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Measurable Dependence of Convex Sets and Functions on Parameters

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1. INTRODUCTION

Various developments in mathematical economics and optimal control have led to the study of the measurability of multivalued mappings. Castaing, in his recent thesis [2] (partly published in [3]), has presented a broad new theory of measurable multivalued mappings, built upon foundations laid by Aumann [1], Debreu [4], Kuratowski and Ryll-Nardzewski [5], Olech [7], Plis [8], von Neumann [11], and others. The context chosen by Castaing is that of mappings from a locally compact topological space to the subsets of another topological space, usually assumed to be metrizable. This stands in contrast to the measurability theory of Debreu, which treats mappings from a measurable space (without topological structure) to the compact subsets of a metric space. Castaing's approach, however, is to prove an interesting generalization of Lusin's theorem, which requires topological structure of the domain space in order to make sense, and to use this theorem as the basic theoretical tool. The results of Castaing are, for the most part, limited to compact-valued mappings.

In this paper, the theory of measurable multivalued mappings will be elaborated in the case of mappings from a general measurable space to the closed (but not necessarily bounded) subsets of n-dimensional Euclidean space R^n. Emphasis will be placed on measurable convex-set-valued mappings and their relationship with the normal convex integrands studied in [9]. Our goal will be to establish several convenient characterizations of measurability and normality, and to use these to show that measurability or normality is preserved when certain operations, such as addition, are performed on multivalued mappings or convex integrands. The consideration of multivalued mappings which are not necessarily compact-valued is important for for this purpose, since in dealing with a convex function on R^n one is auto-

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matically dealing with an unbounded convex subset of R^{n+1}, the epigraph of the function.

Let T be a set, and let $\mathscr{T}$ be a σ-field of subsets of T. A multivalued mapping K from T to R^n will be called *measurable* if, for every closed subset S of R^n, the set

$$K^{-1}(S) = \{t \in T \mid K(t) \cap S \neq \phi\}$$

is measurable in T. Note that, for K to be measurable, it suffices actually if $K^{-1}(S)$ is measurable for every *compact* $S \subset R^n$, since any closed S can be expressed as the union of a sequence of compact sets $S_1, S_2, \ldots$, and the set

$$K^{-1}\left(\bigcup_{m=1}^{\infty} S_m\right) = \bigcup_{m=1}^{\infty} K^{-1}(S_m)$$

is measurable if each $K^{-1}(S_m)$ is measurable.

By a *convex integrand* on $T \times R^n$, we shall mean a function

$$f : T \times R^n \to (-\infty, +\infty]$$

such that $f(t, x)$ is a convex function of x for each t. A convex integrand will be called *normal* if it satisfies the following conditions:

(a) for each $t \in T$, the function $f(t, \cdot)$ is lower semi-continuous on R^n and not identically $+\infty$, and

(b) there exists a *countable* collection U of measurable functions $u: T \to R^n$, such that $f(t, u(t))$ is measurable in t for each $u \in U$, and $U(t) \cap D(t)$ is dense in $D(t)$ for each $t \in T$, where

$$D(t) = \{x \in R^n \mid f(t, x) < +\infty\}, \tag{1.1}$$

$$U(t) = \{u(t) \mid u \in U\}. \tag{1.2}$$

Normal convex integrands were used in [9] to define convex integral functionals of the form

$$I_f(u) = \int_T f(t, u(t))\, dt, \qquad u \in L \tag{1.3}$$

where dt is a positive measure on $(T, \mathscr{T})$ and L is a linear space of measurable functions $u : T \to R^n$. Normality was shown to guarantee, among other things, that the (extended-real-valued) function $f(t, u(t))$ is measurable in t for every measurable u (not just for $u \in U$).

An important advantage of the normality condition, as opposed to various simpler measurability conditions which would suffice for the definition of functionals of the form (1.3), is that *normality is preserved under duality:* as proved in [9], if f is a normal convex integrand, then f^* is also normal, where f^* is the convex integrand on $T \times R^n$ *conjugate* to f, defined by

$$f^*(t, x^*) = \sup\{\langle x, x^*\rangle - f(x) \mid x \in R^n\} \tag{1.4}$$

($\langle x, x^* \rangle$ being the ordinary inner product of two vectors $x \in R^n$ and $x^* \in R^n$). This implies, for example, that the function

$$p(t) = \inf\{f(t, x) \mid x \in R^n\}$$

is measurable when f is normal, since

$$p(t) = -f^*(t, u(t)),$$

where $u(t) = 0$.

A convex integrand f is normal in particular whenever f satisfies condition (a) of normality, $f(t, x)$ is measurable in t for every $x \in R^n$, and the (convex) set $D(t)$ has a nonempty interior for every $t \in T$. (Let Z be a countable dense subset of R^n, and let U be the collection of constant functions on T with values in Z [9, Lemma 2].)

The concept of a normal convex integrand attempts to describe a certain kind of measurable dependence of a convex function $f(t, \cdot)$ on R^n upon an abstract parameter t. The concept of a measurable multivalued mapping $K : T \rightarrow R^n$ describes a kind of measurable dependence of a subset $K(t)$ of R^n on a parameter t. What are the relationships between these concepts?

The following relationships, among others, will be demonstrated below. When f is a convex integrand satisfying condition (a) of normality, f is normal (i.e. also satisfies condition (b)) if and only if the multivalued mapping $K : T \rightarrow R^{n+1}$ is measurable, where

$$K(t) = \operatorname{epi} f(t, \cdot) = \{(x, \mu) \mid x \in R^n, \mu \in R^1, \mu \geqslant f(t, x)\}. \tag{1.5}$$

(The set $\operatorname{epi} f(t, \cdot)$ is called the *epigraph* of the function $f(t, \cdot)$ on R^n; it is convex and closed if and only if $f(t, \cdot)$ is convex and lower semi-continuous.) On the other hand, when $K : T \rightarrow R^n$ is a multivalued mapping such that $K(t)$ is a nonempty closed convex set for each t, K is measurable if and only if the convex integrand f is normal, where

$$f(t, x) = \delta(x \mid K(t)) = \begin{cases} 0 & \text{if} \quad x \in K(t), \\ +\infty & \text{if} \quad x \notin K(t). \end{cases} \tag{1.6}$$

(The function $\delta(\cdot \mid K(t))$ is called the *indicator* of $K(t)$; it is a lower semi-continuous convex function, not identically $+\infty$, if and only if $K(t)$ is a nonempty closed convex set.)

2. Measurability of Multivalued Mappings

The closure, interior and convex hull of a subset S of R^n will be denoted by cl S, int S and conv S, respectively. The following general measurability criterion, essentially due to Castaing [2], will be employed in proving our main results, Theorem 3 and Theorem 4.

THEOREM 1. *Let $K : T \to R^n$ be a multivalued mapping such that $K(t)$ is a non-empty closed set for every $t \in T$. In order that K be measurable, it is necessary and sufficient that there exist a countable collection U of measurable functions $u : T \to R^n$ such that, for every $t \in T$,*

$$K(t) = \text{cl } U(t) = \text{cl}\{u(t) \mid u \in U\}. \tag{2.1}$$

PROOF. Castaing established the necessity of the condition in [2, Section 5] in the case where T is a locally compact topological space and $\mathscr{T}$ = all μ-measurable sets for a Radon measure μ on T (R^n being replaced by a separable complete metric space). He also proved the sufficiency of the condition in this case, but under the further assumption that every compact subset of T is metrizable. Castaing's necessity argument does not in fact make use of any topological structure of T, so it may be carried over directly to the present context. But this sufficiency argument is topological in a fundamental way and therefore cannot be invoked in any form here. Actually, however, since the sets $K(t)$ lie in R^n, the sufficiency in Theorem 1 can be established by the following elementary argument, which does not involve any compactness in T.

Let U be a collection of measurable functions such that (2.1) holds. To prove that K is measurable, it is enough, as observed in Section 1, to show that $K^{-1}(S)$ is measurable for every compact $S \subset R^n$. Given a compact S, define S_m for $m = 1, 2, \ldots$, by

$$S_m = \left\{ x \in R^n \,\middle|\, \exists y \in S, \mid x - y \mid \leqslant \frac{1}{m} \right\} \tag{2.2}$$

(where $| \cdot |$ denotes the Euclidean norm). Since $K(t) = \text{cl } U(t)$, and each S_m is compact, we have $K(t) \cap S \neq \phi$ if and only if $U(t) \cap S_m \neq \phi$ for every m. Thus

$$\begin{aligned} K^{-1}(S) &= \bigcap_{m=1}^{\infty} \{t \mid U(t) \cap S_m \neq \phi\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{u \in U} u^{-1}(S_m). \end{aligned} \tag{2.3}$$

The sets $u^{-1}(S_m)$ are measurable, because the functions $u \in U$ are measurable, and, since U is a countable collection, it follows from (2.3) that $K^{-1}(S)$ is measurable. This finishes the proof.

Although Theorem 1 only characterizes measurable multivalued mappings K such that $K(t)$ is nonempty and closed for every t, it does have some bearing on more general mappings, in view of the fact that, if $K : T \to R^n$ is any measurable multivalued mapping, then the multivalued mapping

$$K' : t \to \text{cl } K(t) \tag{2.4}$$

is measurable. The latter is true because, for any compact $S \subset R^n$, one has

$$(K')^{-1}(S) = \bigcap_{m=1}^{\infty} K^{-1}(S_m),$$

where S_m is given by (2.2). It should be observed further that, if $K : T \to R^n$ is any measurable multivalued mapping, then the set

$$T_0 = \{t \in T \mid K(t) \neq \phi\} = K^{-1}(R^n) \tag{2.5}$$

is measurable. Thus, if $K(t)$ is closed for every t, the restriction of K to T_0 is a measurable multivalued mapping $K_0 : T_0 \to R^n$ of the type to which Theorem 1 (and Corollary 1.1 below) are applicable.

Here are some useful facts implied by Theorem 1.

COROLLARY 1.1 (Kuratowski and Ryll-Nardzewski [5]). *Let $K : T \to R^n$ be a measurable multivalued mapping such that $K(t)$ is a non-empty closed set for every $t \in T$. Then there exists a measurable selector for K, i.e. a measurable function $u : T \to R^n$ such that $u(t) \in K(t)$ for every $t \in T$.*

COROLLARY 1.2. *Let K_1 and K_2 be measurable multivalued mappings from T to R^n. Then the multivalued mapping*

$$K : t \to \mathrm{cl}[K_1(t) + K_2(t)] = \mathrm{cl}\{x_1 + x_2 \mid x_1 \in K_1(t), x_2 \in K_2(t)\}$$

is measurable.

PROOF. The set T_0 defined by (2.5) is measurable, since

$$T_0 = K_1^{-1}(R^n) \cap K_2^{-1}(R^n).$$

Restricting K to T_0 if necessary, we can reduce the assertion to the case where $T_0 = T$, i.e., $K_1(t) \neq \phi$ and $K_2(t) \neq \phi$ for every $t \in T$. We can also assume that $K_1(t)$ and $K_2(t)$ are always closed. Then there exist by Theorem 1 countable collections U_1 and U_2 of measurable functions from T to R^n such that

$$K_i(t) = \mathrm{cl}\{u_i(t) \mid u_i \in U_i\}, \qquad i = 1, 2.$$

Then (2.1) holds for

$$U = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\},$$

so K is measurable by Theorem 1.

COROLLARY 1.3. *Let $(K_i \mid i \in I)$ be a finite or countable family of measurable multivalued mappings from T to R^n, such that $K_i(t)$ is closed for every $t \in T$ and $i \in I$. Then the multivalued mapping*

$$K : t \to \bigcap_{i \in I} K_i(t)$$

is measurable. In particular, the set

$$\left\{t \in T \mid \bigcap_{i \in I} K_i(t) \neq \phi\right\} = K^{-1}(R^n)$$

is measurable in T.

PROOF. Let S be any compact subset of R^n. We must show that $K^{-1}(S)$ is measurable. Consider first the case where $I = \{1, 2\}$, and let

$$K_1'(t) = K_1(t) \cap S, \qquad K_2'(t) = -K_2(t). \tag{2.6}$$

clearly the multivalued mappings K_1' and K_2' defined by (2.6) are again measurable, and since $K_1'(t)$ is compact the set $K_1'(t) + K_2'(t)$ is closed for every t. The mapping

$$K' : t \to K_1'(t) + K_2'(t)$$

is therefore measurable by Corollary 1.2. Moreover

$$K^{-1}(S) = \{t \mid 0 \in K_1'(t) + K_2'(t)\} = (K')^{-1}(\{0\}),$$

so $K^{-1}(S)$ is measurable as claimed.

Since the assertion is true for $I = \{1, 2\}$, it is true for any finite index set I by induction. Consider now the case where I is countable, and let J be the collection of all finite subsets of I. For each $j \in J$ and $t \in T$, let $H_j(t)$ be the intersection of the $K_i(t)$ for $i \in j$. The multivalued mappings H_j so defined for $j \in J$ are measurable by what has already been established. The compactness of S implies that

$$K^{-1}(S) = \{t \mid \forall j \in J, H_j(t) \cap S \neq \phi\} = \bigcap_{j \in J} H_j^{-1}(S).$$

Since $H_j^{-1}(S)$ is measurable and J is countable, $K^{-1}(S)$ is measurable and the proof of Corollary 1.3 is complete.

Corollary 1.2 and Corollary 1.3 were proved by Castaing in the case where T is a locally compact topological space (with $\mathscr{T}$ = all μ-measurable sets for a Radon measure μ) under the assumption that the sets $K_i(t)$ are all compact

[2, Corollary 1 to Theorem 4.4 and Theorem 4.10]. Castaing showed, however, that the compactness assumption on the $K_i(t)$ in Corollary 1.3, at least, could be avoided if every compact subset of T were metrisable [2, Corollary to Theorem 3.3].

Another general measurability fact, which needs to be mentioned for use in Section 4, concerns the *graph* of the multivalued mapping $K : T \to R^n$, i.e., the set

$$G(K) = \{(t, x) \in T \times R^n \mid x \in K(t)\}. \tag{2.7}$$

Let $\mathscr{S}$ be the σ-field in $T \times R^n$ generated by all the subsets of the form $A \times B$, where $A \in \mathscr{T}$ and B is a Borel subset of R^n. The elements of $\mathscr{S}$ will be called the *measurable* subsets of $T \times R^n$. We shall say that the measurable space $(T, \mathscr{T})$ is *complete* if there exists at least one σ-finite (nonnegative) measure μ on $\mathscr{T}$ which is complete (i.e., such that, if $A \in \mathscr{T}$ is a set of measure zero with respect to μ, then every subset of A belongs to $\mathscr{T}$).

THEOREM 2 (Debreu [4, p. 360)]. *If $K : T \to R^n$ is a measurable multivalued mapping such that $K(t)$ is a closed set for every t, then the graph of K is a measurable subset of $T \times R^n$. On the other hand, if $K : T \to R^n$ is a multivalued mapping whose graph is a measurable subset of $T \times R^n$, and if the measurable space $(T, \mathscr{T})$ is complete in the above sense, then K is a measurable multivalued mapping.*

PROOF. The arguments of Debreu are actually applicable if R^n is replaced by any separable complete metric space. The argument given for the first assertion, however, is couched in terms of compact-valued mappings K (and a different but equivalent definition of measurability for that case—see Castaing [2, p. 25]), so some minor changes are necessary. The modified argument is this. Let Z be a countable dense subset of R^n, and for each $z \in Z$ and each positive integer m let $S_{z,m}$ denote the closed ball in R^n with center z and radius $1/m$. Since $K(t)$ is a closed set, one has $x \in K(t)$ if and only if, for every $m > 0$, there exists a $z \in Z$ such that $x \in S_{z,m}$ and

$$K(t) \cap S_{z,m} \neq \phi, \qquad \text{i.e.,} \qquad t \in K^{-1}(S_{z,m}).$$

Therefore the graph of K is given by the formula

$$G(K) = \bigcap_{m=1}^{\infty} \bigcup_{z \in Z} [K^{-1}(S_{z,m}) \times S_{z,m}]. \tag{2.8}$$

Each of the sets

$$K^{-1}(S_{z,m}) \times S_{z,m}$$

is measurable in $T \times R^n$, because $S_{z,m}$ is closed in R^n and K is measurable. Since Z is countable, (2.8) implies then that $G(K)$ is a measurable set.

[illegible], then the graph of K is a Borel subset of $R^m \times R^n$. On the other hand, if the graph of K is a Borel subset of $R^m \times R^n$, then K is Lebesgue measurable.

3. Measurability in the Presence of Convexity

We turn now to special criteria for the measurability of multivalued mappings $K : T \to R^n$ such that $K(t)$ is a closed convex set (not necessarily bounded) for every $t \in T$.

Given any $z \in R^n$ and any lower semi-continuous convex function h from R^n to $(-\infty, +\infty]$ which is not identically $+\infty$, we denote by $\operatorname{prox}(z \mid h)$ the unique point x of R^n where the function

$$x \to h(x) + \tfrac{1}{2} |x - z|^2$$

attains its minimum. The mapping $\text{prox}(\cdot \mid h)$ from R^n into itself is called the *proximation* associated with h. The general theory of proximations has been developed by Moreau [6]. It is known in particular that $\text{prox}(\cdot \mid h)$ is a continuous mapping whose range is dense in the convex set

$$\text{dom}\, h = \{x \in R^n \mid h(x) < +\infty\}.$$

If h is the indicator of a nonempty closed convex set $C \subset R^n$, i.e.,

$$h(x) = \delta(x \mid C) = \begin{cases} 0 & \text{if} \quad x \in C, \\ +\infty & \text{if} \quad x \notin C, \end{cases}$$

then $\text{prox}(z \mid h)$ is the unique point of C nearest to z, and it will also be denoted by $\text{prox}(z \mid C)$.

Proximations were a convenient tool in the study of normal convex integrands in [9], and they will again be helpful here. We shall need two lemmas.

LEMMA 1 [9]. *If f is a normal convex integrand on $T \times R^n$ and $u : T \to R^n$ is a measurable function, then the function*

$$t \to \text{prox}(u(t) \mid f(t, \cdot)) \in R^n$$

is measurable.

LEMMA 2. *If $C_1 \supset C_2 \supset \cdots$ is a non-increasing sequence of closed convex subsets of R^n and*

$$C_0 = \bigcap_{m=1}^{\infty} C_m \neq \phi,$$

then for every $z \in R^n$

$$\lim_{m \to \infty} \text{prox}(z \mid C_m) = \text{prox}(z \mid C_0).$$

PROOF. Fix $z \in R^n$, and for notational simplicity set $z_m = \text{prox}(z \mid C_m)$ for every m. Let $r_m = | z_m - z |$. The sequence $r_1, r_2, \ldots$, is nondecreasing and bounded above by r_0. Let $\bar{r} = \lim_{m \to \infty} r_m$. The sequence $z_1, z_2, \ldots$, is bounded in R^n, and all of its cluster points belong to the set

$$\{x \in C_0 \mid | x - z | \leqslant \bar{r}\} = \bigcap_{m=1}^{\infty} \{x \in C_m \mid | x - z | \leqslant \bar{r}\}.$$

The latter set is nonempty by compactness, so it must contain the unique point z_0 of C_0 nearest to z. Since $\bar{r} \leqslant r_0$, it can contain no other points. Thus z_0 is the only cluster point of the sequence $z_1, z_2, \ldots$ (and $\bar{r} = r_0$).

Our main result about convex-set-valued mappings can now be proved.

THEOREM 3. *Let $K : T \to R^n$ be a multivalued mapping such that $K(t)$ is a nonempty closed convex set for every $t \in T$. Then the following conditions are equivalent:*

(a) *K is a measurable multivalued mapping;*

(b) *the indicator of K, i.e., the function f on $T \times R^n$ defined by (1.6), is a normal convex integrand; in other words, there exists a countable collection U of measurable functions $u : T \to R^n$ such that $\{t \mid u(t) \in K(t)\}$ is a measurable subset of T for each $u \in U$, and $U(t) \cap K(t)$ is dense in $K(t)$ for each $t \in T$, where $U(t)$ is given by (1.2);*

(c) *the support function of K, i.e. the function g on $T \times R^n$ defined by*

$$g(t, y) = \sup\{\langle x, y\rangle \mid x \in K(t)\}, \tag{3.1}$$

is a normal convex integrand;

(d) *there exists a finite or countable family $(u_i \mid i \in I)$ of measurable functions from T to R^n such that, for every $t \in T$,*

$$K(t) = \text{cl conv}\{u_i(t) \mid i \in I\}; \tag{3.2}$$

(e) *there exists a finite or countable family $(v_i \mid i \in I)$ of measurable functions from T to R^n, and a corresponding family $(\alpha_i \mid i \in I)$ of measurable functions from T to R^1, such that, for every $t \in T$,*

$$K(t) = \{x \mid \forall i \in I, \langle x, v_i(t)\rangle \leqslant \alpha_i(t)\}; \tag{3.3}$$

(f) *for each $z \in R^n$, the function*

$$t \to \text{prox}(z \mid K(t)) \tag{3.4}$$

is measurable from T to R^n.

The equivalence of (a) and (f) has already been demonstrated by Castaing in case of T locally compact [2, p. 16]. The equivalence of (a), (c), (d), and (e) in this case could also be derived from Castaing's results, assuming that $K(t)$ is compact for every $t \in T$.

PROOF OF THEOREM 3. The equivalence of (b) and (c) is immediate from the fact that normality is preserved under duality: the convex integrands f and g are conjugate to each other, i.e., one has

$$g(t, y) = \sup\{\langle x, y\rangle - f(t, x) \mid x \in R^n\},$$
$$f(t, x) = \sup\{\langle x, y\rangle - g(t, y) \mid y \in R^n\}.$$

To prove the remaining equivalences, we shall show that

$$(a) \Rightarrow (b) \Rightarrow (e) \Rightarrow (f) \Rightarrow (d) \Rightarrow (a).$$

(a) implies (b). This is clear from Theorem 1.

(b) implies (e). Let $(z_i \mid i \in I)$ be a countable family of points which is dense in R^n. For each $i \in I$ and $t \in T$, let

$$u_i(t) = \text{prox}(z_i \mid K(t)),$$
$$v_i(t) = z_i - u_i(t),$$
$$\alpha_i(t) = \langle u_i(t), v_i(t) \rangle.$$

The functions u_i, v_i and α_i are measurable on T by Lemma 1. For each $i \in I$ and $t \in T$ the set

$$\{x \in R^n \mid \langle x, v_i(t) \rangle \leqslant \alpha_i(t)\} \tag{3.5}$$

is either a closed half-space supporting $K(t)$ at $u_i(t)$, or it is all of $R^n (v_i(t) = 0, \alpha_i(t) = 0)$. Therefore $K(t)$ is contained in the intersection of the sets (3.5) as i ranges over I. On the other hand, $K(t)$ cannot be properly smaller than this intersection. To see this, let $x \notin K(t)$ and $y = \text{prox}(x \mid K(t))$, so that

$$\langle x - y, x - y \rangle > 0.$$

Since $(z_i \mid i \in I)$ is dense, and $\text{prox}(\cdot \mid K(t))$ is a continuous mapping of R^n into itself, there exists an $i \in I$ such that

$$0 < \langle x - u_i(t), z_i - u_i(t) \rangle = \langle x, v_i(t) \rangle - \alpha_i(t).$$

Thus x fails to belong to one of the sets (3.5), and (e) holds as claimed.

(e) implies (f). We may suppose without loss of generality that the index set I is the set of all positive integers. For $m = 1, 2, \ldots$, let

$$K_m(t) = \{x \in R^n \mid \langle x, v_i(t) \rangle \leqslant \alpha_i(t) + 2^{-m}, i = 1, \ldots, m\}.$$

Each $K_m(t)$ is a closed convex set such that

$$\text{int } K_m(t) \neq \phi, \tag{3.6}$$

and we have

$$K_1(t) \supset K_2(t) \supset \cdots \supset K(t) = \bigcap_{m=1}^{\infty} K_m(t).$$

For each m the indicator function

$$f_m(t, x) = \delta(x \mid K_m(t))$$

is a normal convex integrand by (3.6) and the measurability of the functions v_i and α_i. (Let Z be a countable dense subset of R^n, and let U be the collection of all constant functions from T to R^n with values in Z.) Hence, for each m and each $z \in R^n$, the function

$$t \to \text{prox}(z \mid f_m(t, \cdot)) = \text{prox}(z \mid K_m(t))$$

is measurable from T to R^n by Lemma 1. Since this function converges pointwise to (3.4) as $m \to \infty$ by Lemma 2, condition (f) holds.

(f) implies (d). Let $\{z_i \mid i \in I\}$ be a countable family of points dense in R^n, and for each $i \in I$ and $t \in T$ let

$$u_i(t) = \operatorname{prox}(z_i \mid K(t)).$$

The functions u_i are then measurable, and (3.2) holds (where the convex hull operation can be omitted).

(d) implies (a). According to Carathéodory's theorem, the convex hull of $\{u_i(t) \mid i \in I\}$ is the set of all points of the form

$$\lambda_0 u_{i_0}(t) + \cdots + \lambda_n u_{i_n}(t),$$

where $i_k \in I$, $\lambda_k \geqslant 0$ for $k = 0, 1, \ldots, n$, and $\lambda_0 + \cdots + \lambda_n = 1$. Therefore

$$K(t) = \operatorname{cl}\{u(t) \mid u \in U\},$$

where U is the (countable) collection consisting of all functions of the form

$$u = \lambda_0 u_{i_0} + \cdots + \lambda_n u_{i_n}, \qquad \lambda_k \geqslant 0, \qquad \sum_{k=0}^{n} \lambda_k = 1,$$

where $i_k \in I$ and λ_k is *rational* for $k = 0, \ldots, n$. The functions $u \in U$ are measurable, so K is a measurable multivalued mapping by Theorem 1.

Corollary 3.1. *Let $K : T \to R^n$ be a multivalued mapping such that, for every $t \in T$, $K(t)$ is a closed convex set with a nonempty interior. Then K is a measurable multivalued mapping if and only if, for each $x \in R^n$,*

$$\{t \in T \mid x \in K(t)\}$$

is a measurable subset of T.

Proof. The necessity of the condition is immediate from the definition of the measurability of K. The sufficiency follows from the equivalence of (a) and (b) in Theorem 3. (Let Z be a countable dense subset of R^n, and let U be the collection of all constant functions from T to R^n with values in Z.)

Corollary 3.2. *Let $K : T \to R^n$ be a multivalued mapping such that, for every $t \in T$, $K(t)$ is a nonempty closed convex set containing no (whole) lines. Then K is measurable if and only if its support function g, defined by (3.1), is measurable in t for each fixed y.*

Proof. The condition that $K(t)$ contain no whole lines is equivalent to the condition that the convex set

$$\operatorname{dom} g(t, \cdot) = \{y \in R^n \mid g(t, y) < +\infty\}$$

have a nonempty interior for every t. Under the latter condition, g is normal if (and only if) $g(t, y)$ is measurable in t for each y, as noted in Section 1. The result therefore follows from the equivalence of (a) and (c) in Theorem 3.

The line condition in Corollary 3.2 is satisfied in particular if $K(t)$ is compact for every t. Corollary 3.2 has been deduced in this special case by Castaing [2, p. 52] with T locally compact (but with R^n replaced any separable Fréchet space).

COROLLARY 3.3. *If $K : T \to R^n$ is any measurable multivalued mapping from T to R^n, then the multivalued mapping*

$$K' : t \to \text{cl conv } K(t)$$

is measurable.

PROOF. Restricting K to the measurable set T_0 defined by (2.5) if necessary, we can assume that $K(t)$ is nonempty for every t. Let $K''(t) = \text{cl } K(t)$. Since the multivalued mapping $K'' : T \to R^n$ is measurable (as observed in Section 2), there exists by Theorem 1 a countable collection U of measurable functions $u : T \to R$ such that, for every $t \in T$,

$$\text{cl } K(t) = \text{cl}\{u(t) \mid u \in U\}.$$

We have

$$K'(t) = \text{cl conv}\{u(t) \mid u \in U\},$$

so K' is measurable by criterion (d) of Theorem 3.

Corollary 3.3 has previously been proved by Castaing in the case of T locally compact under the assumption that $K(t)$ is compact for every t [2, p. 27].

COROLLARY 3.4. *If $(K_i \mid i \in I)$ is any finite or countable family of measurable multivalued mappings from T to R^n, then the multivalued mapping*

$$K : t \to \text{cl conv} \bigcup_{i \in I} K_i(t)$$

is measurable.

PROOF. Let

$$K'(t) = \bigcup_{i \in I} K_i(t).$$

It follows trivially from the definition of the measurability of the K_i that K' is measurable. Since

$$K(t) = \text{cl conv } K'(t),$$

K is measurable by Corollary 3.3.

COROLLARY 3.5. *Let $K : T \to R^n$ be a measurable multivalued mapping such that $K(t) \neq \phi$ for every t. Then the multivalued mapping*

$$K^0 : t \to K(t)^0 = \{y \in R^n \mid \forall x \in K(t), \langle x, y \rangle \leqslant 1\}$$

is measurable.

PROOF. Let $K'(t) = \mathrm{cl}\, K(t)$ for every t. Since K' is another measurable multivalued mapping, there exists by Theorem 1 a countable family $(u_i \mid i \in I)$ of measurable functions from T to R^n such that, for every $t \in T$,

$$K'(t) = \mathrm{cl}\{u_i(t) \mid i \in I\},$$

we have

$$K^0(t) = K'(t)^0 = \{y \mid \forall i \in I, \langle u_i(t), y \rangle \leqslant 1\}.$$

Hence K^0 is measurable by criterion (e) of Theorem 3.

COROLLARY 3.6. *Let $K : T \to R^n$ be a multivalued mapping such that $K(t)$ is a subspace of R^n for every $t \in T$. Then the following conditions are equivalent:*

(a) *K is a measurable multivalued mapping;*

(b) *the multivalued mapping $K^\perp : T \to R^n$ is measurable, where $K^\perp(t)$ is the orthogonal complement of $K(t)$ for every t;*

(c) *there exist measurable functions $a_i : T \to R^n$, $i = 1,\ldots, m$, such that $K(t)$ is the subspace generated by the vectors $a_1(t),\ldots, a_m(t)$ for every t;*

(d) *there exist measurable functions $a_i : T \to R^n$, $i = 1,\ldots, m$, such that, for every t,*

$$K(t) = \{x \mid \langle x, a_i(t) \rangle = 0, i = 1,\ldots, m\}.$$

PROOF. The equivalence of (a) and (b) follows from the equivalence of conditions (a), (b) and (c) of Theorem 3, because the support function of a subspace is the indicator of its orthogonal complement. Furthermore, it is clear that (c) holds for K if and only if (d) holds for $K^\perp$, and that (d) holds for K if and only if (c) holds for $K^\perp$. Therefore, in view of the equivalence of (a) and (b), to complete the proof we need only show that (a) is equivalent to (c). To see that (c) implies (a), one can apply Theorem 1 to the collection U consisting of all linear combinations of the functions $a_1 ,\ldots, a_m$ with rational coefficients. To see that (a) implies (c), let $e_1 ,\ldots, e_n$ be a basis for R^n, and for each $t \in T$ let $a_1(t),\ldots, a_n(t)$ be the orthogonal projections of $e_1 ,\ldots, e_n$ on $K(t)$. The functions $a_i : t \to a_i(t)$ are measurable by (f) of Theorem 3, so (c) is satisfied (with $m = n$).

Note in Corollary 3.6 that the dimension of $K(t)$ must be a measurable function of t when K is measurable, since by condition (c) this dimension is the rank of a certain matrix $A(t)$ whose rows $a_1(t),\ldots, a_m(t)$ are measurable functions of t. (The rank of $A(t)$ can be expressed in terms of the vanishing of certain determinants which are measurable functions of t.)

4. Normality of Convex Integrands

The results of Section 2 and Section 3 will now be applied to convex integrands, which may be regarded of course as correspondences associating with each $t \in T$, not a subset of R^n, but a convex function $f(t, \cdot)$ on R^n.

Theorem 4. *Let f be a function on $T \times R^n$ with values in $(-\infty, +\infty]$ such that, for each $t \in T$, $f(t, x)$ is a lower semi-continuous convex function of x which is not identically $+\infty$. Then the following conditions are equivalent:*

(a) *f is a normal convex integrand;*

(b) *the epigraph mapping of f, i.e., the multivalued mapping $K : T \to R^{n+1}$ defined by* (1.5), *is measurable;*

(c) *there exists a finite or countable family $(v_i \mid i \in I)$ of measurable functions from T to R^n, and a corresponding family $(\alpha_i \mid i \in I)$ of measurable functions from T to R^1, such that, for every $t \in T$ and $x \in R^n$,*

$$f(t, x) = \sup\{\langle x, v_i(t)\rangle - \alpha_i(t) \mid i \in I\}. \tag{4.1}$$

Proof. Let f^* be the conjugate convex integrand defined by (1.4), and let $K^* : T \to R^{n+1}$ be the epigraph mapping of f^*. Let (b*) and (c*) denote conditions (b) and (c) for f^* in place of f. We shall show that

$$(a) \Rightarrow (b) \Rightarrow (c^*) \Rightarrow (b^*) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).$$

(a) implies (b). Let Z be a countable dense subset of R^n, and let U be the (countable) collection of all functions $u : T \to R^{n+1}$ of the form

$$u(t) = (w(t), f(t, w(t)) + \epsilon), \qquad w(t) = \text{prox}(z \mid f(t, \cdot)),$$

where $z \in Z$ and ϵ is a positive rational number. Each $u \in U$ is measurable by Lemma 1. Since for each t the range of the mapping

$$z \to \text{prox}(z \mid f(t, \cdot))$$

is dense in the set $D(t) = \text{dom} f(t, \cdot)$, as noted in Section 3, the set of points $w(t)$ as z ranges over Z is likewise dense in $D(t)$. Moreover $D(t)$ is the image of $K(t)$ under the projection $(x, \mu) \to x$, and $K(t)$ is a closed convex set. Therefore (2.1) holds, and K is measurable by Theorem 1.

(b) implies (c*). If K is measurable, there exists by Theorem 1 a countable family $(v_i \mid i \in I)$ of measurable functions from T to R^n, and a corresponding family $(\alpha_i \mid i \in I)$ of measurable functions from T to R^1, such that

$$K(t) = \text{cl}\{(v_i(t), \alpha_i(t)) \mid i \in I\}$$

for every $t \in T$. Since

$$f^*(t, x^*) = \sup\{\langle x, x^* \rangle - \mu \mid (x, \mu) \in K(t)\},$$

one has

$$f^*(t, x^*) = \sup\{\langle v_i(t), x^* \rangle - \alpha_i(t) \mid i \in I\} \tag{4.2}$$

for every $t \in T$, so that (c*) holds.

(c*) implies (b*). Given measurable functions $v_i : T \to R^n$ and $\alpha_i : T \to R^1$ such that (4.2) is satisfied, we have

$$\begin{aligned} K^*(t) &= \{(x^*, \mu^*) \mid x^* \in R^n, \mu^* \in R^1, \mu^* \geqslant f^*(t, x^*)\} \\ &= \{y \in R^{n+1} \mid \forall i \in I, \langle y, w_i(t) \rangle \leqslant \alpha_i(t)\}, \end{aligned}$$

where $w_i(t) = (v_i(t), -1)$. Therefore K^* is a measurable multivalued mapping by criterion (e) of Theorem 3.

(b*) implies (c). This follows by the same argument which showed that (b) implies (c*), because f is in turn the convex integrand conjugate to f^*.

(c) implies (b). Same as the argument that (c*) implies (b*).

(b) implies (a). By Theorem 1, the measurability of K implies the existence of a countable family $(u_j \mid j \in J)$ of measurable functions from T to R^n, and a corresponding family $(\beta_j \mid j \in J)$ of measurable functions from T to R^1, such that

$$K(t) = \text{cl}\{(u_j(t), \beta_j(t)) \mid j \in J\}$$

for every $t \in T$. Let $U = \{u_j \mid j \in J\}$. Inasmuch as the set $D(t)$ is just the image of $K(t)$ under the projection $(x, \mu) \to x$, the set

$$U(t) \cap D(t) = \{u_j(t) \mid j \in J\}$$

is dense in $D(t)$ for every $t \in T$. We shall show that $f(t, u_j(t))$ is measurable in t for every $j \in J$, and this will complete the proof that f is normal. It has already been verified that (b) implies (c). Let $(v_i \mid i \in I)$ and $(\alpha_i \mid i \in I)$ be countable families of measurable functions as described in (c). Then

$$f(t, u_j(t)) = \sup\{\langle u_j(t), v_i(t) \rangle - \alpha_i(t) \mid i \in I\}.$$

Thus $f(\cdot, u_j(\cdot))$ is the supremum of a countable family of measurable functions on T and hence is measurable.

COROLLARY 4.1. *Let $(f_i \mid i \in I)$ be a finite or countable collection of normal convex integrands on $T \times R^n$, and let*

$$f(t, x) = \sup\{f_i(t, x) \mid i \in I\}.$$

Then the set

$$T_0 = \{t \in T \mid \exists x \in R^n, f(t, x) < +\infty\}$$

is measurable in T. If $T_0 = T$, f is another normal convex integrand.

PROOF. Let $K_i(t) = \operatorname{epi} f_i(t, \cdot)$. Then

$$\operatorname{epi} f(t, \cdot) = \bigcap_{i \in I} K_i(t).$$

Each of the multivalued mappings $K_i : T \to R^{n+1}$ is measurable by Theorem 4, so the multivalued mapping

$$K : t \to \operatorname{epi} f(t, \cdot)$$

is measurable by Corollary 1.3. Since

$$T_0 = K^{-1}(R^{n+1}),$$

T_0 is measurable. If $T_0 = T$, f is a normal convex integrand by criterion (b) of Theorem 4.

COROLLARY 4.2. *Let f_1 and f_2 be normal convex integrands on $T \times R^n$, and let*

$$f(t, x) = f_1(t, x) + f_2(t, x).$$

Then the set

$$T_0 = \{t \in T \mid \exists x \in R^n, f(t, x) < +\infty\}$$

is measurable in T. If $T_0 = T$, f is another normal convex integrand.

PROOF. The T_0 here is the same as the T_0 in Corollary 4.1 for $I = \{1, 2\}$; hence it is measurable. Assuming that $T_0 = T$, $f(t, \cdot)$ is for each $t \in T$ a lower semi-continuous convex function on R^n which is not identically $+\infty$. In fact, let

$$f_1(t, x) = \sup\{\langle x, v_j^1(t)\rangle - \alpha_j^1(t) \mid j \in I_1\},$$
$$f_2(t, x) = \sup\{\langle x, v_k^2(t)\rangle - \alpha_k^2(t) \mid k \in I_2\},$$

be representations of f_1 and f_2 as in (c) of Theorem 4. Let $I = I_1 \times I_2$, and for each $i = (j, k)$ let

$$v_i(t) = v_j^1(t) + v_k^2(t), \qquad \alpha_i(t) = \alpha_j^i(t) + \alpha_k^2(t).$$

The functions $v_i : T \to R^n$ and $\alpha_i : T \to R^1$ are then measurable, and (4.1) holds. Therefore f is a normal convex integrand by criterion (c) of Theorem 4.

Corollary 4.3. *Let f be a normal convex integrand on $T \times R^n$, and let $C : T \to R^n$ be a measurable multivalued mapping such that, for every $t \in T$, $C(t)$ is a closed convex set. Then the extended real-valued function p on T defined by*

$$p(t) = \inf\{f(t, x) \mid x \in C(t)\} \tag{4.3}$$

is measurable (where $\inf \phi = +\infty$ by convention). Moreover, for any measurable function $\alpha : T \to R^1$ the multivalued mapping

$$K : t \to \{x \in C(t) \mid f(t, x) \leqslant \alpha(t)\} \tag{4.4}$$

is measurable.

In particular, if for every t the infimum in (4.3) is finite and attained, Corollary 1.1 is applicable to the K in (4.4) with $\alpha(t) = p(t)$, and it follows that there exists a measurable function $u : T \to R^n$ such that, for every t,

$$u(t) \in C(t) \qquad \textit{and} \qquad f(t, u(t)) = p(t).$$

Proof. There is no loss of generality in assuming that $C(t) \neq \phi$ for every t. Then

$$g(t, x) = \delta(x \mid C(t))$$

is a normal convex integrand by (b) of Theorem 3. Let

$$h(t, x) = f(t, x) + g(t, x).$$

By Corollary 4.2, the set

$$T_0 = \{t \mid \exists x, h(t, x) < +\infty\} = \{t \mid p(t) \neq +\infty\}$$

is measurable. Thus, to prove the first assertion, it is enough to consider the the case where $T_0 = T$. In this case h is a normal convex integrand by Corollary 4.2. We have

$$p(t) = \inf\{h(t, x) \mid x \in R^n\},$$

so p is measurable (see Section 1).

In proving the second assertion of the corollary, we can assume that $\alpha(t) = 0$ for every t, since otherwise f could be replaced by the convex integrand

$$k(t, x) = f(t, x) - \alpha(t),$$

which would trivially again be normal. Then, given any closed subset S of R^n, we have

$$K^{-1}(S) = (K')^{-1}(S'), \tag{4.5}$$

where

$$K'(t) = \operatorname{epi} h(t, \cdot), \qquad S' = \{(x, \mu) \in R^{n+1} \mid x \in S, \mu \leqslant 0\}.$$

The multivalued mapping $K' : T \to R^{n+1}$ is measurable by (b) of Theorem 4, so the set in (4.5) is measurable. This shows that K is a measurable multivalued mapping.

Results similar to Corollary 4.3 have been proved by Castaing [2, Section 4], but, while these do not assume convexity, they require T to be locally compact and $C(t)$ to be compact for every $t \in T$. They also require f either to be continuous in x for each t (as well as measurable in t for each x), or to be lower semi-continuous as a function of t and x jointly.

COROLLARY 4.4. *Let $(f_i \mid i \in I)$ be a finite or countable collection of normal convex integrands on $T \times R^n$, and let $(\alpha_i \mid i \in I)$ be a corresponding family of measurable functions from T to R^1. Then the multivalued mapping*

$$K : t \to \{x \in R^n \mid \forall i \in I, f_i(t, x) \leqslant \alpha_i(t)\}$$

is measurable. In particular, the set

$$\{t \mid \exists x \in R^n, \forall i \in I, f_i(t, x) \leqslant \alpha_i(t)\} \tag{4.6}$$

is measurable in T.

PROOF. Let

$$K_i(t) = \{x \mid f_i(t, x) \leqslant \alpha_i(t)\}.$$

Each $K_i : T \to R^n$ is measurable by the preceding corollary (with $C(t) = R^n$ for every t), so K is measurable by Corollary 1.3. The set in (4.6) is just $K^{-1}(R^n)$.

COROLLARY 4.5. *Let f be a normal convex integrand on $T \times R^n$, and let $u : T \to R^q$ be a measurable function, where $1 \leqslant q < n$. Let*

$$g(t, y) = f(t, u(t), y)$$

for every $t \in T$ and $y \in R^m$, where $m = n - q$. Then the set

$$T_0 = \{t \mid \exists y \in R^m, g(t, y) < +\infty\}$$

is measurable in T. If $T_0 = T$, g is a normal convex integrand on $T \times R^m$.

PROOF. Let $K : T \to R^n$ be defined by

$$K(t) = \{(u(t), y) \mid y \in R^m\}.$$

This K is a measurable multivalued mapping by Corollary 1.2, in view of the fact that

$$K(t) = \{(0, y) \in R^n \mid y \in R^m\} + (u(t), 0).$$

The indicator

$$k(t, x) = \delta(x \mid K(t))$$

is therefore a normal convex integrand by (b) of Theorem 3. The result now follows from applying Corollary 4.2 to $f + k$.

COROLLARY 4.6. *Let f be a normal convex integrand on $T \times R^n$, and for each $t \in T$ and $x \in R^n$ let $\partial f(t, x)$ be the subdifferential of $f(t, \cdot)$ at x, i.e.,*

$$\partial f(t, x) = \{x^* \in R^n \mid \forall y \in R^n, f(y) \geqslant f(x) + \langle y - x, x^* \rangle\}.$$

Then, for any measurable function $u : T \to R^n$, the multivalued mapping

$$K : t \to \partial f(t, u(t))$$

is measurable.

PROOF. We have

$$K(t) = \{x^* \mid f^*(t, x^*) - \langle u(t), x^* \rangle \leqslant -f(t, u(t)\},$$

where f^* is the normal convex integrand on $T \times R^n$ conjugate to f. The convex integrand

$$g(t, x^*) = f^*(t, x^*) - \langle u(t), x^* \rangle$$

is again normal, and the function

$$\alpha(t) = -f(t, u(t))$$

is measurable. Therefore K is measurable by Corollary 4.3 (with $C(t) = R^n$ for every t).

Finally, we apply Theorem 2 to get criteria for the normality of f in terms of the measurability of $f(t, x)$ in t and x jointly. (For the terminology, see Section 2.)

THEOREM 5. *Let f be a convex integrand on $T \times R^n$ such that, for each $t \in T$, $f(t, x)$ is a lower semicontinuous function of x which is not identically $+\infty$. If f is normal, then f is a measurable function on $T \times R^n$. On the other hand, if f is a measurable function on $T \times R^n$ and the measurable space $(T, \mathscr{T})$ is complete, then f is normal.*

PROOF. For each real number α, let $K_\alpha : T \to R^n$ be the multivalued mapping defined by

$$K_\alpha(t) = \{x \mid f(t, x) \leqslant \alpha\}.$$

The graph of K_α is thus the set

$$G(K_\alpha) = \{(t, x) \in T \times R^n \mid f(t, x) \leqslant \alpha\},$$

and $K_\alpha(t)$ is closed for every $t \in T$ by the lower semicontinuity of $f(t, x)$ in x. If f is normal, every K_α is measurable by Corollary 4.3. Then the sets $G(K_\alpha)$ are all measurable in $T \times R^n$ by Theorem 2, implying that f is measurable.

Suppose now that f is measurable, and that $(T, \mathscr{T})$ is complete in the sense of Section 2. The sets $G(K_\alpha)$ are measurable in $T \times R^n$, so the multivalued mappings K_α are measurable. The multivalued mappings $K'_\alpha : T \to R^{n+1}$ defined by

$$K'_\alpha(t) = \{(x, \mu) \mid x \in K_\alpha(t), \mu = \alpha\}$$

are then measurable too. For the epigraph mapping K of f defined by (1.5), we have

$$K(t) = \mathrm{cl} \bigcup \{K'_\alpha(t) \mid \alpha \text{ rational}\},$$

so K is measurable. Hence f is normal by criterion (b) of Theorem 4.

In the case where T is a Lebesgue (resp. Borel) subset of R^m, let us call a convex integrand f on $T \times R^n$ *Lebesgue* (resp. *Borel*) *normal* if f satisfies the definition of normality with the functions $u \in U$ Lebesgue (resp. Borel) measurable. Then we have the following analogues of Corollaries 2.1 and 2.2.

COROLLARY 5.1. *Suppose that T is a Lebesgue measurable subset of R^m, and let f be a convex integrand on $T \times R^n$ such that, for every $t \in T$, $f(t, x)$ is a lower semi-continuous function of x which is not identically $+\infty$. In order that f be Lebesgue normal, it is necessary and sufficient that f be measurable with respect to the σ-ring in $T \times R^n$ generated by all the sets of the form $A \times B$ such that A is a Lebesgue measurable subset of T and B is a Borel measurable subset of R^n.*

COROLLARY 5.2. *Suppose that T is a Borel subset of R^m, and let f be a convex integrand on $T \times R^n$ such that, for every $t \in T$, $f(t, x)$ is a lower semi-continuous function of x which is not identically $+\infty$. If f is Borel normal, then f is a Borel measurable function on $T \times R^n$. On the other hand, if f is a Borel measurable function on $T \times R^n$, then f is Lebesgue normal.*

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