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1. INTRODUCTION

Let X denote a locally convex Hausdorff (topological vector) space over the reals R. Let X^* denote the dual of X, and write $\langle x, x^* \rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

A multivalued mapping $T: X \to X^*$ is called a *monotone operator* if

$$(1.1) \qquad \langle x - y, x^* - y^* \rangle \geq 0$$

whenever $x^* \in T(x)$ and $y^* \in T(y)$. It is called a *maximal* monotone operator if, in addition, the graph of T, in other words, the set

$$(1.2) \qquad \{(x, x^*) \mid x^* \in T(x)\} \subset X \times X^*,$$

is not properly contained in the graph of any other monotone operator $T': X \to X^*$. It is said to be *locally bounded* at x if there exists a neighborhood U of x such that the set

$$(1.3) \qquad T(U) = \bigcup \{T(y) \mid y \in U\}$$

is an equicontinuous subset of X^*. (Of course, if X is a Banach space, then the equicontinuous subsets of X^* coincide with the bounded subsets.)

In the case where X is a Banach space, it follows from a result of T. Kato [7] that a monotone operator $T: X \to X^*$ is locally bounded at a point x if x is an interior point of the set

$$(1.4) \qquad D(T) = \{x \in X \mid T(x) \neq \emptyset\}$$

and T is locally hemibounded at x (in other words, for each $u \in X$ there exists an $\varepsilon > 0$ such that the set

$$\bigcup \{T(x + \lambda u) \mid 0 \leq \lambda \leq \varepsilon\}$$

is equicontinuous in X^*). Moreover, Kato showed in [6] that the assumption of local hemiboundedness is redundant when X is finite-dimensional.

In this note, we establish the following more general result, which implies, among other things, that the assumption of local hemiboundedness is redundant even when X is an infinite-dimensional Banach space. (The abbreviations conv, int, and cl denote convex hull, interior, and (strong) closure, respectively.)

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THEOREM 1. *Let* X *be a Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose either that*

$$\operatorname{int}(\operatorname{conv} D(T)) \neq \emptyset \tag{1.5}$$

or that X *is reflexive and there exists a point of* $D(T)$ *at which* T *is locally bounded. Then* $\operatorname{int} D(T)$ *is a nonempty convex set whose closure is* $\operatorname{cl} D(T)$. *Furthermore,* T *is locally bounded at each point of* $\operatorname{int} D(T)$, *whereas* T *is not locally bounded at any boundary point of* $D(T)$.

We shall prove Theorem 1 and its corollaries in Section 3.

The following corollary corresponds to the result of Kato [7] that a single-valued, monotone operator T on an open subset of a Banach space X is demicontinuous if it is hemicontinuous.

COROLLARY 1.1. *Suppose the hypothesis of Theorem* 1 *is satisfied, and let* D_0 *denote the subset of* $D(T)$ *where* T *is single-valued. Then* $D_0 \subset \operatorname{int} D(T)$, *and* T *is demicontinuous on* D_0, *in other words, continuous as a single-valued mapping from* D_0 *in the strong topology to* X^* *in the weak** *topology.*

The convexity assertion of Theorem 1 is also worth noting. We have shown elsewhere [11] that if X is a *reflexive* Banach space and $T: X \to X^*$ is a maximal monotone operator, then $\operatorname{cl} D(T)$ is a convex set. In fact, if X is also separable, then $D(T)$ itself is a *virtually convex* set, in the sense that for each relatively (strongly) compact subset K of $\operatorname{conv} D(T)$ and each $\varepsilon > 0$ there exists a strongly continuous mapping p of K into $D(T)$ such that $\|p(x) - x\| \leq \varepsilon$ for every $x \in K$. In this context, Theorem 1 contributes a condition under which $D(T)$ is virtually convex even though X may not be reflexive.

COROLLARY 1.2. *Under the hypothesis of Theorem* 1, $D(T)$ *is virtually convex, and in particular* $\operatorname{cl} D(T)$ *is convex. If in addition* $D(T)$ *is dense in* X, *then* $D(T)$ *must be all of* X.

When X is reflexive, we can apply Theorem 1 to the maximal monotone operator T^{-1}, where

$$T^{-1}(x^*) = \{x \mid x^* \in T(x)\}. \tag{1.6}$$

Since $D(T^{-1})$ is the same as the *range* of T, that is, the set

$$R(T) = \bigcup \{T(x) \mid x \in X\}, \tag{1.7}$$

one can obtain various corollaries concerning the range of T.

COROLLARY 1.3. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Then* $0 \in \operatorname{int} R(T)$ *if and only if* $0 \in \operatorname{cl} R(T)$ *and there exist positive numbers* α *and* ε *such that*

$$\|x\| \geq \alpha \implies \|x^*\| \geq \varepsilon \qquad (\forall\, x^* \in T(x)). \tag{1.8}$$

COROLLARY 1.4. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists a subset* $B \subset X$ *such that*

$$0 \in \operatorname{int}(\operatorname{conv} T(B)).$$

Then there exists an $x \in X$ *such that* $0 \in T(x)$.

COROLLARY 1.5. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a monotone operator (not necessarily maximal). Suppose there exist* $x_i^* \in T(x_i)$ $(i = 1, 2, \cdots)$ *such that*

$$\lim_{i \to \infty} \|x_i\| = \infty \quad \text{and} \quad \lim_{i \to \infty} \|x_i^* - x^*\| = 0.$$

Then x^* *is a boundary point of* $R(T)$.

COROLLARY 1.6. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. In order that* $R(T)$ *be all of* X^*, *it is necessary and sufficient that the sequence* $x_1^*, x_2^*, \cdots$ *have no strongly convergent subsequence whenever* $x_i^* \in T(x_i)$ $(i = 1, 2, \cdots)$ *and* $\lim_{i \to \infty} \|x_i\| = \infty$.

We remark that according to [11, Corollary 1 to Theorem 2], the condition $0 \in \mathrm{cl}\, R(T)$ in Corollary 1.3 is equivalent to the nonexistence of a $u \in X$ and a $\delta > 0$ such that $\langle u, x^* \rangle \leq -\delta$ for every $x^* \in R(T)$.

Corollary 1.4 is a generalization of the main existence theorem of G. J. Minty [8], which requires that the unit ball of X is smooth and that, in effect,

$$0 \in \mathrm{int}(\mathrm{conv}\, T_0(B)),$$

where T_0 is some mapping with the properties that $T_0(x) \subset T(x)$ for every x and

$$\sup_{x \in B} \sup_{x^* \in T_0(x)} \langle x, x^* \rangle < \infty.$$

The necessary and sufficient condition in Corollary 1.6 is satisfied, in particular, if the following condition is satisfied:

$$\text{if } x_i^* \in T(x_i) \ (i = 1, 2, \cdots) \text{ and } \lim_{i \to \infty} \|x_i\| = \infty, \text{ then } \lim_{i \to \infty} \|x_i^*\| = \infty.$$

(The two conditions are equivalent, of course, when X is finite-dimensional.) Under the additional assumption that X is uniformly convex and X^* is strictly convex, F. Browder [4, Theorem 4] established that the latter condition is sufficient for $R(T)$ to be all of X^*.

In the case where T is the subdifferential of a lower-semicontinuous, proper convex function f on X (see [10], [12]), Theorem 1 reduces to known results (see [1], [9]), provided $\mathrm{int}(\mathrm{conv}\, D(T))$ is nonempty; but the fact that the local boundedness of T at some point of $D(T)$ implies the nonemptiness of $\mathrm{int}(\mathrm{conv}\, D(T))$ has not been pointed out previously. (Theorem 1 gives this implication only for reflexive X, but reflexivity is used in the proof only to ensure that $\mathrm{cl}\, D(T)$ is convex, and the latter is true for subdifferential mappings even if X is a nonreflexive Banach space [3].)

2. GENERAL BOUNDEDNESS THEOREM

Theorem 1 will be deduced from a broader result, which is applicable even when X is not a Banach space.

THEOREM 2. *Let* X *be a locally convex (real) Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exist a subset* $S \subset D(T)$ *and an equicontinuous subset* $A \subset X^*$ *such that* $T(x)$ *meets* A *for every* $x \in S$ *and one of the following two conditions holds:*

(a) $\operatorname{int}(\operatorname{cl} S) \neq \emptyset$,

(b) $\operatorname{int}(\operatorname{cl}(\operatorname{conv} S)) \neq \emptyset$ *and* $\sup_{x \in S} \sup_{x^* \in A} |\langle x, x^* \rangle| < \infty$.

Then $\operatorname{int} D(T)$ *is a nonempty, open, convex set whose closure is* $\operatorname{cl} D(T)$. *Furthermore,* T *is locally bounded at each point of* $\operatorname{int} D(T)$, *whereas* T *is not locally bounded at any boundary point of* $D(T)$.

In proving Theorem 2, we shall use three lemmas.

LEMMA 1. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a monotone operator. Let* B *be an equicontinuous subset of* X^*. *Then, for every* $x \in X$, *there exists an* $x^* \in X^*$ *such that*

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall\, u \in X,\ \forall\, u^* \in T(u) \cap B).$$

Proof. Give X^* the weak* topology. Then B is a relatively compact subset of X^*, and the dual of X^* can be identified with X. Let S be the restriction of T^{-1} to B; thus

$$S(u^*) = \{u \in X \mid u^* \in T(u)\}$$

if $u^* \in B$, while $S(u^*) = \emptyset$ if $u^* \notin B$. Then S is a monotone operator from X^* to X with $D(S) \subset B$. According to the theorem of H. Debrunner and P. Flor [5], there exists for every $x \in X$ an $x^* \in X^*$ such that

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall\, u^* \in B,\ \forall\, u \in S(u^*)).$$

The latter relation is identical to the one in the lemma, in view of the definition of S.

COROLLARY. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. If* T *is globally bounded, in other words, if* $R(T)$ *is an equicontinuous subset of* X^*, *then* $D(T)$ *is all of* X.

LEMMA 2. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Then, for each weak***-closed, equicontinuous subset* B *of* X^*, *the set*

$$T^{-1}(B) = \{x \mid B \cap T(x) \neq \emptyset\}$$

is closed in X.

Proof. Let $y \in \operatorname{cl} T^{-1}(B)$. For each neighborhood U of y, the intersection $T(U) \cap B$ is nonempty; denote the weak* closure of this intersection by B_U. Since B is weak*-closed and equicontinuous, each B_U is weak*-compact. The collection of sets B_U, as U ranges over all neighborhoods U of y, has the property that every finite subcollection has a nonempty intersection, and hence this collection as a whole has a nonempty intersection. Thus there exists some $y^* \in B$ such that y^* belongs to the weak* closure of $T(U)$ for every neighborhood U of y. We shall show that the latter implies that $y^* \in T(y)$, and this will prove the lemma.

Consider any $u \in X$ and $u^* \in T(u)$. For each $\varepsilon > 0$, we can find a neighborhood U of y and a weak* neighborhood U^* of y^* such that

$$|\langle x - y, u^* \rangle| \leq \varepsilon \qquad (\forall x \in U), \tag{2.1}$$

$$|\langle u - y, x^* - y^* \rangle| \leq \varepsilon, \qquad (\forall x^* \in U^*), \quad \text{and} \tag{2.2}$$

$$|\langle x - y, x^* \rangle| \leq \varepsilon \qquad (\forall x \in U, \ \forall x^* \in B). \tag{2.3}$$

(Condition (2.3) can be met, because B is equicontinuous.) Let x^* be an element of the set $T(U) \cap U^* \cap B$, which is nonempty by the choice of y^*. Let x be an element of U such that $x^* \in T(x)$. The monotonicity of T gives the relation

$$\langle u - x, u^* - x^* \rangle \geq 0,$$

and hence

$$\begin{aligned} \langle u - y, u^* - y^* \rangle &= \langle u - x, u^* - x^* \rangle + \langle x - y, u^* \rangle \\ &\quad + \langle u - y, x^* - y^* \rangle - \langle x - y, x^* \rangle \geq 0 - \varepsilon - \varepsilon - \varepsilon = -3\varepsilon . \end{aligned} \tag{2.4}$$

Since (2.4) holds for arbitrary $\varepsilon > 0$, we must have that

$$\langle u - y, u^* - y^* \rangle \geq 0. \tag{2.5}$$

Furthermore, inequality (2.5) holds for each $u \in X$ and $u^* \in T(u)$. Therefore the maximality of T implies that $y^* \in T(y)$.

LEMMA 3. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose that* $\mathrm{cl}(\mathrm{conv}\, D(T))$ *has a nonempty interior and that* x *is a point of* $D(T)$ *not belonging to this interior. Then the set* $T(x)$ *contains at least one half-line (and consequently* $T(x)$ *is not equicontinuous).*

Proof. Since x is a boundary point of $\mathrm{cl}(\mathrm{conv}\, D(T))$, which is a closed, convex set with a nonempty interior, there exists a supporting hyperplane to $\mathrm{cl}(\mathrm{conv}\, D(T))$ at x. Thus there exists a $y^* \in X^*$ ($y^* \neq 0$) such that

$$\langle x, y^* \rangle \geq \langle u, y^* \rangle \qquad (\forall u \in D(T)). \tag{2.6}$$

Let x^* be an element of $T(x)$. By (2.6) and the monotonicity of T, each vector $x^* + \lambda y^*$ ($\lambda \geq 0$) satisfies the condition

$$\begin{aligned} \langle u - x, u^* - (x^* + \lambda y^*) \rangle &= \langle u - x, u^* - x^* \rangle + \lambda \langle x - u, y^* \rangle \geq 0 \\ &\qquad (\forall u \in D(T), \ \forall u^* \in T(u)). \end{aligned} \tag{2.7}$$

Since T is maximal, (2.7) implies that $x^* + \lambda y^* \in T(x)$. Thus $T(x)$ contains the half-line

$$\{x^* + \lambda y^* \mid \lambda \geq 0\}.$$

Proof of Theorem 2. If condition (a) holds for S, then condition (b) holds for $S_\alpha = S \cap \alpha A_1^0$, where α is a sufficiently large, positive number and A_1^0 is the polar of $A_1 = A \cup (-A)$. (Since A_1 is equicontinuous, A_1^0 is a neighborhood of the origin in X.) Thus it suffices to prove the theorem with condition (b).

Condition (b) implies in particular that cl(conv D(T)) has a nonempty interior. Let $\bar{x}$ be a point in this interior. We shall prove that T is locally bounded at $\bar{x}$ and that $\bar{x} \in D(T)$. This will establish Theorem 2 except for the assertion that T is not locally bounded at boundary points of D(T).

We deal first with the case where

$$\bar{x} \in \operatorname{int}(\operatorname{cl}(\operatorname{conv} S)). \tag{2.8}$$

For each equicontinuous subset B of X^*, we let $T_B(x)$ denote the set of all $x^* \in X^*$ such that

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall\, u \in X,\ \forall\, u^* \in T(u) \cap B). \tag{2.9}$$

By Lemma 1 and the monotonicity of T, we have that

$$T(x) \subset T_B(x) \neq \emptyset \qquad (\forall\, x \in X). \tag{2.10}$$

Note that $T_B(x)$ is always a weak*-closed set, since by definition it is the intersection of a certain collection of weak*-closed half-spaces in X^*.

To prove that T is locally bounded at $\bar{x}$, we consider the mapping $T_B\colon x \to T_B(x)$ in the case where B = A. Choose a convex neighborhood V of the origin in X such that

$$\bar{x} + 2V \subset \operatorname{cl}(\operatorname{conv} S), \tag{2.11}$$

as is possible by (2.8). Let

$$\mu = \sup_{x \in S}\ \sup_{u^* \in A} |\langle x, u^* \rangle|. \tag{2.12}$$

(Note that μ is finite by hypothesis.) For each $u^* \in A$, the closed, convex set

$$\{x \mid |\langle x, u^* \rangle| \leq \mu\}$$

contains S, and hence it contains cl(conv S). Thus (2.12) actually implies the inequality

$$|\langle x, u^* \rangle| \leq \mu \qquad (\forall\, x \in \operatorname{cl}(\operatorname{conv} S),\ \forall\, u^* \in A). \tag{2.13}$$

Select an $x \in (\bar{x} + V)$ and an $x^* \in T_A(x)$. Relations (2.11) and (2.13) imply that

$$\langle u - x, x^* \rangle \leq \langle u - x, u^* \rangle \leq |\langle u, u^* \rangle| + |\langle x, u^* \rangle| \leq 2\mu,$$

for every $u \in S$ and $u^* \in T(u) \cap A$. Thus $S \subset \{u \mid \langle u - x, x^* \rangle \leq 2\mu\}$, and it follows that

$$x + V \subset \bar{x} + 2V \subset \operatorname{cl}(\operatorname{conv} S) \subset \{u \mid \langle u - x, x^* \rangle \leq 2\mu\}.$$

Therefore $\langle v, x^* \rangle \leq 2\mu$ for every $v \in V$; hence

$$x^* \in (2\mu + 1)V^0,$$

where V^0, the polar of a neighborhood of the origin in X, is an equicontinuous subset of X^*. Since x was any element of $\bar{x} + V$ and x^* was any element of $T_A(x)$, we may conclude from (2.10) that

$$T(\bar{x} + V) = \bigcup \{T(x) \mid x \in (\bar{x} + V)\}$$

$$\subset \bigcup \{T_A(x) \mid x \in (\bar{x} + V)\} \subset (2\mu + 1)V^0 .$$

Thus $T(\bar{x} + V)$ is equicontinuous, and by definition T is locally bounded at $\bar{x}$.

To show that in fact $\bar{x} \in D(T)$, we consider the collection of all the (nonempty, weak*-closed) sets $T_B(\bar{x})$, where B is an equicontinuous subset of X^* containing A. This collection has the property that every finite subcollection has a nonempty intersection. Moreover, every $T_B(\bar{x})$ in the collection is contained in $T_A(\bar{x})$, which is equicontinuous (and hence weak*-compact) according to the preceding paragraph. The collection therefore has a nonempty intersection. Let $\bar{x}^*$ be an element in the intersection. By the definition of the sets $T_B(\bar{x})$, we must have that

$$\langle u - \bar{x}, u^* - \bar{x}^* \rangle \geq 0 \qquad (\forall\, u \in X,\ \forall\, u^* \in T(u)).$$

But T is a maximal monotone operator, so this implies $\bar{x}^* \in T(\bar{x})$. Thus $T(\bar{x}) \neq \emptyset$ and $\bar{x} \in D(T)$.

We deal now with the general case where $\bar{x}$ is an interior point of cl (conv D(T)), not necessarily satisfying (2.8). We shall reduce this case to the previous case by demonstrating the existence of a subset $S' \subset D(T)$ and an equicontinuous subset $A' \subset X^*$ such that T(x) meets A' for every $x \in S'$, condition (b) holds, and

$$\bar{x} \in \operatorname{int}(\operatorname{cl}(\operatorname{conv} S')) . \tag{2.14}$$

According to the argument already given, D(T) contains a nonempty, open set on which T is locally bounded, namely the interior of cl (conv S). Thus there exists a nonempty, open, convex set $W \subset D(T)$ such that T(W) is equicontinuous. Let

$$E = \bigcup_F \operatorname{int}(\operatorname{conv}(W \cup F)),$$

where the union is taken over all finite subsets F of D(T). Then E is a nonempty, open, convex set whose closure contains D(T). It follows that

$$\operatorname{int}(\operatorname{cl}(\operatorname{conv} D(T))) \subset \operatorname{int}(\operatorname{cl} E) = E,$$

and hence that $\bar{x} \in E$. Thus there exist elements $x_1, \cdots, x_n$ of D(T) such that

$$\bar{x} \in \operatorname{int}(\operatorname{conv}(W \cup \{x_1, \cdots, x_n\})). \tag{2.15}$$

Choose an arbitrary $x_i^* \in T(x_i)$ $(i = 1, \cdots, n)$, and let

$$A' = T(W) \cup \{x_1^*, \cdots, x_n^*\} .$$

Then A' is equicontinuous. Since W is convex, relation (2.15) implies the existence of an $x_0 \in W$ such that

$$\bar{x} \in \operatorname{int}(\operatorname{conv}(U \cup \{x_1, \cdots, x_n\})),$$

for every neighborhood U of x_0. Take U to be a neighborhood of x_0, contained in W, on which the linear functionals in A' are uniformly bounded, and let

$$S' = U \cup \{x_1, \cdots, x_n\} \subset D(T).$$

Then (2.14) is satisfied, $T(x)$ meets A' for every $x \in S'$, and condition (b) holds for S' and A' as desired.

The proof of Theorem 2 will be complete as soon as we demonstrate that T is not locally bounded at any boundary point of $D(T)$. Let y be a boundary point, and suppose that U is a neighborhood of y such that $T(U)$ is equicontinuous. We shall derive a contradiction. Let B be the weak* closure of $T(U)$. According to Lemma 2, $T^{-1}(B)$ is closed. Since

$$D(T) \cap U \subset T^{-1}(B) \subset D(T)$$

and $y \in \mathrm{cl}\, D(T)$, it follows that actually $y \in D(T)$. Now $y \notin \mathrm{int}\, D(T)$, and we have shown above that

$$\mathrm{int}\, D(T) = \mathrm{int}(\mathrm{conv}\, D(T)) \neq \emptyset.$$

Lemma 3 then implies that $T(y)$ is not an equicontinuous set, contrary to the assumption that $T(U)$ is equicontinuous. This proves Theorem 2.

COROLLARY 2.1. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists an* $x \in \mathrm{int}(\mathrm{cl}\, D(T))$ *such that* T *is locally bounded at* x. *Then the conclusions of Theorem* 2 *hold.*

Proof. Let U be a neighborhood of x, contained in $\mathrm{cl}\, D(T)$, such that $T(U)$ is an equicontinuous subset of X^*. Then the sets $S = U \cap D(T)$ and $A = T(U)$ satisfy the hypothesis of Theorem 2.

COROLLARY 2.2. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a monotone operator (not necessarily maximal). Let* C *be an open subset of* $\mathrm{cl}\, D(T)$. *If* T *is locally bounded at some point of* C, *then* T *is locally bounded at every point of* C.

Proof. By Zorn's Lemma, there exists a maximal monotone operator $T': X \to X^*$ such that $T'(x) \supset T(x)$ for every x. Let U be a nonempty, open subset of C such that $T(U)$ is an equicontinuous subset of X^*. Then $S = U \cap D(T)$ and $A = T(U)$ satisfy the hypothesis of Theorem 2, with T' in place of T. It follows that T' is locally bounded on the interior of $\mathrm{cl}\, D(T')$. In particular, T is locally bounded throughout C.

Remark. Corollary 2.2 is the analogue for monotone operators of a familiar result about convex functions: if f is a real-valued, convex function on an open, convex subset C of X, and if f is continuous at some point of C, then f is continuous at every point of C. For the connection between these results in the case of subdifferential mappings, see a result of J. J. Moreau [9, p. 79] and its formulation as Theorem 2 of [1].

COROLLARY 2.3. *Let* X *be a locally convex, reflexive Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists a bounded subset* A *of* X *such that one of the following two conditions holds:*

(a) $0 \in \mathrm{int}(\mathrm{cl}\, T(A))$,

(b) *for some* $S \subset T(A)$, *one has* $0 \in \operatorname{int}(\operatorname{cl}(\operatorname{conv} S))$ *and*

$$\sup_{x \in A} \sup_{x^* \in S} |\langle x, x^* \rangle| < \infty .$$

Then there exists an $x \in X$ *such that* $0 \in T(x)$.

Proof. Apply Theorem 2 to T^{-1}. Since X is reflexive, the bounded subsets of X are equicontinuous subsets of X^{**}.

3. PROOFS OF THEOREM 1 AND ITS COROLLARIES

Proof of Theorem 1. We consider first the case where the set

$$C = \operatorname{int}(\operatorname{conv} D(T))$$

is nonempty. For each positive integer n, let S_n denote the set of all $x \in D(T)$ such that $\|x\| \leq n$ and $T(x)$ contains an x^* with $\|x^*\| \leq n$. Since $D(T) = \bigcup_{n=1}^{\infty} S_n$, we have the inclusion

$$C \subset \bigcup_{n=1}^{\infty} \operatorname{conv} S_n . \tag{3.1}$$

Of course C, being a nonempty, open, convex subset of a Banach space, is of the second Baire category. The sets $C \cap \operatorname{conv} S_n$ therefore cannot all be nowhere-dense. Thus

$$\operatorname{int}(\operatorname{cl}(\operatorname{conv} S_n)) \neq \emptyset$$

for some n. The sets $S = S_n$ and

$$A = \{x^* \in X^* \mid \|x^*\| \leq n\}$$

then satisfy the hypothesis of Theorem 2, and the conclusion of Theorem 1 follows.

Next we consider the case where X is reflexive and T is locally bounded at some point of D(T). In this case, there exists an open, convex set U, meeting D(T), such that T(U) is norm-bounded in X^*. If $U \subset D(T)$, the hypothesis of Theorem 2 is satisfied and the conclusions of Theorem 1 again follow. Suppose therefore that U is not included in D(T). Then U contains a boundary point of D(T). We shall show that this is impossible.

Let B be the weak* closure of T(U). By Lemma 2, $T^{-1}(B)$ is closed. We have that

$$U \cap D(T) \subset T^{-1}(B) \subset D(T),$$

and this implies the inclusion

$$\operatorname{cl}[U \cap D(T)] \subset D(T).$$

Therefore

$$U \cap D(T) = U \cap \operatorname{cl} D(T), \tag{3.2}$$

in other words, every boundary point of $D(T)$ in U belongs to $D(T)$. Since X is a reflexive Banach space and T is a maximal monotone operator, $\mathrm{cl}\, D(T)$ is a convex set (Rockafellar [11, Theorem 2]); hence the set of points where $\mathrm{cl}\, D(T)$ has a supporting hyperplane is dense in the boundary of $\mathrm{cl}\, D(T)$ (E. Bishop and R. R. Phelps [2, Theorem 1]). Since U contains a boundary point of $D(T)$, it follows from (3.2) that U actually contains a point $x \in D(T)$ such that $\mathrm{cl}\, D(T)$ has a supporting hyperplane at x, in other words, such that (2.6) holds for some nonzero $y^* \in X^*$. The argument given to prove Lemma 3 now implies that $T(x)$ is unbounded, contrary to the assumption that $T(U)$ is bounded. This completes the proof of Theorem 1.

Proof of Corollary 1.1. Since $\mathrm{int}\, D(T) = \mathrm{int}(\mathrm{conv}\, D(T)) \neq \emptyset$ by Theorem 1, Lemma 3 implies that $T(x)$ is an unbounded set for each $x \in D(T) \setminus \mathrm{int}\, D(T)$. Therefore $D_0 \subset \mathrm{int}\, D(T)$, and, by Theorem 1, T is locally bounded at each point of D_0. Let x_0 be a point of D_0, and let x_0^* be the unique element of $T(x_0)$. Let U be a neighborhood of x_0 such that $T(U)$ is equicontinuous, and let U^* be some weak*-open neighborhood of x_0^*. Let B be the intersection of the weak* closure of $T(U)$ with the complement of U^* in X^*. Thus B is a weak*-closed, equicontinuous set, so that $T^{-1}(B)$ is closed by Lemma 2. If there did not exist a neighborhood W of x_0 such that $T(W) \subset U^*$, then $T^{-1}(B)$ would meet every neighborhood W of x_0. This would imply that $x_0 \in T^{-1}(B)$, contrary to the fact that $T(x_0)$ contains no element of B. Thus $T(W) \subset U^*$ for some neighborhood W of x_0. This shows, in particular, that the restriction of T to D_0 is strong-to-weak continuous at x_0.

Proof of Corollary 1.2. Every set containing the interior of its convex hull is virtually convex—see the proof of the lemma in [11]. According to Theorem 1, we have that

$$\mathrm{cl}\, D(T) = \mathrm{cl}(\mathrm{int}\, D(T)),$$

where $\mathrm{int}\, D(T)$ is convex. Thus

$$\mathrm{int}(\mathrm{cl}\, D(T)) = \mathrm{int}\, D(T),$$

and if $\mathrm{cl}\, D(T) = X$, it follows that

$$X = \mathrm{int}\, D(T) \subset D(T).$$

Proof of Corollary 1.3. Applying Theorem 1 to T^{-1}, we obtain that $0 \in \mathrm{int}\, D(T^{-1})$ if and only if $0 \in \mathrm{cl}\, D(T^{-1})$ and T^{-1} is locally bounded at 0. The latter means that there exist positive numbers α and ε such that $\|x\| < \alpha$ whenever $x \in T^{-1}(x^*)$ and $\|x^*\| < \varepsilon$.

Proof of Corollary 1.4. Since $T(B) \subset R(T) = D(T^{-1})$, the corollary follows immediately from applying Theorem 1 to T^{-1}.

Proof of Corollary 1.5. Because $x_i \in T^{-1}(x_i^*)$ for every i, the set $T^{-1}(U)$ is a nonempty, unbounded subset of X for every neighborhood U of x^* in X^*. Thus x^* belongs to the closure of $D(T^{-1})$, but T^{-1} is not locally bounded at x^*. By Zorn's Lemma, there exists a maximal monotone operator $S\colon X^* \to X$ such that $S(y^*) \supset T^{-1}(y^*)$ for every $y^* \in X^*$. This operator S cannot be locally bounded at x^*, and hence Theorem 1 implies that $x^* \notin \mathrm{int}\, D(S)$ and, in particular, that $x^* \notin \mathrm{int}\, D(T^{-1})$. Thus x^* is a boundary point of $D(T^{-1}) = R(T)$.

Proof of Corollary 1.6. The stated condition means that T^{-1} is locally bounded at every point of X^*. By Theorem 1 (applied to T^{-1}), this is equivalent to $D(T^{-1})$ being an open, convex subset of X^* with no boundary points, and the only such nonempty subset is X^* itself.

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