside. A very different pair of examples, likewise oriented to downside or upside, is furnished by the *lower range* and the *upper range*,

$$\mathcal{D}(X) = EX - \inf X, \qquad \mathcal{D}(X) = \sup X - EX, \tag{3}$$

where $\inf X$ and $\sup X$ denote the "essential" infimum and supremum of $X(\omega)$ over $\omega \in \Omega$ (obtained by disregarding subsets of Ω having probability 0). For either of these, it is possible for some r.v.'s X that $\mathcal{D}(X) = \infty$, which is allowed by the axioms. Of course, both are sure to be finite in the case of a finite, discrete probability space Ω.

Another class of deviation measures, of increasing interest now in applications, arises from *conditional value-at-risk*, CVaR, as an alternative to *value-at-risk*, VaR. A brief discussion of risk measures, in contrast to deviation measures, will lay the platform for introducing this class properly.

By a *strictly expectation bounded risk measure* will be meant any functional $\mathcal{R}$ that assigns values $\mathcal{R}(X)$ to random variables X in such a way that

(R1) $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C,
(R2) $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$,
(R3) $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ for all X and X',
(R4) $\mathcal{R}(X) > E[-X]$ for all nonconstant X, whereas $\mathcal{R}(X) = E[-X]$ for constant X.

A strictly expectation bounded risk measure $\mathcal{R}$ will be called *coherent* if, in addition to these four axioms, it satisfies

(R5) $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \geq X'$.

Axiom R4 is the property we explicitly mean by "strict expectation boundedness". The equation part of R4 is already a consequence of R1, so the *strict* inequality for *nonconstant* X is the chief assertion. (In [23], we spoke of this simply as expectation boundedness, but now add "strict" as a safeguard against misunderstandings.)[7]

Artzner, Delbaen, Eber and Heath in their landmark contribution to risk theory in [5] were the first to consider risk measures from a broad perspective, but they concentrated instead on functionals $\mathcal{R}$ satisfying R1, R2, R3 and, instead of R4, the monotonicity axiom R5. They called these *coherent risk measures*. (Actually, they posed R5 in a seemingly weaker form, namely $\mathcal{R}(X) \leq 0$ when $X \geq 0$, which is equivalent to the present R5 under the other axioms. Also, they had a somewhat different version R1, tailored to the use of an investment instrument, but the version used here was subsequently adopted by Delbaen [7].) Property R5 is natural and even critical for many purposes, and we fully believe in its importance. However, we forgo it in our basic *definition* of a strictly expectation bounded risk measure in order to capture a fundamental pairing between risk measures and deviation measures.

Risk measure axiom R4, first spelled out in our own earlier work, is needed for the following result, where it emerges as the counterpart to deviation axiom D4. As background, it should be noted that the weak inequality $\mathcal{R}(X) \geq E[-X]$, in place of the strict one in R4, is known to hold always when $\mathcal{R}$ is a coherent risk measure that depends only on the distribution of an r.v. and in addition has a certain convergence property, and furthermore X is an r.v. having no atoms in its distribution; cf.

[7] The weak-inequality version has recently been dubbed "risk relevance" in [14].

[8, Lemma 4.45]. That comes nowhere near to furnishing R4 in our general setting, however, so this really is a property that has to be brought in separately.

Theorem 1 [23] (deviation vs. risk). *Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations*

(a) $\mathcal{D}(X) = \mathcal{R}(X - EX)$,

(b) $\mathcal{R}(X) = E[-X] + \mathcal{D}(X)$.

Specifically, if $\mathcal{R}$ is an strictly expectation bounded risk measure and $\mathcal{D}$ is defined by (a), then $\mathcal{D}$ is a deviation measure that yields back $\mathcal{R}$ through (b). On the other hand, if $\mathcal{D}$ is any deviation measure and $\mathcal{R}$ is defined by (b), then $\mathcal{R}$ is a risk measure that yields back $\mathcal{D}$ through (a). In this correspondence, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is coherent.

The fact in the last part of Theorem 1 explains, of course, why we introduced D5 and used it to define coherency as a concept for deviation measures. We did not require this property in the basic definition of a deviation measure, because that would have left out standard deviation, $\mathcal{D}(X) = \sigma(X)$, thereby making nonsense of the terminology and excluding classical portfolio theory from our setting. The cogent arguments in favor of coherency made in [5] and elsewhere are nonetheless a prime motivation for what we are undertaking here.

In the pairing of Theorem 1, the deviation measure $\mathcal{D}(X) = \rho\sigma(X)$ for any $\rho \in (0, \infty)$ corresponds to $\mathcal{R}(X) = \rho\sigma(X) - \mu(X)$, whereas the lower-range deviation measure $\mathcal{D}(X) = EX - \inf X$ corresponds to the *maximum loss* risk measure $\mathcal{R}(X) = \sup[-X]$. Coherency is present in the second example, although absent in the first.

Even more of interest here is the utilization of Theorem 1 in deriving deviation measures of CVaR type. Recall that for any $\alpha \in (0, 1)$ the *value-at-risk* of X at level α is defined by

$$\mathrm{VaR}_\alpha(X) = -\inf\{\, z \mid P\{X \leq z\} > \alpha\}. \tag{4}$$

The corresponding *conditional value-at-risk* is then

$$\mathrm{CVaR}_\alpha(X) = -[\text{expectation of } X \text{ in its lower } \alpha\text{-tail distribution}], \tag{5}$$

where the expectation is the same as the conditional expectation of X subject to $X \leq -\mathrm{VaR}_\alpha(X)$ when $P\{\, X \mid X = -\mathrm{VaR}_\alpha(X)\} = 0$, but in general refers to the expectation of the r.v. whose cumulative distribution function F_α is obtained from the cumulative distribution function F for X by taking $F_\alpha(z) = F(z)/\alpha$ when $z < -\mathrm{VaR}_\alpha(X)$ and $F_\alpha(z) = 1$ when $z \geq -\mathrm{VaR}_\alpha(X)$.[8] The important thing is that a *coherent* risk-deviation pair is obtained by taking

$$\mathcal{R}(X) = \mathrm{CVaR}_\alpha(X), \qquad \mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX). \tag{6}$$

The functional $\mathcal{R}(X) = \mathrm{VaR}_\alpha(X)$, in contrast, fails to satisfy axioms R3, R4 and R5, so correspondingly the functional $\mathcal{D}(X) = \mathrm{VaR}_\alpha(X - EX)$ fails to satisfy D3, D4 and D5 and is not a deviation measure in our sense, much less a coherent one. Worse, it lacks the property of *convexity* that is crucial to our developments.

Beyond "pure" measures of CVaR type that come out of the correspondence in Theorem 1, there also "mixed" CVaR measures cf. [22, 23] having a "spectral representation" as in [1].

[8] This form of the definition of $\mathrm{CVaR}_\alpha(X)$ corresponds to the development of the concept in [22]; earlier, in [21], we concentrated only on the case of continuous distribution functions. Acerbi [1] independently arrived at this risk measure by an integral formula, calling it "expected shortfall," and that term has subsequently been used also in [3, 28]. A alternative minimization expression for $\mathrm{CVaR}_\alpha(X)$ in [21, 22] provides a powerful approach to computations. Other background on VaR and CVaR can be found in [8].

A final example, which deserves mention because of its modeling implications and the theoretical insights it provides, is the one in which Ω is just a finite set of future scenarios, with scenario ω having probability $P(\omega) > 0$. Consider any collection $\{P_j\}_{j\in J}$ of probability measures on Ω,

$$P_j(\omega) \geq 0, \qquad \sum_{\omega\in\Omega} P_j(\omega) = 1, \tag{7}$$

for an index set J (finite or infinite). Denote the expectation with respect to P_j by E_j, so that $E_j X = \sum_{\omega\in\Omega} X(\omega)P_j(\omega)$ in contrast to $EX = \sum_{\omega\in\Omega} X(\omega)P(\omega)$. Suppose the collection has the "richness" property that for every nonconstant X there is at least one $j \in J$ such that $E_j[-X] > E[-X]$, i.e., the expected loss incurred by X would be worse if the probability measure were P_j instead of the reference probability measure P. Then a coherent risk-deviation pair is obtained by defining

$$\mathcal{R}(X) = \sup_{j\in J} E_j[-X], \qquad \mathcal{D}(X) = \sup_{j\in J}\Big\{E_j[-X] - E[-X]\Big\}. \tag{8}$$

In this setting, P_j might be viewed as an alternative to P that an investor has selected for prudent comparisons, in case P just represents an educated guess about what the future will bring and may not be completely reliable. The deviation $\mathcal{D}(X)$ expressed by (8) identifies the worst discrepancy that could occur between the expected losses under the specified alternatives and the nominal expected loss $E[-X]$.

Besides being of interest for practical purposes in portfolio optimization, deviation measures of the kind in (8) serve as examples where differentiability can fail. Indeed, when the index set J is finite, $\mathcal{D}$ is the pointwise maximum of a finite collection of linear functions of X and therefore is *piecewise linear* on $\mathcal{L}^2(\Omega)$. It fails to be differentiable on the joins between the different "pieces", and its lower level sets $\{\, X \mid \mathcal{D}(X) \leq c\}$ are *polyhedral* convex sets with ridges and flat sides.

3 Portfolio Framework

To proceed with our effort to extend the classical results in portfolio theory for standard deviation to general deviation measure $\mathcal{D}$, we must provide a market setting. The market will be taken, for model purposes, to consist of instruments $i = 0, 1, \ldots, n$ having rates of return r_i. The first of these instruments, for $i = 0$, is risk-free; its rate of return r_0 is a constant. The other instruments, for $i = 1, \ldots, m$, are risky; their rates of return r_i are r.v.'s in $\mathcal{L}^2(\Omega)$. A dollar invested in instrument i brings back $1 + r_i$, for a gain (or profit) of r_i dollars at the end of the time period under consideration.

We will be concerned with portfolios that can be put together by investing an amount x_i in each instrument i. These amounts, which we can take to be in dollars,[9] can be positive, zero or negative. (A negative investment corresponds to a short position.) Such a portfolio has the present price $x_0 + x_1 + \cdots + x_n$ and the uncertain future value $x_0(1+r_0) + x_1(1+r_1) + \cdots + x_n(1+r_n)$. The associated gain is thus the r.v. X in $\mathcal{L}^2(\Omega)$ described by

$$X = x_0 r_0 + x_1 r_1 + \cdots + x_n r_n. \tag{9}$$

Here we are using "gain" in the sense that a loss is a negative gain. Costs, too, might be negative as well as positive, or zero.

[9] More typically, these amounts are viewed as "fractions," even if greater than 1, but this interpretation will facilitate discussion of portfolio prices.

To facilitate our work with these r.v.'s X while taking into account the special role of the risk-free instrument and keeping notation simple, we introduce

$$r = (r_1, \dots, r_n) \text{ (vector r.v.)}, \qquad \bar{r} = (\bar{r}_1, \dots, \bar{r}_n) \text{ for } \bar{r}_i = Er_i,$$

along with the vectors

$$x = (x_1, \dots, x_n), \qquad e = (1, \dots, 1).$$

The general r.v. X in (9) is then

$$X = x_0 r_0 + x^\top r, \quad \text{with expected gain } \; x_0 r_0 + x^\top \bar{r} \; \text{ and price } \; x_0 + x^\top e.$$

We speak of $x = (x_1, \dots, x_n)$ itself as giving the *x-portfolio* for which the gain is the r.v. $x^\top r$ in $\mathcal{L}^2(\Omega)$, the expected gain is $x^\top \bar{r}$, and the price is $x^\top e$.

The following assumptions on instruments $i = 1, \dots, n$ in the model will henceforth be in effect. The rest of this section will be devoted to elucidating their immediate consequences.

Basic Assumptions.

(A1) *No x-portfolio with $x \neq 0$ is risk-free.*

(A2) *The expected rates of return $\bar{r}_1, \dots, \bar{r}_n$ are not all the same.*

(A3) *$\mathcal{D}(r_i) < \infty$ and $\mathcal{D}(-r_i) < \infty$ for all i.*

Assumption A1 is harmless and merely underscores our aim of letting the $i = 0$ instrument do all the risk-free service. A notion of redundancy will help in understanding why this is true. Let us say that an instruments i is *redundant* in the model if the associated r.v. r_i, which gives the gain from investing one dollar in instrument i, can exactly be replicated by the gain r.v. of a portfolio put together from the other instruments. Note that such replication, if possible at all, would have to be achieved at price 1, or an arbitrage opportunity would exist, thereby undermining our intent of starting from a market in which prices are in equilibrium.

Proposition 1 (elimination of redundancy)**.** *Assumption A1 is fulfilled if and only if none of the instruments i in the model is redundant.*

Proof. If some x-portfolio with $x \neq 0$ were risk-free, we could find a value x_0 such that the r.v. $X = x_0 r_0 + x_1 r_1 + \ldots + x_n r_n$ is identically 0. One of the coefficients $x_1, \dots, x_n$ would be nonzero; suppose for purposes of illustration that is x_1. We would then have $r_1 = x_0' r_0 + x_2' r_2 + \cdots + x_n' r_n$ for $x_i' = -x_i / x_1$, which would mean that the $i = 1$ instrument is redundant.

For the converse, suppose some instrument i is redundant. If that holds for $i = 0$, then by the definition of redundancy there must be a nonzero x-portfolio that is risk-free. Otherwise, we can suppose for simplicity of notation that $i = 1$ is redundant. This refers to the existence of coefficients $x_0, x_2, \dots, x_n$ such that $r_1 = x_0 r_0 + x_2 r_2 + \cdots + x_n r_n$. Then $-r_1 + x_2 r_2 + \cdots + x_n r_n = -x_0 r_0$, so the x-portfolio for $x = (-1, x_2, \dots, x_n)$ would be risk-free. □

Redundant instruments offer nothing new, so we could always eliminate them from the model one by one until nothing redundant was left. Then A1 would hold.

Another insight into A1 can be obtained through consideration of distribution functions.

Proposition 2 (continuous distributions)**.** *Assumption A1 is satisfied when the r.v. r is continuously distributed (i.e., the multivariate distribution function for $r_1, \dots, r_n$ is continuous on $I\!R^n$), thus guaranteeing that the gain $x^\top r$ of any x-portfolio with $x \neq 0$ is continuously distributed as well.*

Proof. The well known fact about $x^\top r$ being continuously distributed in these circumstances precludes $x^\top r$ from being a constant r.v., of course. □

Assumption A2 is needed to sidestep special circumstances which have little interest for us here. If it did not hold, there would be a value ρ such that $\bar{r}_i = \rho$ for $i = 1, \ldots, n$; then the expected gain $x^\top \bar{r}$ of an x-portfolio would always be ρ times its price $x^\top e$.

Both A1 and A2 seem to be taken for granted by many in finance, even though they are essential to the validity of commonly made assertions.[10] The desire to maintain mathematical rigor in our development of portfolio theory requires us to make these assumptions, and others, explicit.

Proposition 3 (richness of price-gain combinations)**.** *For every choice of $(\pi, \zeta) \in I\!R^2$, there is an x-portfolio having price $x^\top e = \pi$ and expected gain $x^\top \bar{r} = \zeta$.*

Proof. This is the main consequence of A2. The set of pairs (π, ζ) coming from portfolios in this way constitutes a subspace of $I\!R^2$, so if it were not all of $I\!R^2$, these pairs would be collinear, and we would be in the lockstep situation excluded by A2. □

Assumption A3, which will guarantee the finiteness of $\mathcal{D}(x^\top r)$ according to the next proposition, is certainly satisfied when $\mathcal{D}$ is a deviation measure that is finite on all of $\mathcal{L}^2(\Omega)$, and many measures with that property have already been indicated beyond $\mathcal{D}(X) = \sigma(X)$, a major example being $\mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX)$. But A3 may also be satisfied for some deviation measures that are not finite on all of $\mathcal{L}^2(\Omega)$. An example is $\mathcal{D}(X) = EX - \inf X$ when the rates of return r_i are bounded. Note that we are obliged to require the finiteness of $\mathcal{D}(r_i)$ and $\mathcal{D}(-r_i)$ separately, because $\mathcal{D}$ need not be symmetric.

Proposition 4 (portfolio deviations)**.** *The deviation function*

$$f_{\mathcal{D}}(x) = \mathcal{D}(x^\top r)$$

is finite everywhere and convex on $I\!R^n$ (hence also continuous), moreover with the properties that

(a) $f_{\mathcal{D}}(0) = 0$, *but* $f_{\mathcal{D}}(x) > 0$ *when* $x \neq 0$,
(b) $f_{\mathcal{D}}(\lambda x) = \lambda f_{\mathcal{D}}(x)$ *when* $\lambda > 0$,
(c) $f_{\mathcal{D}}(x + x') \leq f_{\mathcal{D}}(x) + f_{\mathcal{D}}(x')$,
(d) $\{\, x \mid f_{\mathcal{D}}(x) \leq \delta \}$ *is a bounded set for every* $\delta > 0$.

Proof. In view of axiom D4 on $\mathcal{D}$, the strict inequality in (a) is equivalent to A1. Properties (b) and (c), together with the fact in (a) that $f_{\mathcal{D}}(0) = 0$, follow immediately from axioms D2 and D3 on $\mathcal{D}$. They imply in particular that $f_{\mathcal{D}}$ is a convex function. The set of x for which $f_{\mathcal{D}}(x) < \infty$ is then a convex subset of $I\!R^n$. Because of A3, that set includes the vectors,

$$(\pm 1, 0, \ldots, 0),\ \ (0, \pm 1, \ldots, 0),\ \ (0, 0, \ldots, \pm 1),$$

which correspond to portfolios consisting of just one of the instruments $i = 1, \ldots, n$, either in unit long position or unit short position. It must also then include all positive multiples of those vectors, through (b), as well as all sums generated from those, through (c). Thus, it has to be all of $I\!R^n$.

For the fact that finite convex functions on $I\!R^n$ are continuous, see [19, Theorem 10.1]. On the principle of [19, Corollary 8.7.1] and $f_{\mathcal{D}}$ being convex and continuous, if any set of form $\{\, x \mid f_{\mathcal{D}}(x) \leq \delta \}$ is bounded, then all sets of that form must be bounded. By (a), the set $\{\, x \mid f_{\mathcal{D}}(x) \leq 0 \}$ is the singleton $\{0\}$, so (d) is correct. □

Readers familiar with the emphasis on VaR in much of finance nowadays should note that if we were to take $\mathcal{D}(x^\top r) = \mathrm{VaR}_\alpha(x^\top r - x^\top \bar{r})$, which does not conform to our assumptions because it merely

[10] For instance, in the text [12, p. 159], the $n+2$ linear equations in $n+2$ unknowns that describe the weights for an efficient risky portfolio in the Markowitz model are said to have a unique solution, but really that is only true when the coefficient matrix is nonsingular. The matrix in question fails to be nonsingular if A1 and A2 do not hold.

satisfies D1 and D2, a number of the crucial properties of the deviation function $f_{\mathcal{D}}$ in Proposition 4 would drop away. Certainly (c) would be gone, and with it the convexity of $f_{\mathcal{D}}$. Even the strict inequality in (a) would be violated in some situations, and the boundedness of the level sets in (d) would then be lost. Moreover, $f_{\mathcal{D}}$ could no longer even be counted on to be continuous.

4 Basic Funds and Master Funds

The fundamental problem of optimization that we wish to examine closely with respect to the portfolio r.v.'s X in (9) is

$$\mathcal{P}(\Delta) \qquad \text{minimize } \mathcal{D}(x_0 r_0 + x^\top r) \text{ subject to } x_0 + x^\top e = 1 \text{ and } x_0 r_0 + x^\top \bar{r} \geq r_0 + \Delta,$$

where $\mathcal{D}(x_0 r_0 + x^\top r)$ is actually just the deviation $f_{\mathcal{D}}(x)$ in Proposition 4, of course. The price constraint $x_0 + x^\top e = 1$ signifies (in our mode of interpreting the x_i's as dollar amounts) that the price of the portfolio nets out to exactly one dollar; that is how much is to be invested initially. The gain constraint $x_0 r_0 + x^\top \bar{r} \geq r_0 + \Delta$ requires that this unit investment should result in an expected future value of at least $1 + r_0 + \Delta$ dollars. The parameter Δ gives the *risk premium* — the extra amount being demanded over the gain associated with investing at the risk-free rate r_0. The gain constraint has been written as an inequality instead of an equation because there should not be any objection if some portfolio, without worsening the deviation or costing more, might have an expected gain that is more than $r_0 + \Delta$. It will come out below, however, that any portfolio solving problem $\mathcal{P}(\Delta)$ must satisfy this constraint with equality, when $\Delta > 0$.

The unit price constraint in $\mathcal{P}(\Delta)$ can be used to eliminate x_0 by assigning it the value $x_0 = 1 - x^\top e$. The problem statement comes down then to:

$$\mathcal{P}_0(\Delta) \qquad \text{minimize } f_{\mathcal{D}}(x) \text{ subject to } x^\top[\bar{r} - r_0 e] \geq \Delta.$$

Adopting this framework in terms of x-portfolios alone, we let

$$\begin{cases} d_0(\Delta) = & \text{optimal value (the infimum of the deviation) in } \mathcal{P}_0(\Delta), \\ S_0(\Delta) = & \text{optimal solution set (the minimizing vectors } x\text{) in } \mathcal{P}_0(\Delta). \end{cases} \tag{10}$$

Proposition 5 (solution existence and homogeneity)**.** *An optimal solution to problem $\mathcal{P}_0(\Delta)$ is sure to exist (not necessarily uniquely), no matter what the choice of Δ. Indeed, the optimal solution set $S_0(\Delta)$ is always convex, closed and bounded, in addition to being nonempty. Moreover,*

$$\begin{cases} \text{for } \Delta \leq 0: & d_0(\Delta) = 0 \text{ and } S_0(\Delta) = \{0\} \text{ (put all in the risk-free instrument)}, \\ \text{for } \Delta > 0: & d_0(\Delta) > 0, \text{ with } d_0(\Delta) = \Delta \cdot d_0(1) \text{ and } S_0(\Delta) = \{\, \Delta \cdot x \mid x \in S_0(1)\}. \end{cases} \tag{11}$$

Additionally, when $\Delta > 0$ the gain constraint is always active in $\mathcal{P}_0(\Delta)$, i.e., every $x \in S_0(\Delta)$ satisfies $x^\top[r - r_0 e] = \Delta$.

Proof. In view of Proposition 3, the constraint in problem $\mathcal{P}_0(\Delta)$ can be satisfied regardless of the choice of r_0 and Δ. The sets of type

$$\{\, x \mid f_{\mathcal{D}}(x) \leq \delta,\ x^\top[r - r_0 e] \geq \Delta\} \quad \text{for } \delta > d_0(\Delta) \tag{12}$$

are nonempty by the definition of $d_0(\Delta)$ as well as compact because of the continuity of $f_{\mathcal{D}}$ and the boundedness in Proposition 4(d). Any nest of nonempty compact sets has a nonempty intersection. In this case, moreover, the sets are convex by virtue of the convexity of $f_{\mathcal{D}}$, so the intersection is likewise a convex set. This confirms that $S_0(\Delta)$ is nonempty, convex and compact.

The special assertions about $\mathcal{P}_0(\Delta)$ in the case of $\Delta \leq 0$ are evident from Proposition 4(a). They rely also on the constraint having been stated as an inequality rather than an equation. In the case where $\Delta > 0$, the relationships involving $d_0(1)$ and $S_0(1)$ are immediate from the positive homogeneity of $f_{\mathcal{D}}$ in Proposition 4(b), according to which anything optimal for $\Delta = 1$ can be rescaled to be optimal for other Δ.

The constraint in $\mathcal{P}_0(\Delta)$ has to be active when $\Delta > 0$, because if x has $x^\top[r - r_0 e] > \Delta$, there is a factor $\theta \in (0,1)$ such that the vector $x' = \theta x$ satisfies the same inequality and yet yields a deviation amount that is smaller than the one for x by the same factor. This is incompatible with x being optimal. Note that here we are invoking Proposition 4(a) once more, since this argument would fall through if the deviation in question were 0. □

Proposition 5 sets the stage for a complete understanding of the efficient frontier for portfolios that include the risk-free instrument, $i = 0$. It will lead quickly to the classical concept of a "master fund," with certain extensions. However, for the sake of capturing at once what it says about the best way to invest, regardless of the technical complications that will soon come up, another "fund" concept will be helpful.

Definition 1 (basic funds and basic deviation value). *For any $\bar{x} \in S_0(1)$, providing in problem $\mathcal{P}_0(1)$ the minimum portfolio deviation for a gain of exactly 1 over the risk-free rate, the $\bar{x}$-portfolio will be said to furnish a* basic fund. *The minimum deviation amount will be called the* basic deviation value *and denoted by $\bar{\delta}$:*

$$\bar{\delta} = d_0(1) = f_{\mathcal{D}}(\bar{x}).$$

Theorem 2 (generalized one-fund theorem in basic fund form). *Let $\bar{x}$ furnish a basic fund, achieving the basic deviation value $\bar{\delta}$. Then, for any $\Delta > 0$, a solution to the fundamental portfolio problem $\mathcal{P}(\Delta)$ is obtained by investing the amount $\Delta[\bar{x}^\top e]$ in the $\bar{x}$-portfolio and the amount $1 - \Delta[\bar{x}^\top e]$ in the risk-free instrument. That solution portfolio has deviation $\Delta\bar{\delta}$. In further detail, this prescription for investment comes down to the following three cases, depending on the price $\bar{x}^\top e$ of the basic fund:*

(a) Positive case: price $\bar{x}^\top e > 0$. *Invest the* positive *amount $\Delta[\bar{x}^\top e]$ in the $\bar{x}$-fund while investing the amount $1 - \Delta[\bar{x}^\top e]$ (possibly positive, negative or zero) in the risk-free instrument.*

(b) Negative case: price $\bar{x}^\top e < 0$. *Invest the* negative *amount $\Delta[\bar{x}^\top e]$ in the $\bar{x}$-fund (i.e., take a short position of this magnitude in it), while investing the* positive *amount $1 - \Delta[\bar{x}^\top e] > 1$ in the risk-free instrument.*

(c) Threshold case: price $\bar{x}^\top e = 0$. *Invest the original 1 dollar entirely in the risk-free instrument while assuming a position in $\bar{x}$ of magnitude Δ, which nets out (through longs and shorts) to price 0.*

The three cases arise, in principle at least, because there is no *a priori* restriction in our optimization problems on the prices of the x-portfolios under consideration. This price is left to the optimization outcome itself. Understanding the extent to which these different cases can truly occur, or conceivably even overlap, will require serious effort in the rest of this paper. Some insights are immediately available, however.

In the positive case, one is investing in risky instruments in the classical way and making up the difference with the original 1 dollar by buying, or borrowing, a quantity of the risk-free instrument, as needed. In the negative case, the scheme effectively involves borrowing from the market in the risky instruments and putting the proceeds into the risk-free instrument. Clearly, that would not be interesting unless the risk-free return r_0 is attractively high from the investor's point of view. Much more will be said about this circumstance as we go on, but observe that it does not imply the presence