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ON THE MAXIMAL MONOTONICITY OF SUBDIFFERENTIAL MAPPINGS

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The subdifferential of a lower semicontinuous proper convex function on a Banach space is a maximal monotone operator, as well as a maximal cyclically monotone operator. This result was announced by the author in a previous paper, but the argument given there was incomplete; the result is proved here by a different method, which is simpler in the case of reflexive Banach spaces. At the same time, a new fact is established about the relationship between the subdifferential of a convex function and the subdifferential of its conjugate in the nonreflexive case.

Let E be a real Banach space with dual E^*. A *proper convex function* on E is a function f from E to $(-\infty, +\infty]$, not identically $+\infty$, such that

$$f((1-\lambda)x + \lambda y) \leqq (1-\lambda)f(x) + \lambda f(y)$$

whenever $x \in E$, $y \in E$ and $0 < \lambda < 1$. The *subdifferential* of such a function f is the (generally multivalued) mapping $\partial f: E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* \mid f(y) \geqq f(x) + \langle y - x, x^* \rangle, \forall y \in E\} ,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between E and E^*.

A multivalued mapping $T: E \to E^*$ is said to be a *monotone operator* if

$$\langle x_0 - x_1, x_0^* - x_1^* \rangle \geqq 0 \quad \text{whenever} \quad x_0^* \in T(x_0), x_1^* \in T(x_1) .$$

It is said to be a *cyclically monotone operator* if

$$\langle x_0 - x_1, x_0^* \rangle + \cdots + \langle x_{n-1} - x_n, x_{n-1}^* \rangle + \langle x_n - x_0, x_n^* \rangle \geqq 0$$
$$\text{whenever} \quad x_i^* \in T(x_i), i = 0, \cdots, n .$$

It is called a *maximal* monotone operator (resp. maximal cyclically monotone operator) if, in addition, its graph

$$G(T) = \{(x, x^*) \mid x^* \in T(x)\} \subset E \times E^*$$

is not properly contained in the graph of any other monotone (resp. cyclically monotone) operator $T': E \to E^*$.

This note is concerned with proving the following theorems.

THEOREM A. *If f is a lower semicontinuous proper convex function on E, then ∂f is a maximal monotone operator from E to E^*.*

THEOREM B. *Let $T: E \rightarrow E^*$ be a multivalued mapping. In order that there exist a lower semicontinuous proper convex function f on E such that $T = \partial f$, it is necessary and sufficient that T be a maximal cyclically monotone operator. Moreover, in this case T determines f uniquely up to an additive constant.*

These theorems have previously been stated by us in [4] as Theorem 4 and Theorem 3, respectively. However, a gap occurs in the proofs in [4], as has kindly been brought to our attention recently by H. Brézis. (It is not clear whether formula (4.7) in the proof of Theorem 3 of [4] will hold for ε sufficiently small, because x_i^* depends on ε and could conceivably increase unboundedly in norm as ε decreases to 0. The same oversight appears in the penultimate sentence of the proof of Theorem 4 of [4]). In view of this oversight, the proofs in [4] are incomplete; further arguments must be given before the maximality in Theorem A, the maximality in the necessary condition in Theorem B, and the uniqueness in Theorem B can be regarded as established. Such arguments will be given here.

2. Preliminary result. Let f be a lower semicontinuous proper convex function on E. (For proper convex functions, lower semicontinuity in the strong topology of E is the same as lower semicontinuity in the weak topology.) The conjugate of f is the function f^* on E^* defined by

$$f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x) \mid x \in E\} . \tag{2.1}$$

It is known that f^* is a weak* lower semicontinuous (and hence strongly lower semicontinuous) proper convex function on E^*, and that

$$f(x) + f^*(x^*) - \langle x, x^* \rangle \geqq 0,\ \forall x \in E,\ \forall x^* \in E^* , \quad \text{with equality if and only if } x^* \in \partial f(x) \tag{2.2}$$

(see Moreau [3, § 6]). The subdifferential ∂f^*, which is a multivalued mapping from E^* to the bidual E^{**}, can be compared with the subdifferential ∂f from E to E^*, when E is regarded in the canonical way as a weak** dense subspace of E^{**} (the weak** topology being the weak topology induced on E^{**} by E^*). Facts about the relationship between ∂f^* and ∂f will be used below in proving Theorems A and B.

In terms of the conjugate f^{**} of f^*, which is the weak** lower semicontinuous proper convex function on E^{**} defined by

(2.3) $$f^{**}(x^{**}) = \sup\{\langle x^{**}, x^*\rangle - f^*(x^*) \mid x^* \in E^*\},$$

we have, as in (2.2),

(2.4) $$f^{**}(x^{**}) + f^*(x^*) - \langle x^{**}, x^*\rangle \geqq 0, \ \forall x^{**} \in E^{**}, \forall x^* \in E^*,$$
with equality if and only if $x^{**} \in \partial f^*(x^*)$.

Moreover, the restriction of f^{**} to E is f (see [3, § 6]). Thus, if E is reflexive, we can identify f^{**} with f, and it follows from (2.2) and (2.4) that ∂f^* is just the "inverse" of ∂f, in other words one has $x \in \partial f^*(x^*)$ if and only if $x^* \in \partial f(x)$. If E is not reflexive, the relationship between ∂f^* and ∂f is more complicated, but ∂f^* and ∂f still completely determine each other, according to the following result.

PROPOSITION 1. *Let f be a lower semicontinuous proper convex function on E, and let $x^* \in E^*$ and $x^{**} \in E^{**}$. Then $x^{**} \in \partial f^*(x^*)$ if and only if there exists a net $\{x_i^* \mid i \in I\}$ in E^* converging to x^* in the strong topology and a bounded net $\{x_i \mid i \in I\}$ in E (with the same partially ordered index set I) converging to x^{**} in the weak** topology, such that $x_i^* \in \partial f(x_i)$ for every $i \in I$.*

Proof. The sufficiency of the condition is easy to prove. Given nets as described, we have

$$f(x_i) + f^*(x_i^*) = \langle x_i, x_i^*\rangle, \ \forall i \in I$$

by (2.2), where $f(x_i) = f^{**}(x_i)$. Then by the lower semicontinuity of f^* and f^{**} we have

$$\begin{aligned} f^{**}(x^{**}) + f^*(x^*) &\leqq \liminf\{f^{**}(x_i) + f^*(x_i^*)\} \\ &= \lim\langle x_i, x_i^*\rangle = \langle x^{**}, x^*\rangle . \end{aligned}$$

(The last equality makes use of the boundedness of the norms $\|x_i\|$, $i \in I$.) Thus $x^{**} \in \partial f^*(x^*)$ by (2.4).

To prove the necessity of the condition, we demonstrate first that, given any $x^{**} \in E^{**}$, there exists a *bounded* net $\{y_i \mid i \in I\}$ in E such that y_i converges to x^{**} in the weak** topology and

(2.5) $$\lim f(y_i) = f^{**}(x^{**}) .$$

Consider $f + h_\alpha$, where α is a positive real number and h_α is the lower semicontinuous proper convex function on E defined by

(2.6) $$h_\alpha(x) = 0 \quad \text{if} \quad \|x\| \leqq \alpha, \ h_\alpha(x) = +\infty \quad \text{if} \quad \|x\| > \alpha .$$

Assuming that α is sufficiently large, there exist points x at which f and h_α are both finite and h_α is continuous (i.e., points x such that $f(x) < +\infty$ and $\|x\| < \alpha$). Then, by the formulas for conjugates of sums of convex functions (see Moreau [3, pp. 38, 56, 57] or Rockafellar [5, Th. 3]), we have $(f + h_\alpha)^* = f^* \square h_\alpha^*$ (infimal convolution), and consequently

$$(2.7) \qquad (f + h_\alpha)^{**} = (f^* \square h_\alpha^*)^* = f^{**} + h_\alpha^{**} .$$

Moreover $h_\alpha^*(x^*) = \alpha \| x^* \|$ for ever $x^* \in E^*$, so that

$$h_\alpha^{**}(x^{**}) = \sup \{\langle x^{**}, x^* \rangle - \alpha \| x^* \| \mid x^* \in E^*\}$$
$$= \begin{cases} 0 & \text{if } \| x^{**} \| \leqq \alpha , \\ +\infty & \text{if } \| x^{**} \| > \alpha . \end{cases}$$

Hence by (2.7), given any $x^{**} \in E^{**}$, we have

$$(2.8) \qquad f^{**}(x^{**}) = (f + h_\alpha)^{**}(x^{**})$$

for sufficiently large $\alpha > 0$. On the other hand, it is known that, for any lower semicontinuous proper convex function g on E, g^{**} is the greatest weak** lower semicontinuous function on E^{**} majorized by g on E (see [3, § 6]), so that

$$(2.9) \qquad g^{**}(x^{**}) = \liminf_{y \to x^{**}} g(y) ,$$

where the "lim inf" is taken over all nets in E converging to x^{**} in the weak** topology. Taking $g = f + h_\alpha$, we see from (2.8) and (2.9) that

$$f^{**}(x^{**}) = \liminf_{y \to x^{**}} [f(y) + h_\alpha(y)] ,$$

implying that (2.5) holds as desired for some net $\{y_i \mid i \in I\}$ in E such that y_i converges to x^{**} in the weak** topology and $\| y_i \| \leqq \alpha$ for every $i \in I$.

Now, given any $x^* \in E^*$ and $x^{**} \in \partial f^*(x^*)$, let $\{y_i \mid i \in I\}$ be a bounded net in E such that y_i converges to x^{**} in the weak** topology and (2.5) holds. Define $\varepsilon_i \geqq 0$ by

$$\varepsilon_i^2 = f(y_i) + f^*(x^*) - \langle y_i, x^* \rangle .$$

Note that $\lim \varepsilon_i = 0$ by (2.5) and (2.4). According to a lemma of Brøndsted and Rockafellar [1, p. 608], there exist for each $i \in I$ an $x_i \in E$ and an $x_i^* \in E^*$ such that

$$x_i^* \in \partial f(x_i),\ \| x_i - y_i \| \leqq \varepsilon_i,\ \| x_i^* - x^* \| \leqq \varepsilon_i .$$

The latter two conditions imply that the net $\{x_i^* \mid i \in I\}$ converges to x^* in the strong topology of E^*, while the net $\{x_i \mid i \in I\}$ is bounded and converges to x^{**} in the weak** topology of E^{**}. This completes the proof of Proposition 1.

3. Proofs of Theorems A and B. In the sequel, f denotes a lower semicontinuous proper convex function on E, and j denotes the continuous convex function E defined by $j(x) = (1/2)\|x\|^2$. We shall make use of the fact that, for each $x \in E$, $\partial f(x)$ is by definition a certain (possibly empty, possibly unbounded) weak* closed convex subset of E^*, whereas $\partial j(x)$ is (by the finiteness and continuity of j, see [3, p. 60]) a certain nonempty weak* compact convex subset of E^*. Furthermore

$$(3.1) \qquad \partial(f + j) = \partial f(x) + \partial j(x),\ \forall x \in E$$

(see [3, p. 62] or [5, Th. 3]). The conjugate of j is given by $j^*(x^*) = (1/2)\|x^*\|^2$, and since

$$(f + j)^*(x^*) = (f^* \,\Box\, j^*)(x^*) = \min_{y^* \in E^*} \{f^*(y^*) + j^*(x^* - y^*)\}$$

([3, § 9] or [5, Th. 3]) the conjugate function $(f + j)^*$ is finite and continuous throughout E^*.

Proof of Theorem A. Theorem A has already been established by Minty [2] in the case of convex functions which, like j, are everywhere finite and continuous. Applying Minty's result to the function $(f + j)^*$, we may conclude that $\partial(f + j)^*$ is a maximal monotone operator from E^* to E^{**}. We shall show this implies that ∂f is a maximal monotone operator from E to E^*.

Let T be a monotone operator from E to E^* such that the graph of T includes the graph of ∂f, i.e.,

$$(3.2) \qquad T(x) \supset \partial f(x),\ \forall x \in E\ .$$

We must show that equality necessarily holds in (3.2).

The mapping $T + \partial j$ defined by

$$\begin{aligned}(T + \partial j)(x) &= T(x) + \partial j(x)\\ &= \{x_1^* + x_2^* \mid x_1^* \in T(x),\ x_2^* \in \partial j(x)\}\end{aligned}$$

is a monotone operator from E to E^*, since T and ∂j are, and by (3.1) and (3.2) we have

$$(3.3) \qquad (T + \partial j)(x) \supset \partial(f + j)(x),\ \forall x \in E\ .$$

Let S be the multivalued mapping from E^* to E^{**} defined as follows: $x^{**} \in S(x^*)$ if and only if there exists a net $\{x_i^* \mid i \in I\}$ in E^* converging to x^* in the strong topology, and a bounded net $\{x_i \mid i \in I\}$ in E (with the same partially ordered index set I) converging to x^{**} in the weak** topology, such that

$$x_i^* \in (T + \partial j)(x_i),\ \forall i \in I\ .$$

It is readily verified that S is a monotone operator. (The boundedness of the nets $\{x_i \mid i \in I\}$ enters in here.) Moreover

$$S(x^*) \supset \partial(f + j)^*(x^*),\ \forall x^* \in E^* , \tag{3.4}$$

by (3.3) and Proposition 1. Since $\partial(f + j)^*$ is a maximal monotone operator, equality must actually hold in (3.4). This shows that one has $x \in \partial(f + j)^*(x^*)$ whenever $x \in E$ and $x \in S(x^*)$, hence in particular whenever $x^* \in (T + \partial j)(x)$. On the other hand, one always has $x^* \in \partial(f + j)(x)$ if $x \in \partial(f + j)^*(x^*)$ and $x \in E$. (This follows from applying (2.2) and (2.4) to $f + j$ in place of f.) Thus one has $x^* \in \partial(f + j)(x)$ if $x^* \in (T + \partial j)(x)$, implying by (3.3) and (3.1) that

$$T(x) + \partial j(x) = \partial f(x) + \partial j(x),\ \forall x \in E . \tag{3.5}$$

We shall show now from (3.5) that actually

$$T(x) = \partial f(x),\ \forall x \in E ,$$

so that ∂f must be a maximal monotone operator as claimed. Suppose that $x \in E$ is such that the inclusion in (3.2) is proper. This will lead to a contradiction. Since $\partial f(x)$ is a weak* closed convex subset of E^*, there must exist some point of $T(x)$ which can be separated strictly from $\partial f(x)$ be a weak* closed hyperplane. Thus, for a certain $y \in E$, we have

$$\sup \{\langle y, x^*\rangle \mid x^* \in T(x)\} > \sup \{\langle y, x^*\rangle \mid x^* \in \partial f(x)\} .$$

But then

$$\begin{aligned} &\sup \{\langle y, z^*\rangle \mid z^* \in T(x) + \partial j(x)\} \\ &= \sup \{\langle y, x^*\rangle \mid x^* \in T(x)\} + \sup \{\langle y, y^*\rangle \mid y^* \in \partial j(x)\} \\ &> \sup \{\langle y, x^*\rangle \mid x^* \in \partial f(x)\} + \sup \{\langle y, y^*\rangle \mid y^* \in \partial j(x)\} \\ &= \sup \{\langle y, z^*\rangle \mid z^* \in \partial f(x) + \partial j(x)\} , \end{aligned}$$

inasmuch as $\partial j(x)$ is a nonempty bounded set, and this inequality is incompatible with (3.5).

Proof of Theorem B. Let g be a lower semicontinuous proper convex function on E such that

$$\partial g(x) \supset \partial f(x),\ \forall x \in E . \tag{3.6}$$

As noted at the beginning of the proof Theorem 3 of [4], to prove Theorem B it suffices, in view of Theorem 1 of [4] and its Corollary 2, to demonstrate that $g = f + \text{const.}$

We consider first the case where f and g are everywhere finite and continuous. Then, for each $x \in E$, $\partial f(x)$ is a nonempty weak* compact set, and

(3.7) $$f'(x; u) = \max\{\langle u, x^*\rangle \mid x^* \in \partial f(x)\},\ \forall u \in E,$$

where

$$f'(x; u) = \lim_{\lambda \downarrow 0} [f(x + \lambda u) - f(x)]/\lambda$$

[3, p. 65]. Similarly, $\partial g(x)$ is a nonempty weak* compact set, and

(3.8) $$g'(x; u) = \max\{\langle u, x^*\rangle \mid x^* \in \partial g(x)\},\ \forall u \in E.$$

It follows from (3.6), (3.7) and (3.8) that

(3.9) $$f'(x; u) \leqq g'(x; u),\ \forall x \in E,\ \forall u \in E.$$

On the other hand, for any $x \in E$ and $y \in E$, we have

$$f(y) - f(x) = \int_0^1 f'((1 - \lambda)x + \lambda y; y - x)d\lambda,$$

$$g(y) - g(x) = \int_0^1 g'((1 - \lambda)x + \lambda y; y - x)d\lambda$$

(see [6, § 24]), so that by (3.9) we have

$$f(y) - f(x) \leqq g(y) - g(x),\ \forall x \in E,\ \forall y \in E.$$

Of course, the latter can hold only if $g = f +$ const.

In the general case, we observe from (3.6) that

$$\partial g(x) + \partial j(x) \supset \partial f(x) + \partial j(x),\ \forall x \in E,$$

and consequently

$$\partial(g + j)(x) \supset \partial(f + j)(x),\ \forall x \in E,$$

by (3.1)(and its counterpart for g). This implies by Proposition 1 that

(3.10) $$\partial(g + j)^*(x^*) \supset \partial(f + j)^*(x^*),\ \forall x^* \in E^*.$$

The functions $(f + j)^*$ and $(g + j)^*$ are finite and continuous on E^*, so we may conclude from (3.10) and the case already considered that

$$(g + j)^* = (f + j)^* + \alpha$$

for a certain real constant α. Taking conjugates, we then have

(3.11) $$(g + j)^{**} = (f + j)^{**} - \alpha.$$

Since $(g + j)^{**}$ and $(f + j)^{**}$ agree on E with $g + j$ and $f + j$, respectively, (3.11) implies that

$$g + j = f + j - \alpha,$$

and hence that $g = f + \text{const}$.

REMARK. The preceding proofs become much simpler if E is reflexive, since then ∂f^* and $\partial(f + j)^*$ are just the "inverses" of ∂f and $\partial(f + j)$, respectively, and Proposition 1 is superfluous. In this case, S may be replaced by the inverse of $T + \partial j$ in the proof of Theorem A.

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