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DUAL PROBLEMS OF OPTIMAL CONTROL

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The close connection between duality and convexity is well known. It is no surprise, therefore, that the strongest properties of duality in optimal control are displayed by problems which are especially "convex" in nature. Such problems arise commonly in various situations, for example economic applications, where duality may be interpreted in terms of pricė behavior. They can also arise theoretically as local or global convexifications of more general problems. The possible use of duality in the construction of algorithms is another motivation for studying them.

In what follows, we indicate briefly some of the main results that have been obtained for convex problems of Bolza [1,2,3,4,5]. To simplify the discussion and to make clearer the relationship with control problems as they are usually formulated, we limit ourselves here to the "autonomous" case and choose a model in which the control variables appear explicitly. Nevertheless we use the device of incorporating constraints into the cost functions by means of infinite penalties, since this is not only very convenient in theory, but essential if the basic ideas are not to be obscured.

The model problem consists of minimizing

$$(1) \qquad \int_0^1 f(x(t),u(t))dt + \ell(x(0),x(1))$$

over all the absolutely continuous arcs $x:[0,1] \to R^n$ and measurable functions $u:[0,1] \to R^n$ satisfying

$$(2) \qquad \dot{x}(t) = Ax(t) + u(t) \quad \text{a.e.},$$

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where it is assumed that f and l are lower semicontinuous, convex functions from $R^n \times R^n$ to $(-\infty,+\infty]$, not identically $+\infty$. Note that $f(x,u)$ is to be convex jointly in x and u, rather than just convex in u, as would be a more common assumption. The problem can be described in the context of functional analysis as that of minimizing a certain Bolza functional

$$(3) \qquad \Phi(x) = \int_0^1 f(x(t), \dot{x}(t) - Ax(t))dt + l(x(0),x(1))$$

over a Banach space $\mathcal{A}$ consisting of all the absolutely continuous functions $x:[0,1] \to R^n$.

Since in a problem of minimization the points where the cost function has the value $+\infty$ do not compete for the optimum, our model contains implicit constraints on the controls, states and endpoints. For almost every t, the control vector $u(t)$ should belong to the set $U(x(t))$, where

$$(4) \qquad U(x) = \{u \varepsilon R^n \mid f(x,u) < +\infty\}$$

(implicit control region), and thus the state vector $x(t)$ should belong to the set

$$(5) \qquad X = \{x \varepsilon R^n \mid \exists u, f(x,u) < +\infty\}.$$

The endpoint pair $(x(0),x(1))$ should belong to the set

$$(6) \qquad C = \{(a_0,a_1) \varepsilon R^n \times R^n \mid l(a_0,a_1) < +\infty\}.$$

As an illustration, consider the problem of minimizing (for some $\lambda \geq 0$)

$$\lambda \int_0^1 |x(t)|dt + \int_0^1 |u(t)|^2 dt$$

subject to $\dot{x} = Ax + u$, $|u| \leq 1$, $x(0) = a$ and $x(1) \varepsilon R^n_+$ (nonnegative orthant). This corresponds to

$$\begin{aligned} f(x,u) &= \lambda|x| + |u|^2 \quad \text{if} \quad |u| \leq 1, \\ &= +\infty \quad \text{if} \quad |u| > 1, \end{aligned}$$

$$\begin{aligned} l(a_0,a_1) &= 0 \quad \text{if} \quad a_0 = a \quad \text{and} \quad a_1 \varepsilon R^n_+, \\ &= +\infty \quad \text{if} \quad a_0 \neq a \quad \text{or} \quad a_1 \notin R^n_+. \end{aligned}$$

The example serves to emphasize that no differentiability is assumed in the cost functions, even with respect to x. It shows also that, although we speak formally of problems of Bolza, other classes of problems, such as those of Lagrange, are aptly covered by the same notation.

In working with $+\infty$, it is necessary to take a somewhat different approach than usual to a number of technical questions concerning measurability, integrability, and so forth. This is particularly true in the case, not discussed here, where f and A depend on t. Fortunately, the theory of measurable multifunctions, as developed extensively by Castaing and others, comes to our aid. At the same time, convexity leads to many simplifications. Thus it can be shown under our assumptions that the Bolza functional $\Phi: \mathcal{A} \to (-\infty, +\infty]$ is not only well-defined and convex, but lower semicontinuous in the weak (and strong) topologies [1]. In fact, if the generalized Hamiltonian function

$$(7) \qquad H(x,p) = \sup_{u} \{p\cdot(Ax+u) - f(x,u)\}$$

nowhere has the value $+\infty$, then Φ has the remarkable property that its level sets

$$\{x \in \mathcal{A} \mid \Phi(x) \leq \alpha\}, \quad \alpha \text{ real},$$

are (closed and) locally compact in the weak topology [3]. This can be used to deduce the existence of optimal arcs in the control problem [4]. However, the existence also follows from duality theorems stated below. Results on necessary conditions for optimality also follow from the duality theorems, so that, for convex problems of Bolza, the latter really play the central role.

Duality is obtained by passing to the convex functions f^* and ℓ^* conjugate to f and ℓ, or rather to slightly modified forms of these functions. We define the dual cost functions g and m by

$$(8) \qquad g(p,w) = f^*(w,p) = \sup_{x,u} \{w\cdot x + p\cdot u - f(x,u)\},$$

$$(9) \qquad m(b_0,b_1) = \ell^*(b_0,-b_1) = \sup_{a_0,a_1} \{a_0 \cdot b_0 - a_1 \cdot b_1 - \ell(a_0,a_1)\}.$$

Then g and m are again lower semicontinuous, convex functions from $R^n \times R^n$ to $(-\infty,+\infty]$, not identically $+\infty$. Furthermore, f and ℓ may be recovered in turn as the duals of g and m:

$$f(x,u) = \sup_{p,w}\{u \cdot p + x \cdot w - g(p,w)\},$$

$$\ell(a_0,a_1) = \sup_{b_0,b_1} \{a_0 \cdot b_0 - a_1 \cdot b_1 - m(b_0,b_1)\}.$$

The dual control problem is taken to be that of minimizing

$$(10) \qquad \int_0^1 f(p(t), w(t)dt + m(p(0), p(1))$$

over all the absolutely continuous arcs $p:[0,1] \to R^n$ and measurable functions $w:[0,1] \to R^n$ satisfying

$$(11) \qquad \dot{p}(t) = -A^*p(t) + w(t) \qquad \text{a.e.},$$

where A^* is the transpose of the matrix A. In view of the symmetric relationship between f and g, and between ℓ and m, the problem which is dual to the dual problem is the primal (i.e. original) problem. The dual, like the primal, contains implicit constraints on the controls $w(t)$, states $p(t)$ and endpoint pair $(p(0),p(1))$. It can be regarded as the problem of minimizing the lower semicontinuous, convex Bolza functional

$$(12) \qquad \Psi(p) = \int_0^1 g(p(t), \dot{p}(t) + A^*p(t))dt + m(p(0),p(1))$$

over the Banach space $\mathcal{A}$.

In the example given earlier, one calculates easily from (8) and (9) that the dual problem consists of minimizing

$$\int_0^1 \theta(|p(t)|)dt + a \cdot p(0)$$

subject to $\dot{p} = -A^*p + w$, $|w| \leq \lambda$ and $p(1) \in R^n_+$, where

$$\theta(s) = s^2/4 \quad \text{if } s \leq 2,$$
$$= s - 1 \quad \text{if } s \geq 2.$$

The definitions of the Bolza functionals imply that the inequality

$$\Phi(x) + \Psi(p) \geq 0$$

is valid, and consequently that

(13) [inf in primal] $\geq$ - [inf in dual].

A fundamental question in duality theory is whether, or rather under what conditions, equality holds in (13). It usually happens that, when equality can be established, one obtains by the same argument the existence of a minimizing arc for Φ or Ψ, so that "inf" can be replaced by "min" in one of the two problems. In general, the study of (13) involves the convex functionals on $\mathcal{A}^*$ conjugate to Φ and Ψ, and these can be described in terms of the behavior of the control problems with respect to certain perturbations of the data [1]. To obtain sharper results, which proceed from "readily verifiable" assumptions on the functions f and l and the matrix A, an argument based on the separation of convex sets has been devised [3]. This argument is made complicated by the fact that the convex sets belong to the space $\mathcal{A}^* \times R^2$. One must show, despite the underlying nonreflexivity, that the separating hyperplane corresponds to an element of $\mathcal{A} \times R^2$, rather than $\mathcal{A}^{**} \times R^2$.

The main duality theorems depend on finiteness assumptions on the Hamiltonian function (7) and attainability assumptions on the control systems. An endpoint pair $(x(0), x(1))$ is said to be <u>attainable</u> for the primal problem if it arises from an arc satisfying (2) and

$$u(t) \in U(x(t)) \quad \text{a.e.},$$

where $U(x(t))$ is the implicit control region in (4). The set of all such pairs is convex in $R^n \times R^n$. The <u>attainability condition</u> is said to be satisfied if the relative interior of this set meets the relative interior of the convex set of all feasible endpoint pairs, that is, the set C in (6).

DUALITY THEOREM 1[3]. If the primal problem satisfies the attainability contion and $H(x,p) > -\infty$ everywhere (in other words, there are no real state constraints on x), one has

(14) $\quad$ [inf in primal] = -[min in dual].

DUALITY THEOREM 2[3]. If the dual problem satisfies the attainability condition and $H(x,p) < +\infty$ everywhere (a growth condition of Nagumo-Tonelli type on the function $u \to f(x,u)$), one has

(15) $\quad$ [min in primal] = -[inf in dual].

The dual attainability condition on endpoint pairs $(p(0),p(1))$ can be expressed equivalently as a growth condition on the Bolza functional Φ in the primal problem [3].

The fact that the attainment of the infimum in the dual problem seems to require the absence of state constraints in the primal problem can be explained from the role that dual optimal arcs have in the statement of necessary conditions for the primal, as seen below. It is known that, when state constraints are present, the necessary conditions ought to involve jumps in the costate vector $p(t)$. However, an optimal arc for the dual problem is by definition absolutely continuous. This suggests that, in order to obtain a better duality theory in the case of state constraints, one should pass to a generalized dual problem where the arcs p are allowed to be discontinuous. The idea has been worked out in [4] for a large class of problems in which, roughly speaking, the effects of the state constraints on x can be kept separate from the effects of the other constraints. The dual problem then consists of minimizing an extended Bolza functional, not over $\mathcal{A}$, but over a Banach space $\mathcal{B}$ consisting of all the functions $p:[0,1] \to R^n$ of bounded variation.

Nevertheless, the theory is still incomplete, since it does not cover many problems, encountered for example in economics, where the effects of the state constraints cannot be kept separate. Also, it would seem desirable ultimately, at least for the sake of symmetry, to formulate the primal, as well as the dual, in terms of functions of bounded variation, deriving continuity properties of the optimizing arcs in a given case from the necessary conditions for optimality.

We conclude by describing the necessary conditions that correspond to the situation in Theorem 1. These depend on the fact that the Hamiltonian $H(x,p)$ is not only convex in p, as follows immediately from the definition (7), but also concave in x. The latter property is a consequence (indeed, virtually an equivalent form) of our assumption that $f(x,u)$ is convex jointly in x and u. Recall that in dealing with convex functions one can replace the usual notion of differentiability, that of a tangent hyperplane to the graph of the function, by the notion of a supporting hyperplane to the epigraph of a function. Specifically, if φ is an extended real-valued, convex function on R^n, we define a subgradient of φ at z to be a vector y such that the inequality

$$\varphi(z') \geq \varphi(z) + y\cdot(z' - z)$$

holds for all $z'\varepsilon R^n$. The set of such subgradients is denoted by $\partial\varphi(z)$. Subgradients of concave fundtions are defined analogously, with the opposite inequality. Applying this idea to $H(x,p)$ as a function of x and p separately, we obtain a generalized Hamiltonian system of differential equations:

$$(16) \qquad \dot{x}\varepsilon\ \partial_p H(x,p) \text{ and } -\dot{p}\varepsilon\ \partial_x H(x,p) \quad \text{a.e.}$$

Subgradients can also be used to formulate a generalized transversality condition:

$$(17) \qquad (-p(0),p(1)\ \varepsilon\ \partial l(x(0),x(1)).$$

We refer to (16) and (17) as the fundamental optimality conditions for a convex problem of Bolza. Of course, in particular cases the many theorems available for the calculation of subgradients can be used to express these conditions in a less abstract manner [1]. The main result is the following:

THEOREM 3. Let the assumptions of Theorem 1 be satisfied. Then, in order that an arc $x \in \mathcal{A}$ be optimal for the primal problem, it is necessary and sufficient that there exist an arc $p \in \mathcal{A}$ for which conditions (16) and (17) are satisfied. Such an arc p is optimal for the dual problem.

The generalized Hamiltonian system is quite amenable to study, despite its "multivaluedness". For example, under the assumption that H is finite, it has been shown that local solutions exist, and along such solutions the value of H is constant [2]. The behavior of the system near a saddle-point of H (in the minimax sense) has also been investigated and shown to be relevant to certain optimal control problems where $[0,1]$ is replaced by an infinite time interval [5].

Theorem 3 has been extended in [4] to a class of problems with state constraints through an appropriate definition of what is meant by the Hamiltonian condition (16) in the case where p is not absolutely continuous, but merely of bounded variation.

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