HELLY'S THEOREM AND MINIMA OF CONVEX FUNCTIONS

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1. Introduction. The object of this paper is to prove an existence theorem for solutions to a very general class of constrained and unconstrained minimization problems involving convex functions on R^n. This theorem is in effect an extension of the classical theorem of Helly, according to which an infinite collection of compact convex sets in R^n has a nonempty intersection if every $n + 1$ of the sets have a point in common. (For the general literature on Helly's theorem see the expository article by Danzer, Grünbaum and Klee in *Convexity*, Proceedings of the Symposium in Pure Mathematics, vol. VI, American Mathematical Society, 1963.)

The idea of extending Helly's theorem to convex functions is not new; such extensions have been given by Bohnenblust, Karlin and Shapley [2; 185] (discussed also in [13]) and by Fenchel in his 1953 lecture notes [7; 96–101]. Both of these, however, are limited essentially to collections of convex functions on a bounded convex set. Our theorem does not have this limitation, and hence it can be used both in the compact case and in the theory of convex programming, where compactness is usually too severe a restriction. It implies, for instance, that a polynomial convex function achieves a minimum on any polyhedral convex set where it is bounded below, a result obtained in the quadratic case by Frank and Wolfe [8]. Yet at the same time it contains, in a direct way, a new generalization of Helly's Theorem in which the sets and their intersections can sometimes all be unbounded.

Our principal device is to replace compactness, wherever this might otherwise be necessary, by "asymptotic regularity conditions" which restrict behavior along certain infinite rays which might be present. This was suggested by Fenchel's work with the asymptotic cones of convex sets [7, 42–44 and 99–101].

Besides applying the existence theorem to ordinary convex programs, we shall derive from it results in the theory of inequalities and Lagrange multipliers complementary to those in [6]. A new general version of von Neumann's minimax theorem, not requiring compactness, will also be deduced.

2. Existence theorem. Throughout this paper P will denote a non-empty polyhedral convex set in R^n, i.e. a set which can be represented as the intersection of finitely many closed half-spaces. The choices of P we have most in mind are: R^n itself, the "non-negative orthant" of R^n, the unit simplex, the product of n closed intervals of R, or some combination of these, such as the

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set of $x = \langle \xi_1, \cdots, \xi_n \rangle$ with $\xi_i \geq 0$ for $i = 1, \cdots, s$ and $\xi_1 + \cdots + \xi_r = 1$, $r \leq s \leq n$.

A function f on P with values in $(-\infty, +\infty]$ is *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{whenever } x \in P,\ y \in P,\ 0 < \lambda < 1. \tag{2.1}$$

It is *affine* if equality always holds in (2.1). It is *lower semi-continuous* (l.s.c.) if all its (sub-) level sets

$$\{x \in P \mid f(x) \leq \mu\}, \quad \text{where} \quad \mu \in R, \tag{2.2}$$

are closed. A convex function f on a convex subset C of P can be extended convexly to all of P by assigning f the value $+\infty$ at points of P not in C. The extension is l.s.c., in particular, when C is closed and f is continuous (or l.s.c.) on C.

Let C be a non-empty closed convex subset of P. We say $u \in R^n$ is a *direction in which C is unbounded* if $\{x + \lambda u \mid \lambda \geq 0\}$ is contained in C for all $x \in C$. ("Direction", as we shall use the term here, is hardly more than a suggestive synonym for "vector"; we do *not* assume u is normalized, and, in particular, the "trivial direction" $u = 0$ is admitted.) If u is a direction in which C is unbounded, and if actually the intersection of $\{x + \lambda u \mid -\infty < \lambda < \infty\}$ with P is contained in C for every $x \in C$, we say u is a *direction in which C is a linear relative to P*.

Let f be l.s.c. convex on P and let u be a direction in which P is unbounded. We say u is a *direction in which f is non-increasing* (respectively *constant*) if $f(x + \lambda u)$ is a non-increasing (respectively constant) function of $\lambda \geq 0$ for every $x \in P$.

EXAMPLE 1. Let C be a non-empty closed convex subset of P and let f be either the *indicator function* of C

$$\delta(x \mid C) = 0 \quad \text{if} \quad x \in C,\ \delta(x \mid C) = \infty \quad \text{if} \quad x \notin C, \tag{2.3}$$

or the *distance function* of C

$$\rho(x \mid C) = \min \{\|x - z\| \mid z \in C\} \tag{2.4}$$

(for any norm). Then f is l.s.c. convex on P. Moreover, the directions in which f is non-increasing (respectively constant) are precisely the directions in which C is unbounded (respectively linear relative to P).

EXAMPLE 2. If P is the last of the sets described in the initial paragraph of this section, then $u = \langle \eta_1, \cdots, \eta_n \rangle$ is a direction in which P is unbounded if and only if $\eta_i = 0$ for $i = 1, \cdots, r$ and $\eta_i \geq 0$ for $i = r + 1, \cdots, s$. In general, if P is the set of vectors x such that $(a_j, x) \geq \alpha_j$ for $j = 1, \cdots, t$ (where $\alpha_j \in R$, $a_j \in R^n$ and (a_j, x) is the scalar product), then the directions in which P is unbounded are the vectors u such that $(a_j, u) \geq 0$ for $j = 1, \cdots, t$.

EXAMPLE 3. Suppose f is a finite convex function which can be extended differentiably to a neighborhood of P. Let $\nabla f(x)$ denote the gradient of f at x. The directions in which f is non-increasing (respectively constant) are then the directions u in which P is unbounded and $(\nabla f(x), u) \leq 0$ (respectively $(\nabla f(x), u) = 0$) for all $x \in P$.

DEFINITION 1. A *convex program* (at least for present purposes) is a problem of the form

$$(\mathfrak{P}) \qquad \text{minimize } f(x) = \sup\{f_i(x) \mid i \in I\} \text{ subject to the constraints: } x \in P \text{ and } f_j(x) \leq 0 \text{ for all } j \in J,$$

where $\{f_k \mid k \in K = I \cup J\}$ is a (finite or infinite) partitioned collection of l.s.c. convex functions on P (with $I \cap J = \phi$, $I \neq \phi$).

If the constraints of $(\mathfrak{P})$ can be satisfied by at least one x for which $f(x) < \infty$, we say $(\mathfrak{P})$ is *consistent*. A vector $\bar{x}$ is a *solution* to $(\mathfrak{P})$ if $\bar{x}$ satisfies the constraints, $f(\bar{x}) = \bar{\mu} < \infty$, and $f(x) \geq \bar{\mu} > -\infty$ for every x satisfying the constraints.

DEFINITION 2. The convex program $(\mathfrak{P})$ is *weakly consistent* if there exists at least one $\mu \in R$ such that, for all $\epsilon > 0$ and for every subset K_1 of K containing $n + 1$ indices or less, the finite system of inequalities

$$(2.5) \qquad f_i(x) \leq \mu + \epsilon \quad \text{for} \quad i \in I \cap K_1 \quad \text{and} \quad f_j(x) \leq \epsilon \quad \text{for} \quad j \in J \cap K_1$$

is satisfied by at least one $x \in P$. (Note that consistency implies weak consistency.)

EXAMPLE 4. Let $\{C_i \mid i \in I\}$ be a finite or infinite collection of closed convex subsets of P, and for each $i \in I$ let $f_i(x) = \delta(x \mid C_i)$ (see (2.3)). Then $(\mathfrak{P})$, for $K = I$, $J = \phi$, is the problem of finding a point common to all the C_i. Solutions exist if and only if $(\mathfrak{P})$ is consistent. On the other hand, $(\mathfrak{P})$ is weakly consistent if and only if every $n + 1$ of the C_i have a point in common.

EXAMPLE 5. Let $\{f_j \mid j \in J\}$ be a finite or infinite collection of l.s.c. convex functions on P. Let $K = \{0\} \cup J$, $I = \{0\}$, where f_0 is identically zero. Then $(\mathfrak{P})$ is consistent whenever the system of inequalities

$$(2.6) \qquad f_j(x) \leq 0 \quad \text{for all} \quad j \in J$$

has a solution in P, and such solutions are the solutions of $(\mathfrak{P})$. It is weakly consistent if merely every subsystem of the form

$$(2.7) \quad f_{j_l}(x) \leq \epsilon \quad \text{for} \quad l = 1, \cdots, s, \quad \text{where} \quad \epsilon > 0, s \leq n + 1, j_l \in J,$$

has a solution in P.

DEFINITION 3. The convex program $(\mathfrak{P})$ is *asymptotically regular* if there are no directions in which P is unbounded and *all* the f_k, $k \in K$, are non-increasing, except perhaps for directions in which all the f_k for $k \in K - K_0$ are constant, where the exceptional set of indices K_0 is allowed to be any finite (or empty) subset of K selected in advance (i.e. independent of the direction), such that $K_0 \not\supseteq I$ and f_k is *affine* for all $k \in K_0$.

If x satisfies the constraints of $(\mathfrak{P})$ with $f(x) < \infty$, and $u \neq 0$ is a direction in which P is unbounded and all the f_k are non-increasing, then every point of the infinite ray $\{x + \lambda u \mid \lambda \geq 0\}$ also satisfies the constraints, and f is non-increasing along this ray. The asymptotic regularity condition ensures that f achieve a minimum along every such special ray. The condition really ensures much more than this, as we shall see in a moment.

EXAMPLE 6. Suppose that $\{f_k \mid k \varepsilon K\}$ is a finite collection of affine functions, and that $(\mathfrak{P})$ is consistent. Then $(\mathfrak{P})$ is asymptotically regular if and only if there does not exist an infinite ray $\{x + \lambda u \mid \lambda \geq 0\}$ of points, all satisfying the constraints, along which f is (uniformly strictly) decreasing.

EXAMPLE 7. An ordinary linear or quadratic program (see [4], [12]) is asymptotically regular if and only if the constraints of its *dual* can be satisfied. (This may be proved by applying the Lemma of Farkas to the system consisting of the constraints of $(\mathfrak{P})$ and the inequalities obtained from the asymptotic regularity condition; cf. Example 3.)

Our general existence theorem for convex programs will now be stated.

THEOREM 1. *If $(\mathfrak{P})$ is weakly consistent and asymptotically regular, then $(\mathfrak{P})$ is consistent and has at least one solution. Moreover, the minimum in $(\mathfrak{P})$ is then the smallest of the real numbers μ for which the weak consistency condition can be satisfied.*

The proof of Theorem 1 will be given in the next section. Observe that Theorem 1 leads immediately to the following extension of Helly's Theorem.

COROLLARY 1. *Let $\{C_i \mid i \varepsilon I\}$ be any finite or infinite collection of non-empty closed convex subsets of a polyhedral convex set P. Suppose there are no directions in which all the C_i are unbounded, except perhaps for directions in which they are all linear relative to P. If every $n + 1$ of the C_i have a point in common, then there exists a point common to all the C_i. The assumption that every $n + 1$ of the C_i have a point in common can even be weakened to the following: for every $\epsilon > 0$ and $\{i_1, \cdots, i_r\} \subseteq I$, $r \leq n + 1$, there exists a point whose distance from each of the sets $C_{i_1}, \cdots, C_{i_r}$ does not exceed ϵ.*

Proof. The first part is obvious from Examples 1 and 4. The ϵ version can be obtained by applying Theorem 1 instead to Example 5 in the case of the distance functions (2.4) of the sets.

If we keep the first two sentences of Corollary 1 and assume that

$$\sup \{\rho(x \mid C_i) \mid i \varepsilon I\} < \infty$$

for at least one $x \varepsilon P$, we can still conclude the existence of a *proximity point* [15; 248] for $\{C_i \mid i \varepsilon I\}$, i.e. a point for which $\sup \{\rho(x \mid C_i) \mid i \varepsilon I\}$ is minimal. This is proved by applying Theorem 1 to $(\mathfrak{P})$ with $f_i(x) = \rho(x \mid C_i)$, $J = \phi$. This result and the ϵ part of Corollary 1 are new even for finite collections of convex sets.

Corollary 1 extends Fenchel's version [7; 101] of Helly's theorem, which says that a collection of non-empty closed convex sets in R^n has a non-empty intersection if every $n + 1$ sets have a point in common and there is no direction $u \neq 0$ in which all the sets are unbounded. Fenchel's hypothesis implies that some finite sub-collection has a (non-empty) compact intersection (see the proof of Lemma 7 in the next section).

Specializing Theorem 1 to the case where $\{f_k \mid k \in K\}$ consists of a single function, we have

COROLLARY 2. *Let f be* l.s.c. *convex on P and not identically $+\infty$. If f is constant in every direction in which P is unbounded and f is non-increasing, then f achieves a (finite) minimum on P.*

It is very important here to understand the role of $+\infty$ as a value of f. Although it would seem that Corollary 2 is applicable only to minimization problems on polyhedral convex sets, this is not really true. Indeed, it contains the following result in particular.

COROLLARY 2′. *Let g be a continuous (or* l.s.c.*) finite-valued convex function on a non-empty closed convex subset C of P. Suppose there are no directions $u \neq 0$ in which C is unbounded and g is non-increasing along the infinite ray $\{x + \lambda u \mid \lambda \geq 0\}$ for every $x \in C$, except perhaps for directions in which C is linear relative to P and g is constant along all the infinite rays in question. Then g achieves a minimum on C.*

Proof. Let $f(x) = g(x)$ when $x \in C$, $f(x) = \infty$ when $x \in P$ but $x \notin C$, and apply Corollary 2 to the equivalent problem of minimizing f on P.

One might suppose that "constant" could be replaced by "eventually constant" in Corollary 2′. The assertion would then be false, as is shown for $C = P = R^2$ by

$$g(x) = g(\xi_1, \xi_2) = \xi_1 + \min \{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 \mid \eta_2 \geq \eta_1^2\}.$$

This is a continuously differentiable convex function which achieves a minimum along each individual line in R^2, and yet is not even bounded below. Another example in R^2 shows why, for the directions in question, C has to be linear relative to P (or something like it). Let $P = \{\langle \xi_1, \xi_2 \rangle \mid \xi_2 \geq 0\}$,

$$C = \{\langle \xi_1, \xi_2 \rangle \in P \mid \xi_1 \xi_2 \geq 1\}, \; g(\xi_1, \xi_2) = \xi_1 .$$

The infinite rays in C along which g is non-increasing are of form

$$\{\langle \xi_1, \xi_2 + \lambda \eta_2 \rangle \mid \lambda \geq 0\},$$

where $\eta_2 \geq 0$ and $\xi_2 \geq 1/\xi_1 > 0$. Along each such ray g is constant, but g does not achieve a minimum on C.

As anticipated in the introduction, Corollary 2 implies that a quadratic convex function f achieves a minimum on any polyhedral convex set P where it is bounded below. This is true because such a function is affine on any infinite ray along which it is non-increasing. The author is grateful to the referee for pointing out that this result is still valid, for the same reasons, if f is any polynomial function which is convex on P.

3. **Proof of existence theorem.** We shall break the proof down into a series of lemmas, the first three of which have independent interest.

LEMMA 1. *Let C be a non-empty convex set in R^n and let g be a convex function on C with values in $(-\infty, \infty]$. Then g is* l.s.c. *on C if and only if g is* l.s.c. *along each line segment $\{\lambda x + (1-\lambda)y \mid 0 \leq \lambda \leq 1\}$ in C.*

Proof. For each $\mu \in R$, let $C_\mu = \{x \in C \mid g(x) \leq \mu\}$. Each C_μ is a convex set, and g is l.s.c. on C if and only if all the C_μ are closed in the relative topology on C. When the latter is true, then the intersection of a line segment in C with each C_μ is closed, so g is l.s.c. along every such segment. On the other hand, take any $\mu \in R$ with $C_\mu \neq \phi$ and suppose the intersection of C_μ with each line segment in C is closed. We shall show that then C_μ is closed in the relative topology of C; this will prove the lemma. Let x be any point in C belonging to the closure of C_μ. Choose an interior point y of C_μ relative to the smallest linear manifold in R^n containing C_μ. (Such a point of y exists because $C_\mu \neq \phi$; see [5; 16]). Then $\lambda x + (1-\lambda)y \in C_\mu$ for $0 \leq \lambda < 1$ (see [5; 9]), so $x \in C_\mu$ by our supposition.

LEMMA 2. *Let g be* l.s.c. *convex on P and let $A(g)$ be the set of directions in which P is unbounded and g is non-increasing. Then*

(a) *$A(g)$ is a non-empty closed convex cone in R^n, i.e. $A(g)$ is closed, $0 \in A(g)$, and $\sum \lambda_j u_j \in A(g)$ whenever $\lambda_j \geq 0$ and $u_j \in A(g)$;*

(b) *in order that $u \in A(g)$, it is sufficient that there exist one $x_0 \in P$ and $\mu_0 \in R$, such that $x_0 + \lambda u \in P$ and $g(x_0 + \lambda u) \leq \mu_0$ for all $\lambda \geq 0$.*

Proof. (a). It is clear from the definition of $A(g)$ that $0 \in A(g)$, and that $\lambda u \in A(g)$ when $u \in A(g)$ and $\lambda \geq 0$. If $u_1 \in A(g)$ and $u_2 \in A(g)$ then, for each $x \in P$ and $\lambda \geq 0$, $x + \lambda(u_1 + u_2) = (x + \lambda u_1) + \lambda u_2 = x' + \lambda u_2 \in P$, $x' \in P$, so $u_1 + u_2$ is a direction in which P is unbounded. Also, for each $x \in P$ and $\lambda_1 \geq \lambda_2 \geq 0$ we have

$$g(x + \lambda_1(u_1 + u_2)) \leq g(x + \lambda_1 u_1 + \lambda_2 u_2) \leq g(x + \lambda_2(u_1 + u_2)),$$

so $u_1 + u_2$ is a direction in which g is non-increasing. To show $A(g)$ is closed, suppose $u_j \in A(g)$ for $j = 1, 2, \cdots$, and $\lim_j u_j = u$. Let $x \in P$ and $\lambda \geq 0$. Then $x + \lambda u = \lim_j \{x + \lambda u_j\} \in P$ since $x + \lambda u_j \in P$ for all j (polyhedral convex sets being closed by definition), and by lower semi-continuity

$$g(x + \lambda u) \leq \liminf_j g(x + \lambda u_j) \leq g(x).$$

Hence for each $x \in P$ and $\lambda_1 \geq \lambda_2 \geq 0$,

$$g(x + \lambda_1 u) = g(x + \lambda_2 u + (\lambda_1 - \lambda_2)u) \leq g(x + \lambda_2 u). \tag{3.1}$$

Thus $u \in A(g)$.

(b). Let x_0 and μ_0 have the property described. Let $x \in P$ and $\lambda \geq 0$. Then, for $0 < \lambda' \leq 1$,

$$(1 - \lambda')(x + \lambda u) + \lambda'(x_0 + \lambda u) = (1 - \lambda')x + \lambda'(x_0 + (\lambda/\lambda')u) \in P,$$

$$g((1 - \lambda')(x + \lambda u) + \lambda'(x_0 + \lambda u)) \leq (1 - \lambda')g(x) + \lambda'\mu_0.$$

Since P is closed and g is l.s.c., it follows that $x + \lambda u \in P$ and $g(x + \lambda u) \leq g(x)$. This shows that u is a direction in which P is unbounded and g is non-increasing (the latter by the argument in (3.1)), so $u \in A(g)$.

COROLLARY. *Let $C \neq \phi$ be a closed convex set in R^n and let $A(C)$ be the set of directions in which C is unbounded. Then $A(C)$ is a non-empty closed convex cone, and $u \in A(C)$ whenever $\{x + \lambda u \mid \lambda \geq 0\} \subseteq C$ for one $x \in C$. Furthermore*

$$A(\cap_i C_i) = \cap_i A(C_i) \quad \text{if} \quad \cap_i C_i \neq \phi, \tag{3.2}$$

when $\{C_i \mid i \in I\}$ is a collection of closed convex sets in R^n.

Proof. Apply Lemma 2 to $P = R^n$ and the indicator function (2.3) of C to obtain the first statement. Equation (3.2) is easy to deduce from this.

Stoker [18] calls $A(C)$ the characteristic cone of C, while Fenchel [7; 42–46] calls it the asymptotic cone. The first part of the corollary, and the next lemma, were proved essentially in [18], but not in a form convenient enough for the present context.

LEMMA 3. *If C is a non-empty closed convex set in R^n, then C is compact if and only if $A(C) = \{0\}$.*

Proof. Obviously $A(C) \neq \{0\}$ implies C is not compact. Conversely, if C is not compact, there exists a sequence $x_1, x_2, \cdots$, in C such that $||x_i||$ increases without bound (given any particular norm). Choosing a subsequence if necessary, we have

$$\lim_i (x_i/||x_i||) = u, \quad ||u|| = 1.$$

We shall show that $u \in A(C)$. Since $u \neq 0$, this will complete the proof. Let $H = \{x \mid (a, x) \leq \alpha\}$ be any closed half-space containing C. Let $x \in H$ and $\lambda \geq 0$. Then

$$\begin{aligned}(a, x + \lambda u) &= (a, x) + \lambda \lim_i [(a, x_i)/||x_i||] \\ &\leq \alpha(1 + \lambda \lim_i (1/||x_i||)) = \alpha\end{aligned}$$

because $x_i \in H$ and $||x_i||$ increases without bound. Thus $x + \lambda u \in H$ whenever $x \in H$ and $\lambda \geq 0$. Since C is the intersection of all the closed half-spaces containing it (by a standard separation theorem), this implies $u \in A(C)$.

We proceed now with the proof of Theorem 1 itself.

LEMMA 4. *Suppose ($\mathfrak{P}$) is weakly consistent, and let $\bar{\mu}$ be the infimum of the real numbers μ for which the weak consistency condition is satisfied, $\infty > \bar{\mu} \geq -\infty$.*

Let

$$\bar{K} = \{\bar{k} = \langle k, \mu, \varepsilon\rangle \mid k \,\varepsilon\, K, \bar{\mu} < \mu \,\varepsilon\, R, 0 < \epsilon\},$$

and for each $\bar{k} \,\varepsilon\, \bar{K}$ *let*

$$C_{\bar{k}} = \{x \,\varepsilon\, P \mid f_k(x) \leq \mu + \epsilon\} \text{ if } k \,\varepsilon\, I, \text{ or } C_{\bar{k}} = \{x \,\varepsilon\, P \mid f_k(x) \leq \epsilon\} \text{ if } k \,\varepsilon\, J.$$

Then $\{C_{\bar{k}} \mid \bar{k} \,\varepsilon\, \bar{K}\}$ *is a collection of closed convex subsets of* P *such that every finite sub-collection has a non-empty intersection. Moreover,* $\bar{x}$ *is a solution to* $(\mathfrak{P})$ *if and only if* $\bar{x}$ *is common to all the* $C_{\bar{k}}$, *and in this case* $\bar{\mu}$ *is finite and is the minimum of* f *in* $(\mathfrak{P})$.

Proof. The $C_{\bar{k}}$ are closed and convex because the f_k are l.s.c. convex. It is easy to see from Definition 2 that every $n + 1$ of the sets have a point in common; hence every m of them have a point in common, for each positive integer m, by the special theorem of Helly for finite collections of convex sets (see [15, Theorem 1]). Now, $\bar{x}$ belongs to all the $C_{\bar{k}}$ if and only if $\bar{x}$ satisfies the constraints of $(\mathfrak{P})$ and $\bar{\mu} \geq f(\bar{x}) > -\infty$. But, if x satisfies the constraints and $\infty > f(x) = \mu$, then the weak consistency condition is satisfied for this μ, and hence $\mu \geq \bar{\mu}$. Thus in this case $\bar{x}$ is a solution and $\bar{\mu}$ is the finite minimum.

LEMMA 5. *If all weakly consistent convex programs which satisfy Definition 3 with* $K_0 = \phi$ *have solutions, then all weakly consistent asymptotically regular convex programs have solutions.*

Proof. Assume $(\mathfrak{P})$ satisfies the asymptotic regularity condition, with K_0 taken to be as small as possible. Let μ_0 be a real number for which $(\mathfrak{P})$ satisfies the weak consistency condition and let $\epsilon_0 > 0$. For each $k \,\varepsilon\, K - K_0 = K'$ let f'_k be the restriction of f_k to the polyhedral convex set

$$P' = \{x \,\varepsilon\, P \mid f_k(x) \leq \mu_0 + \epsilon_0 \text{ for } k \,\varepsilon\, I \cap K_0 \text{ and } f_k(x) \leq \epsilon_0 \text{ for } k \,\varepsilon\, J \cap K_0\},$$

and let $(\mathfrak{P}')$ be the convex program defined by P', $\{f'_k \mid k \,\varepsilon\, K'\}$, $I' = I - K_0$, $J' = J - K_0$. Lemma 4 implies that P' is non-empty and $(\mathfrak{P}')$ is weakly consistent. By Lemma 2b, any direction in which P' is unbounded and all the f'_k, $k \,\varepsilon\, K'$, are non-increasing is a direction in which P is unbounded and all the f_k, $k \,\varepsilon\, K$, are non-increasing. It follows that $(\mathfrak{P}')$ satisfies the asymptotic regularity condition with $K'_0 = \phi$, so that $(\mathfrak{P}')$ has a solution $\bar{x}'$ and a finite minimum $\bar{\mu}'$ by hypothesis. We shall show that $\bar{x}'$ can be modified into a solution to the original problem $(\mathfrak{P})$.

Fix any $k_1 \,\varepsilon\, K_0$. Since f_{k_1} is affine and K_0 is minimal, there exists a direction $u_1 \neq 0$ in which P is unbounded and all the f_k, $k \,\varepsilon\, K$, are non-increasing, and $f_{k_1}(\bar{x}' + \lambda u_1)$ is uniformly strictly decreasing in $\lambda \geq 0$. Now choose $\lambda_1 \geq 0$ large enough that $f_{k_1}(\bar{x}' + \lambda_1 u_1) \leq \bar{\mu}'$ if $k_1 \,\varepsilon\, I \cap K_0$, or $f_{k_1}(\bar{x}' + \lambda_1 u_1) \leq 0$ if $k_1 \,\varepsilon\, J \cap K_0$. Set $\bar{x}'_1 = \bar{x}' + \lambda_1 u_1$. Then

$$\bar{x}'_1 \,\varepsilon\, P \quad \text{and} \quad f_k(\bar{x}'_1) \leq f_k(\bar{x}') \quad \text{for all} \quad k \,\varepsilon\, K,$$

which implies in particular that $\bar{x}'_1$ is another solution to $(\mathfrak{P}')$. Now, however, we have $f_{k_1}(\bar{x}_1') \leq \bar{\mu}$ if k_1 was in $I \cap K_0$, or $f_{k_1}(\bar{x}_1') \leq 0$ if k_1 was in $J \cap K_0$. Next choose another $k_2 \in K_0$ and repeat the same process, replacing $\bar{x}_1'$ by $\bar{x}_2'$, etc. Since K_0 is finite, one eventually obtains a solution $\bar{x}$ to ($\mathfrak{P}'$) such that

$$f_k(\bar{x}) \leq \bar{\mu}' \quad \text{for} \quad k \in I \cap K_0 \quad \text{and} \quad f_k(\bar{x}) \leq 0 \quad \text{for} \quad k \in J \cap K_0 .$$

This $\bar{x}$ satisfies the constraints of the original problem ($\mathfrak{P}$) and has $f(\bar{x}) \leq \bar{\mu}'$. Thus ($\mathfrak{P}$) is consistent and $\bar{\mu} \leq f(\bar{x}) \leq \bar{\mu}' < \infty$, where $\bar{\mu}$ is the infimum of f in ($\mathfrak{P}$). But $\bar{\mu} \geq \bar{\mu}'$ if anything, by the definition of ($\mathfrak{P}'$). Hence $\bar{x}$ is a solution to ($\mathfrak{P}$).

LEMMA 6. *If all weakly consistent convex programs, such that there is no non-zero direction in which the set is unbounded and all the functions are non-increasing, have solutions, then all weakly consistent convex programs which satisfy Definition* 3 *with* $K_0 = \phi$ *have solutions.*

Proof. Let ($\mathfrak{P}$) be a weakly consistent, convex program of the latter sort, i.e. such that

(3.3) every direction in which P is unbounded and all f_k are non-increasing is a direction in which all the f_k are constant.

It will be shown that ($\mathfrak{P}$) is "equivalent" to a problem ($\mathfrak{P}'$) of the first sort.

Let $A(P)$ be the set of directions in which P is unbounded, and let $A(f_k)$ be the set of $u \in A(P)$ for which f_k is non-increasing. Let

$$A_0 = \cap_k A(f_k), \; M = \{v - u \mid v \in A_0, u \in A_0\}.$$

Then A_0 is a non-empty closed convex cone in R^n by Lemma 2a, and M is the subspace generated by A_0. Let M' be a subspace complementary to M, i.e. a subspace of R^n such that

$$M \cap M' = \{0\}, \; M + M' = R^n.$$

($C_1 + C_2$ will denote the set of all sums $x_1 + x_2$ for $x_1 \in C_1$ and $x_2 \in C_2$.) Let P' be the projection of P into M', i.e. $P' = M' \cap (M + P)$. Actually,

$$P' = \{x' \in M' \mid x' + u \in P \quad \text{for some} \quad u \in A_0\} \tag{3.4}$$

by definition of M, inasmuch as $x' + u = (x' + u - v) + v \in P$ when $x' + u - v \in P$ and $v \in A_0 \subseteq A(P)$. It is evident from the characterization of a polyhedral convex set as the "convex hull of finitely many points and rays" [7] that $M + P$ is polyhedral and $A(M + P) = M + A(P)$. Hence P' is a (non-empty) polyhedral convex set in R^n and, using (3.2)

$$A(P') = A(M' \cap (M + P)) = A(M') \cap A(M + P) = M' \cap (M + A(P)). \tag{3.5}$$

Now, for each $x' \in P'$ and $k \in K$, let

$$f_k'(x') = f_k(x' + u) \quad \text{where} \quad u \in A_0, \, x' + u \in P. \tag{3.6}$$

We shall show f_k' is a well-defined l.s.c. convex function on P'.

Suppose first that $x' + u_1 \in P$ and $x' + u_2 \in P$ for $u_1 \in A_0$ and $u_2 \in A_0$. Then $x' + u_1 + u_2 \in P$, since $A_0 \subseteq A(P)$. Also, u_1 and u_2 are directions in which f_k is constant by (3.3). Hence

$$f_k(x' + u_1) = f_k(x' + u_1 + u_2) = f_k(x' + u_2).$$

This demonstrates that the definition of f'_k is unambiguous, i.e. that $f'_k(x')$ does not depend on the particular u chosen in (3.6). Of course, f'_k is defined on all of P' by (3.4). To prove f'_k is l.s.c. convex on P', it is enough, in view of Lemma 1, to prove f'_k is l.s.c. convex on each line segment in P'. Let $x' \in P'$ and $y' \in P'$. Choose $u_1 \in A_0$ and $u_2 \in A_0$ such that $x' + u_1 \in P$ and $y' + u_2 \in P$, as is possible by (3.4). Let $u = u_1 + u_2$. Then $u \in A_0$, $x' + u = x \in P$ and $y' + u = y \in P$, because A_0 is a convex cone and $A_0 \subseteq A(P)$. Moreover

$$(3.7) \qquad f'_k(\lambda x' + (1 - \lambda)y') = f_k(\lambda x + (1 - \lambda)y) \quad \text{for} \quad 0 \leq \lambda \leq 1.$$

by Definition (3.6). Since f_k is l.s.c. convex on P, the right side of (3.7) is l.s.c. convex in λ; hence the left side is also.

Let $(\mathfrak{P}')$ be the convex program corresponding to P' and $\{f'_k \mid k \in K\}$, with I and J as in $(\mathfrak{P})$. The weak consistency of $(\mathfrak{P}')$ follows from that of $(\mathfrak{P})$. Indeed, suppose $x \in P$ satisfies (2.5), and choose any $u \in A_0$ such that $x - u = x' \in P'$. (This is possible by (3.4), P' being the projection of P into M'.) Then $f'_k(x') = f_k(x)$ for all k by definition (3.6), so $x' \in P'$ satisfies (2.5) with f_k replaced by f'_k. A similar argument proves that $(\mathfrak{P})$ has a solution if $(\mathfrak{P}')$ has a solution.

Suppose now that u' is a direction in which P' is unbounded and all the f'_k are non-increasing. We shall show that $u' = 0$. This will finish the proof of Lemma 6, because, by the hypothesis, then $(\mathfrak{P}')$ (and hence $(\mathfrak{P})$) has a solution. According to (3.5) and the definition of M we have

$$(3.8) \qquad u' \in M', u' = v - u + w, \quad \text{where} \quad v \in A_0, u \in A_0, w \in A(P).$$

Take any $x \in P$. Choose $u_0 \in A_0$ such that $x - u_0 = x' \in P'$. Then for each $\lambda \geq 0$

$$(3.9) \qquad (x' + \lambda u') + (u_0 + \lambda u) = x + \lambda(u' + u),$$

where $u_0 + \lambda u \in A_0$ and $u' + u = v + w \in A(P)$, because A_0 and $A(P)$ are convex cones and $A_0 \subseteq A(P)$. From (3.9) and Definition (3.6) we now get

$$(3.10) \quad f'_k(x' + \lambda u') = f_k(x + \lambda(u' + u)) \quad \text{for all} \quad \lambda \geq 0 \quad \text{and} \quad k \in K.$$

(The two sides are defined for all $\lambda \geq 0$ since $x' \in P'$, $u' \in A(P')$, $x \in P$, $u' + u \in A(P)$.) But u' is by assumption a direction in which every f'_k is non-increasing, so (3.10) and the arbitrary choice of x imply $u' + u$ is a direction in which every f_k is non-increasing. Thus $u' + u \in A_0$ and $u' = (u' + u) - u \in M$. But $u' \in M'$ by (3.8), so $u' \in M' \cap M = \{0\}$.

LEMMA 7. *Let $(\mathfrak{P})$ be a convex program such that there is no direction $u \neq 0$ in which P is unbounded and every f_k is non-increasing. If $(\mathfrak{P})$ is weakly consistent, then $(\mathfrak{P})$ has a solution.*

Proof. Consider the family of sets $C_{\bar{k}} = C_{k,\mu,\epsilon}$ defined in Lemma 4. Lemma

2b implies that $A(C_{k,\mu,\epsilon}) \subseteq A(f_k)$. Hence, according to our hypothesis,

$$\bigcap_{\bar{k}} A(C_{\bar{k}}) \subseteq \bigcap_k A(f_k) = \{0\}. \tag{3.11}$$

For each $\bar{k} \in \bar{K}$ let $A_{\bar{k}}$ be the intersection of $A(C_{\bar{k}})$ with the unit sphere $\{x \mid \|x\| = 1\}$ (in some norm). The $A(C_{\bar{k}})$ are closed cones by the corollary to Lemma 2, so $\{A_{\bar{k}} \mid \bar{k} \in \bar{K}\}$ is a collection of compact sets having an empty intersection. According to a standard compactness argument, there must exist $\bar{k}_1, \cdots, \bar{k}_r$ in $\bar{K}$ such that $A_{\bar{k}_1}, \cdots, A_{\bar{k}_r}$ have an empty intersection. Thus

$$A(C_{\bar{k}_1}) \cap \cdots \cap A(C_{\bar{k}_r}) = \{0\}. \tag{3.12}$$

For each $\bar{k} \in \bar{K}$ let

$$C'_{\bar{k}} = C_{\bar{k}} \cap C_{\bar{k}_1} \cap \cdots \cap C_{\bar{k}_r}.$$

From Lemma 4 we know that the $C'_{\bar{k}}$ are non-empty, in fact every finite subcollection of $\{C'_{\bar{k}} \mid \bar{k} \in \bar{K}\}$ has a non-empty intersection. Moreover, the $C'_{\bar{k}}$ are all *compact* by Lemma 3, (3.2), and (3.12). The same fact about compactness used above now implies the existence of some $\bar{x}$ such that

$$\bar{x} \in \bigcap_{\bar{k}} C'_{\bar{k}} = \bigcap_{\bar{k}} C_{\bar{k}}.$$

Such an $\bar{x}$ is a solution to $(\mathfrak{P})$ by Lemma 4.

Proof of Theorem 1. The chain of Lemmas 5, 6 and 7 says that every weakly consistent asymptotically regular convex program has a solution. The second assertion of Theorem 1 is contained in Lemma 4.

4. Inequalities and Lagrange multipliers. The rest of the paper is devoted to deducing consequences of Theorem 1. We begin by showing to what degree the familiar Lagrange multiplier theory for ordinary convex programs [14] is true for the more general problems $(\mathfrak{P})$.

THEOREM 2. Suppose the convex program $(\mathfrak{P})$ is asymptotically regular. Let $\bar{\mu} = \infty$ if $(\mathfrak{P})$ is not consistent; otherwise let $\bar{\mu}$ be the finite minimum in $(\mathfrak{P})$ guaranteed by Theorem 1. Then $\bar{\mu} > \alpha \in R$ if and only if there exist $i_1, \cdots, i_r$ in I and $j_1, \cdots, j_s$ in J with $1 \leq r + s \leq n + 1$, and "Lagrange multipliers"

$$\lambda_{i_1} > 0, \cdots, \lambda_{i_r} > 0, \lambda_{j_1} > 0, \cdots, \lambda_{j_s} > 0, \lambda_{i_1} + \cdots + \lambda_{i_r} = 1, \tag{4.1}$$

such that

$$[\lambda_{i_1} f_{i_1}(x) + \cdots + \lambda_{i_r} f_{i_r}(x)] + [\lambda_{j_1} f_{j_1}(x) + \cdots + \lambda_{j_s} f_{j_s}(x)] > \alpha \tag{4.2}$$

for all $x \in P$.

(It is permissible here that either $r = 0$ or $s = 0$, but not both; in the first case, the summation condition naturally has to be omitted in (4.1). This degenerate case can not arise if the constraints in $(\mathfrak{P})$ can be satisfied.)

Proof. By Theorem 1 and Definition 2, $\bar{\mu} > \alpha$ if and only if, for some $\epsilon > 0$, there exists a subset K_1 of K containing $n + 1$ indices or less, such that the system of inequalities

(4.3) $f_k(x) - \alpha - \epsilon \leq 0$ for $k \varepsilon I \cap K_1$, $f_k(x) - \epsilon \leq 0$ for $k \varepsilon J \cap K_1$

has no solution in P. Let

$$C = \{x \varepsilon P \mid f_k(x) < \infty \quad \text{for all} \quad k \varepsilon K_1\}.$$

This is a convex set. In particular, the version of (4.3) with strict inequalities could have no solution in C. If $C \neq \phi$, it follows from a theorem of Fan, Glicksberg and Hoffman [6, Theorem 1] that

$$(4.4) \qquad \sum_{k \varepsilon I \cap K_1'} \mu_k(f_k(x) - \alpha - \epsilon) + \sum_{k \varepsilon J \cap K_1'} \mu_k(f_k(x) - \epsilon) \geq 0$$

for all $x \varepsilon C$, where the μ_k are certain positive real numbers and K_1' is some non-empty subset of K_1. Actually (4.4) holds on all of P, since the left side is unambiguously $+ \infty$ for x in P but not in C. If $C = \phi$, (4.4) is true in the same way for $K_1' = K_1$ and arbitrary $\mu_k > 0$. Now if $I \cap K_1' \neq \phi$, let μ_0 be the sum of the μ_k for $k \varepsilon I \cap K_1'$ and let $\lambda_k = \mu_k/\mu_0$ for each $k \varepsilon K_1'$. Setting $I \cap K_1' = \{i_1, \cdots, i_r\}$ and $J \cap K_1' = \{j_1, \cdots, j_s\}$, we then have (4.1), and we get (4.2) from (4.4). On the other hand, if $I \cap K_1' = \phi$, no vector x can satisfy the constraints of ($\mathfrak{P}$), since the left side of (4.4) would be negative for such vectors. In this case let $\lambda_{j_l} = \lambda \mu_{j_l}$ for $l = 1, \cdots, s$, where $\lambda > 0$ is large enough so that $\lambda \varepsilon (\mu_{j_1} + \cdots + \mu_{j_s}) > \alpha$. Then (4.2) follows once more from (4.4).

Corollary 1. *Suppose ($\mathfrak{P}$) is an asymptotically regular convex program whose constraints can be satisfied. Let Q be the convex subset of R^K consisting of all $y = \{\lambda_k \geq 0 \mid k \varepsilon K\}$ with $\lambda_k = 0$ for all but finitely many k, $\lambda_i > 0$ for at least one $i \varepsilon I$, $\sum_{i \varepsilon I} \lambda_i = 1$. Let*

$$(4.5) \qquad L(x, y) = \sum_{i \varepsilon I} \lambda_i f_i(x) + \sum_{j \varepsilon J} \lambda_j f_j(x)$$

for each $x \varepsilon P$ and $y \varepsilon Q$, using the convention $0 \infty = 0$. Then

$$(4.6) \qquad \inf_{x \varepsilon P} \sup_{y \varepsilon Q} L(x, y) = \bar{\mu} = \sup_{y \varepsilon Q} \inf_{x \varepsilon P} L(x, y)$$

with $\bar{\mu}$ as in Theorem 2.

Proof. $\sup\{L(x, y) \mid y \varepsilon Q\}$ equals $f(x)$ when x satisfies the constraints of ($\mathfrak{P}$), but equals $+\infty$ otherwise. This implies the first half of (4.6). Theorem 2 gives the second half.

The conclusion of Corollary 1 appears to be true even when the constraints of ($\mathfrak{P}$) can not be satisfied, but we have omitted the proof because of its length.

Corollary 2. *Let $\{f_j \mid j \varepsilon J\}$ be a finite or infinite collection of l.s.c. convex functions on R^n. Assume there exixts a finite (permissibly empty) affine subcollection $\{f_j \mid j \varepsilon J_0\}$, such that there are no directions in which f_j is non-increasing for all $j \varepsilon J$, except perhaps for directions in which all the f_j for $j \varepsilon J - J_0$ are* *constant. Then either the system of inequalities*

$$(4.7)\qquad f_j(x) \leq 0 \quad \textit{for all} \quad j \in J, \quad \textit{where} \quad x \in R^n,$$

has a solution, or there exist $j_1, \cdots, j_s$ *in* J *and* $\lambda_{j_1} > 0, \cdots, \lambda_{j_s} > 0$, $1 \leq s \leq n + 1$, *such that*

$$(4.8)\qquad \lambda_{j_1} f_{j_1}(x) + \cdots + \lambda_{j_s} f_{j_s}(x) \geq \epsilon > 0 \quad \textit{for all} \quad x \in R^n.$$

(The alternatives are mutually exclusive.)

Proof. Define ($\mathfrak{P}$) as in Example 5, with $P = R^n$. Then ($\mathfrak{P}$) is asymptotically regular. Take $\bar{\mu}$ as in Theorem 2. If $\bar{\mu} \leq 0$, (4.7) has a solution by Theorem 1. If $\bar{\mu} > 0$, (4.8) follows from Theorem 2.

Theorem 2 extends the results of Bohnenblust, Karlin and Shapley [2] and Fenchel [7] referred to in the introduction. Both apply to the case where $I = K$, $J = \phi$ and $P = R^n$. The first concerns f_k which are all $+\infty$ outside of a certain compact set $C \neq \phi$ where they are finite and continuous. The second requires that the set of points x for which $\infty > \sup\{f_k(x) \mid k \in K = I\} = f(x)$ be non-empty and bounded. Both results follow from Theorem 2, because the asymptotic regularity condition is then satisfied trivially, there being no infinite ray along which the f_k are all even finite. Fenchel also showed [7; 100] that his boundedness hypothesis would be satisfied when the (closures of the) non-empty convex sets $C_k = \{x \mid f_k(x) < \infty\}$ are not all unbounded in some single non-zero direction. (In this event, finitely many of the C_k have a bounded intersection; see the proof of Lemma 7.)

The results in this section are complementary to those in [6], which deal with *finite* collections of convex functions on spaces of arbitrary dimension, and (roughly speaking) have strict inequalities wherever we have weak inequalities, and vice versa.

5. A new minimax theorem. Let $L(x, y)$ be a real-valued function defined on $C \times D$ in $R^n \times R^m$, where C and D are non-empty convex sets, such that L is convex on C for each $y \in D$ and concave on D for $x \in C$. Assume also that L is *completely closed* in the sense that $\{x \in C \mid L(x, y) \leq \mu\}$ is closed in R^n for each $y \in D$ and $\mu \in R$, and $\{y \in D \mid L(x, y) \geq \mu\}$ is closed in R^m for each $x \in C$ and $\mu \in R$. (This is always true, in particular, when C and D are closed and L is continuous in each argument.)

Let P and Q be non-empty polyhedral convex sets in R^n and R^m respectively, with $P \cap C \neq \phi$ and $Q \cap D \neq \phi$. It is elementary that

$$(5.1)\qquad \inf_{x \in P \cap C} \sup_{y \in Q \cap D} L(x, y) \geq \sup_{y \in Q \cap D} \inf_{x \in P \cap C} L(x, y)$$

(see [12; 22]). The two sides of (5.1) are finite, equal and attained, if and only if there exist $\bar{x} \in P \cap C$ and $\bar{y} \in Q \cap D$ with

$$(5.2)\qquad L(x, \bar{y}) \geq L(\bar{x}, \bar{y}) \geq L(\bar{x}, y) \quad \text{for all} \quad x \in P \cap C \quad \text{and} \quad y \in Q \cap D.$$

Such a pair $\langle \bar{x}, \bar{y} \rangle$ is called a *saddle-point of* L.

It is known that a saddle-point always exists when $P_\cap C$ and $Q_\cap D$ are compact. (See the generalizations of the von Neumann minimax theorem proved by Kakutani [11] and Sion [17].) Our aim here is to extend this theorem to the non-compact case using a condition suggested by the results in previous sections of this paper.

DEFINITION 5. We shall say L *is asymptotically regular relative to* $P_\cap C$ *and* $Q_\cap D$ if L has the following dual properties:

(a) the only infinite rays $\{x + \lambda u \mid \lambda \geq 0\}$ contained in $P_\cap C$, along which *all* the $\{f_y \mid y \in Q_\cap D, f_y(x) = L(x, y)\}$ are non-increasing, are those rays along which all the f_y are constant, and for which the intersection of the line $\{x + \lambda u \mid -\infty < \lambda < \infty\}$ with P is contained in its intersection with C;

(b) the only infinite rays $\{y + \lambda v \mid \lambda \geq 0\}$ contained in $Q_\cap D$, along which *all* the $\{g_x \mid x \in P_\cap C, g_x(y) = L(x, y)\}$ are non-decreasing, are those rays along which all the g_x are constant, and for which the intersection of the line $\{y + \lambda v \mid -\infty < \lambda < \infty\}$ with Q is contained in its intersection with D.

Remark. This asymptotic regularity condition is always satisfied, of course, when there are no infinite rays in $P_\cap C$ along which all the f_y are non-increasing, and no infinite rays in $Q_\cap D$ along which all the g_x are non-decreasing. This is true in particular when $P_\cap C$ and $Q_\cap D$ are bounded, as in the minimax results cited above. When P and Q are contained in the closures of $P_\cap C$ and $Q_\cap D$, the linear intersection condition may be omitted (assuming that L is completely closed).

THEOREM 3. *Let L be completely closed convex-concave on $C \times D$ as described above, and suppose L is asymptotically regular relative to $P_\cap C$ and $Q_\cap D$. Then L has a saddle-point $\langle \bar{x}, \bar{y} \rangle$ relative to $P_\cap C$ and $Q_\cap D$.*

Proof. Let α_0 and β_0 be the values of the two extrema in (5.1), $\infty \geq \alpha_0 \geq \beta_0 \geq -\infty$. We shall show that $\alpha_0 \leq \beta_0$ if $\alpha_0 > -\infty$, and that the "inf" on the left side (5.1) is finite and attained at some $\bar{x}$ if $\alpha_0 < \infty$. This will be enough to prove the theorem. Indeed, by a dual argument we would have $\alpha_0 \leq \beta_0$ if $\beta_0 < \infty$, and the "sup" on the right side of (5.1) would be finite and attained at some $\bar{y}$ if $\beta_0 > -\infty$. By combining these results with the fact that $\alpha_0 \geq \beta_0$, we could then conclude that $\infty > \alpha_0 = \beta_0 > -\infty$, and that both the "inf" and "sup" are attained at $\bar{x}$ and $\bar{y}$, respectively. This $\langle \bar{x}, \bar{y} \rangle$ would be a saddle-point (see [12; 23]).

For each $y \in Q_\cap D$, assign f_y the value $+\infty$ on P outside of C (with f_y as in Definition 4). Then $\{f_y \mid y \in Q_\cap D = K\}$ is a collection of l.s.c. convex functions on R^n which, together with P, defines an asymptotically regular convex program $(\mathfrak{P})$ for $I = K$, $J = \phi$. (cf. Lemma 2b and Corollary 2′ to Theorem 1.) Also, α_0 is the $\bar{\mu}$ in Theorem 2. Hence the "inf" on the left side of (5.1) is attained at some $\bar{x}$ by Theorem 1 if $\alpha_0 < \infty$. On the other hand, suppose $\alpha_0 > \alpha \in R$. By Theorem 2 there then exist $y_1, \cdots, y_r$ in $Q_\cap D$ and $\lambda_1 > 0, \cdots, \lambda_r > 0$ with $\lambda_1 + \cdots + \lambda_r = 1$, such that $\sum \lambda_i f_{y_i}(x) > \alpha$ for all $x \in P$. By the definition of f_y and the concavity of L in y, we now have

$$\alpha < \sum \lambda_i L(x, y_i) \leq L(x, y_0) \quad \text{for all} \quad x \in P \cap C,$$

where $y_0 = \lambda_1 y_1 + \cdots + \lambda_r y_r \in Q \cap D$. This says that $\alpha < \beta_0$. Thus $\beta_0 > \alpha$ whenever $\alpha_0 > \alpha \in R$, so $\alpha_0 = \beta_0$ if $\alpha_0 > -\infty$.

6. **Finite convex programs.** We shall now illustrate some of our results by specializing them to a case of particular interest.

THEOREM 4. *Let $f_1, \cdots, f_r$ and $g_1, \cdots, g_s$ be continuous finite convex functions on a non-empty polyhedral convex set P in R^n, and let $g_{s+1}, \cdots, g_t$ be affine ($r \geq 1$, $t \geq s \geq 0$). Assume that:*

(a) *the only infinite rays $\{x + \lambda u \mid \lambda \geq 0\}$ in P along which all the f_i and g_j are non-increasing for $1 \leq i \leq r$ and $1 \leq j \leq t$, are those along which the f_i and g_j are constant for $1 \leq i \leq r$ and $1 \leq j \leq s$;*

(b) *there exists some x in the relative interior of P such that*

$$g_1(x) < 0, \cdots, g_s(x) < 0, \; g_{s+1}(x) \leq 0, \cdots, g_t(x) \leq 0. \tag{6.1}$$

(The relative interior of P is its interior in the smallest linear manifold containing it.)

Then the convex program

$(\mathfrak{P}_0)$ *minimize* $\max \{f_1(x), \cdots, f_r(x)\}$ *subject to the constraints:* $x \in P$, $g_1(x) \leq 0, \cdots, g_t(x) \leq 0$

has at least one solution $\bar{x}$. Moreover, let

$$Q = \{y = \langle \lambda_1, \cdots, \lambda_r ; \mu_1, \cdots, \mu_t \rangle \in R^{r+t} \mid \lambda_i \geq 0, \mu_j \geq 0, \lambda_1 + \cdots + \lambda_r = 1\},$$

$$L(x, y) = \sum \lambda_i f_i(x) + \sum \mu_j g_j(x) \quad \text{for} \quad x \in P \quad \text{and} \quad y \in Q.$$

Then $\bar{x} \in P$ is a solution to $(\mathfrak{P}_0)$ if and only if there exists some $\bar{y} \in Q$ such that

$$L(x, \bar{y}) \geq L(\bar{x}, \bar{y}) \geq L(\bar{x}, y) \quad \text{for all} \quad x \in P \quad \text{and} \quad y \in Q,$$

where $L(\bar{x}, \bar{y}) = \bar{\mu}$ is the minimum in $(\mathfrak{P}_0)$.

Proof. For $C = P$ and $D = Q$, assumptions (a) and (b) actually guarantee that the hypothesis of Theorem 3 is satisfied; Theorem 4 can then be derived easily from Theorem 3. Rather than verify this, however, we shall take a somewhat different approach. $(\mathfrak{P}_0)$ is a consistent asymptotically regular convex program by (a) and (b) (cf. Lemma 2b), and hence it has a solution $\bar{x}$ by Theorem 1. The "inf sup" of L is finite and attained at $\bar{x}$, as pointed out in the proof of Corollary 1 to Theorem 2. Theorem 4 will therefore follow from the same corollary, as soon as we prove that the (finite) "sup inf" of L is attained at some $\bar{y} \in Q$. By Theorem 1, it is enough to show that the consistent convex program $(\mathfrak{P})$, in which supremum of the collection of (convex, actually linear) functions

$$\{h_x \mid x \in P, \; h_x(y) = -\sum \lambda_i f_i(x) - \sum \mu_j g_j(x)\}$$

is minimized on Q, is asymptotically regular. Suppose, therefore, that v is a direction in which Q is unbounded and all the h_x, $x \in P$, are non-increasing. Then $v = \langle 0, \cdots, 0, \nu_1, \cdots, \nu_t \rangle$ where each $\nu_i \geq 0$ and $0 \leq \sum \nu_i g_i(x) = g(x)$ for all $x \in P$. Since (6.1) can be satisfied, we must have $\nu_j = 0$ for $j = 1, \cdots, s$. Hence g is affine. But (6.1) can actually be satisfied by some x in the relative interior of P, so the non-negative affine function g vanishes at a relative interior point of the convex set P. Therefore $0 = g(x) = \sum \nu_i g_i(x)$ for all $x \in P$. This implies v is a direction in which all the h_x, $x \in P$, are constant, and finishes the proof.

Ghouila-Houri [1; 83] proved the minimax criterion in Theorem 4 for the case where $P = R^n$ and $r = 1$, under assumption (b). A similar theorem of Karlin [12; 201] applies even to non-polyhedral P under (b), provided $t = s$ (i.e. provided all inequalities can be satisfied strictly). The original theorem of this type is, of course, due to Kuhn and Tucker [14]. None of these authors supplies a criterion for the existence of a solution to $(\mathfrak{P}_0)$ (other than compactness of the non-empty set of vectors satisfying the constraints, say, or the minimax criterion itself).

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