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Continuous Versus Measurable Recourse in N-Stage Stochastic Programming

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An optimization model is studied in which a convex functional, giving expected cost subject to convex constraints, is minimized over a class of measurable recourse functions describing decisions that depend nonanticipatively on a sequence of observations of random variables. Conditions are established under which the infimum of the cost functional is not altered if the recourse functions are restricted to be continuous.

1. INTRODUCTION

Let Ξ_k be a Hausdorff topological space for $k = 1,..., N$, and let σ be a (regular Borel) probability measure on $\Xi_1 \times \cdots \times \Xi_N$ with support denoted by Ξ (that is, Ξ is the smallest closed subset of $\Xi_1 \times \cdots \times \Xi_N$ of measure 1). We consider an abstract sequential optimization problem with N stages, where in the kth stage an element of R^{n_k}, called a recourse, is selected. The measure σ gives the probability distribution of all possible outcomes $\xi = (\xi_1 ,..., \xi_N)$, where ξ_k is the element observed in stage k.

Not only are decisions made in a sequential manner, but it is essential to our model that the recourse selected in stage k can only depend on the observations made up to that time. Thus we specifically limit our attention to recourse functions

$$x: \Xi \to R^n \triangleq R^{n_1} \times \cdots \times R^{n_N}$$

of the form

$$x(\xi) = (x_1(\xi_1), x_2(\xi_1 , \xi_2),..., x_N(\xi_1 ,..., \xi_N)), \tag{1.1}$$

where $x_k(\xi_1 ,..., \xi_k) \in R^{n_k}$. Such a function is said to be *nonanticipative*. The

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object of the optimization problem is to minimize a functional of the form

$$F(x) = \int_{\Xi} f(\xi, x(\xi))\, d\sigma = E\{f(\xi, x(\xi))\} \tag{1.2}$$

over a class of nonanticipative recourse functions. Constraints are represented by allowing f to be extended-real-valued (cf. [9]).

This rather abstract model is the prototype of a large class of stochastic optimization problems of the adaptive type. The only significant class which is not included is the one whose dynamics are described by a continuous process and involve continuous feedback. A nonanticipative function of the type (1.1) can naturally be viewed as the feedback control of a stochastic dynamical system with a finite number of observations and decisions (corrections). More specifically, this model includes various classes of inventory problems [5], stochastic dynamic programming problems (with possibly an uncountable number of states) [1, 12], stochastc programming problems [2, 4], and discrete stochastic control problems [3].

For each of these types of problems the question of characterizing the class of admissible and optimal recourse functions has been studied extensively. This was done in order to obtain various results of more or less theoretical interest, as well as to develop useful properties which have direct bearing on devising efficient methods of solution. The first results we know of appear in papers analyzing certain classes of inventory and economic problems [5], where recourse functions are known as *decision rules*. It is also under the name of decision rules that these functions first entered the literature devoted to stochastic programming. Charnes, Cooper and Symonds [2] were the first to be concerned with their properties, in particular for stochastic programs with chance constraints. For a review of the results for stochastic programs, see [4] and the references mentioned there. Recourse functions are known as *policies* in (stochastic) dynamic programming [1, 12]. Of course, given the very specific models studied in that context (usually the sets Ξ_k are finite and the process is Markovian), it has been possible to obtain thorough characterizations, if not the explicit form.

The positive results which are available in the literature rely on the specific nature of the problem being analyzed. As background to the results obtained here, we cite briefly what has already been shown for linear stochatsic programs with $N = 1$, i.e., for the case where $\Xi = \Xi_1$ and $f(\xi, x)$ is a map from $\Xi \times R^n$ to $(-\infty, +\infty]$ which is measurable with respect to ξ for all x and polyhedral convex in x for all ξ. Kall [6] gave a constructive proof of the fact that $x(\xi)$ could be chosen *measurably* to approximate the infimum of $f(\xi, \cdot)$ for each ξ, so as to minimize the functional (1.2). With some further restrictions on the form of f and integrability with respect to ξ, it was shown that the recourse functions could be restricted to the class $\mathscr{L}^2(\Xi, \sigma)$ without affecting the infimum. Specializing f even further, it was shown in [13] that if the problem has a finite infimum, then in fact the minimum is attained by a *continuous* piecewise linear recourse function.

A measurable function $x: \Xi \to R^n$ will be called *essentially nonanticipative* if it can be made into a measurable nonanticipative function by altering its values on a set of measure zero.

Let $\mathcal{N}_\infty$ denote the space of all measurable, essentially nonanticipative functions $x: \Xi \to R^n$ which are essentially bounded, and let $\mathcal{N}_\mathscr{C}$ denote the subspace of $\mathcal{N}_\infty$ consisting of the continuous nonanticipative functions. Our chief aim in this paper is to give conditions on f ensuring that the functional F is welldefined on $\mathcal{N}_\infty$ and has the same infimum over $\mathcal{N}_\infty$ that it has over $\mathcal{N}_\mathscr{C}$. We do not treat possible relationships between $\mathcal{N}_\infty$ and more general spaces of measurable functions, and indeed for $N > 1$ these are not as elementary as one might suppose, because the direct ways of "truncating" a recourse function tend to disrupt either feasibility or the nonanticipative property.

The arguments we furnish about the relationship between $\mathcal{N}_\infty$ and $\mathcal{N}_\mathscr{C}$ depend on E. Michael's theorem [7] on selecting a continuous function from a lower-semicontinuous, *convex*-valued multifunction. Convexity is therefore fundamental, and we use it accordingly to simplify other assumptions as well. Our basic assumptions are the following

(A1) The support Ξ of σ is compact.

(A2) For each $\xi \in \Xi$, the function $f(\xi, \cdot): R^n \mapsto (-\infty, +\infty]$ is convex and lower-semicontinuous.

(A3) For each $\xi \in \Xi$, the convex set $D(\xi) = \{x \in R^n \mid f(\xi, x) < +\infty\}$ has a nonempty interior.

(A4) The multifunction $\xi \mapsto \operatorname{cl} D(\xi)$ is continuous from Ξ to R^n.

(A5) For each $x \in R^n$, the function $\xi \mapsto f(\xi, x)$ is measurable.

(A6) If $U \subset \Xi$ is open (relative to Ξ), $V \subset R^n$ is open, and f is finite throughout $U \times V$, then

$$\int_U |f(\xi, x)|\, d\sigma < +\infty, \qquad \text{for each } x \in V.$$

By the *continuity* of a multifunction $\Gamma: \Xi \mapsto R^n$, we mean of course that the graph set

$$G(\Gamma) = \{(\xi, x) \mid x \in \Gamma(\xi)\} \tag{1.3}$$

is closed, and Γ is *lower-semicontinuous*, i.e., the set $\{\xi \in \Xi \mid \Gamma(\xi) \cap V \neq \varnothing\}$ is open (relative to Ξ) for every open set $V \subset R^n$. We shall denote by $\mathscr{L}^\infty$

the space of all essentially bounded, measurable functions $x: \Xi \to R^n$ equipped with the usual seminorm

$$\| x \|_\infty = \text{ess sup}\{ | x(\xi) | \mid \xi \in \Xi \}, \tag{1.4}$$

where $| \cdot |$ is the euclidean norm in R^n; analogously $\mathscr{L}^1$. Let

$$B = \{x \in R^n \mid | x | \leqslant 1\}. \tag{1.5}$$

THEOREM 1. *Under assumptions* (A1)–(A6), *the functional F in* (1.2) *is well-defined from $\mathscr{L}^\infty$ to $(-\infty, +\infty]$, in the sense that for each $x \in \mathscr{L}^\infty$ the function $\xi \to f(\xi, x(\xi))$ is measurable and majorizes at least one summable function (the meaning of the integral then being unambiguously a real number or $+\infty$). In fact F is convex and lower-semicontinuous, not only relative to $\| \cdot \|_\infty$, but also relative to the weak topology induced on $\mathscr{L}^\infty$ under the natural pairing with $\mathscr{L}^1$. The set*

$$\mathscr{W} = \{x \in \mathscr{L}^\infty \mid \exists \epsilon > 0, x(\xi) + \epsilon B \subset D(\xi), \text{a.e.}\} \tag{1.6}$$

satisfies

$$\text{int}\{x \in \mathscr{L}^\infty \mid F(x) < +\infty\} = \mathscr{W} \neq \varnothing, \tag{1.7}$$

and on $\mathscr{W}$ the functional F is continuous.

Since in the weak topology induced on $\mathscr{L}^\infty$ by $\mathscr{L}^1$ every closed bounded set is compact, Theorem 1 immediately yields a result on the existence of optimal recourse functions.

COROLLARY. *If* (A1)–(A6) *hold and the union of the sets $D(\xi)$, $\xi \in \Xi$, is bounded in R^n, then F attains its minimum over $\mathscr{N}_\infty$.*

Theorem 1 establishes the setting, but for our main result a further restriction on the nature of the probability measure σ and its support Ξ is needed. For any $S \subset \Xi$ and index k, $1 \leqslant k < N$, let

$$S^k = \text{projection of } S \text{ in } \Xi_1 \times \cdots \times \Xi_k, \tag{1.8}$$

$$\Lambda_k^S(\xi_1, \ldots, \xi_k) = \{(\xi_{k+1}, \ldots, \xi_N) \mid (\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_N) \in S\}. \tag{1.9}$$

We shall say that σ is *laminary* if the following two conditions are satisfied.

(a) If S is a measurable subset of Ξ with $\sigma(\Xi \setminus S) = 0$, and if S^k is measurable, then

$$\text{cl}\, \Lambda_k^S(\xi_1, \ldots, \xi_k) = \Lambda_k^\Xi(\xi_1, \ldots, \xi_k) \qquad \text{for almost every } (\xi_1, \ldots, \xi_k) \in S^k; \tag{1.10}$$

(b) The multifunction Λ_k^{Ξ} is lower semicontinuous relative to Ξ^k. (The measure on $\Xi_1 \times \cdots \times \Xi_k$ in (1.10) is of course the "projection" of the probability σ.)

It is an elementary consequence of the theory of product measures (essentially, Fubini's Theorem) that σ is laminary if, for example,

$$\sigma(d\xi) = \rho(\xi_1, ..., \xi_N)\, \pi_1(d\xi_1) \cdots \pi_N(d\xi_N), \tag{1.11}$$

where the density function ρ is (measurable and) positive on the product of the supports of the (regular Borel) measures π_k on the spaces Ξ_k. (This product is then the set Ξ, so that each multifunction Λ_k^{Ξ} is actually constant-valued.) In particular, σ is laminary if the random variables ξ_k are independant. However, the definition also covers various cases where the conditional distribution of $(\xi_{k+1}, ..., \xi_N)$, given the value of $(\xi_1, ..., \xi_k)$, is well-defined but has its support dependent on $(\xi_1, ..., \xi_k)$. Note that, trivially, σ is also laminary if Ξ is an arbitrary finite set.

We shall denote by $\mathcal{N}_{\mathscr{D}}$ the set of all measurable functions $x: \Xi \to R^n$ such that x is (truly, not just essentially) nonanticipative and bounded and satisfies

$$x(\xi) \in \operatorname{cl} D(\xi) \qquad \text{for every } \xi \in \Xi. \tag{1.12}$$

THEOREM 2. *Suppose in addition to* (A1)–(A6) *that σ is laminary. Then each $x \in \mathcal{N}_\infty$ with $F(x) < +\infty$ agrees almost everywhere with some function in $\mathcal{N}_{\mathscr{D}}$. Thus in particular,*

$$\inf\{F(x) \mid x \in \mathcal{N}_\infty\} = \inf\{F(x) \mid x \in \mathcal{N}_{\mathscr{D}}\}, \tag{1.13}$$

where one infimum is attained if and only if the other is. If furthermore $\mathscr{W} \cap \mathcal{N}_\infty \neq \varnothing$, then $\mathscr{W} \cap \mathcal{N}_{\mathscr{C}} \neq \varnothing$ and

$$\inf\{F(x) \mid x \in \mathcal{N}_\infty\} = \inf\{F(x) \mid x \in \mathcal{N}_{\mathscr{C}}\}. \tag{1.14}$$

Without the assumption that σ is laminary, the first conclusion of Theorem 2 may not be valid, and strict inequality may hold in (1.13). Our proof shows however that one would still have $\mathscr{W} \cap \mathcal{N}_{\mathscr{C}} \neq \varnothing$ and

$$\inf\{F(x) \mid x \in \mathcal{N}_{\mathscr{D}}\} = \inf\{F(x) \mid x \in \mathcal{N}_{\mathscr{C}}\}, \tag{1.15}$$

assuming the existence of an $x \in \mathcal{N}_{\mathscr{D}}$ and an $\epsilon > 0$ such that $x(\xi) + \epsilon B \subset D(\xi)$ for every $\xi \in \Xi$. Without the continuity assumption (A4), equality can also fail in (1.15), even in the case of $N = 1$.

COUNTEREXAMPLE 1. In this example all the assumptions are satisfied except condition (a) of the definition of "laminary", but

$$-\infty < \min\{F(x) \mid x \in \mathcal{N}_\infty\} < \min\{F(x) \mid x \in \mathcal{N}_\mathscr{G}\} < +\infty. \qquad (1.16)$$

Let $N = 2$, $\Xi_1 = R^1 = \Xi_2$, $R^{n_1} = R^1 = R^{n_2}$. Define the probability measure σ as follows. Let the interval $[0, 1]$ be expressed as the union of two disjoint subsets A and B of positive measure with A dense and B closed, and set

$$\sigma(S) = \text{mes}(S \cap T)/\text{mes } T \qquad \text{for all Borel sets } S,$$

where $T = (A \times [0, 2]) \cup (B \times [0, 1])$. Then, $\Xi = \text{cl } T = [0, 1] \times [0, 2]$. Observe that σ is absolutely continuous with respect to a product measure (Lebesgue measure), but condition (a) in the definition of "laminary" fails for $S = T$. Define f on $\Xi \times R^2$ by

$$f(\xi_1, \xi_2, x_1, x_2) = \begin{cases} x_1 & \text{if} \quad 0 \leqslant x_2 \leqslant x_1 - \xi_2, \\ +\infty & \text{otherwise,} \end{cases}$$

so that

$$D(\xi_1, \xi_2) = \text{cl } D(\xi_1, \xi_2) = \{(x_1, x_2) \in R^2 \mid 0 \leqslant x_2 \leqslant x_1 - \xi_2\}.$$

Consider now the function $\bar{x} \in \mathcal{N}_\infty$ defined by

$$\bar{x}(\xi) = (\bar{x}_1(\xi_1), \bar{x}_2(\xi_1, \xi_2)) = \begin{cases} (2, 0) & \text{if} \quad \xi_1 \in A, \\ (1, 0) & \text{if} \quad \xi_1 \in B, \end{cases}$$

for which one has

$$F(\bar{x}) = [4 \text{ mes } A + \text{mes } B]/\text{mes } T = 2 - (\text{mes } B/\text{mes } T).$$

If x is any function in $\mathcal{N}_\infty$ with $F(x) < +\infty$, then

$$(x_1(\xi_1), x_2(\xi_1, \xi_2)) \in D(\xi_1, \xi_2) \qquad \text{for almost all } (\xi_1, \xi_2),$$

implying $0 \leqslant x_1(\xi_1) - \xi_2$ for almost all $(\xi_1, \xi_2) \in T$. Hence, $x_1(\xi_1) \geqslant 2$ for almost all $\xi_1 \in A$, while $x_1(\xi_1) \geqslant 1$ for almost all $\xi_1 \in B$. This shows that the function $\bar{x}$ actually gives the minimum of F over $\mathcal{N}_\infty$;

$$\min\{F(x) \mid x \in \mathcal{N}_\infty\} = 2 - (\text{mes } B/\text{mes } T) < 2. \qquad (1.17)$$

Consider now, on the other hand, the function $\bar{\bar{x}} \in \mathcal{N}_\mathscr{G}$ defined by

$$\bar{\bar{x}}(\xi) = (\bar{\bar{x}}_1(\xi_1), \bar{\bar{x}}_2(\xi_1, \xi_2)) \equiv (2, 0),$$

for which one has trivially $F(\bar{\bar{x}}) = 2$. If x is any function in $\mathcal{N}_{\mathscr{D}}$, then $0 \leqslant x_1(\xi_1) - \xi_2$ for all $(\xi_1, \xi_2) \in \Xi$, so that $x_1(\xi_1) \geqslant 2$ for all $\xi_1 \in [0, 1]$. Therefore $\bar{\bar{x}}$ gives the minimum of F over $\mathcal{N}_{\mathscr{D}}$:

$$\min\{F(x) \mid x \in \mathcal{N}_{\mathscr{D}}\} = 2. \tag{1.18}$$

In particular, it is impossible, by modifying the function $\bar{x}$ on a set of measure zero, to obtain a function in $\mathcal{N}_{\mathscr{D}}$. In other words, the constraint

$$0 \leqslant x_2(\xi_1, \xi_2) \leqslant x_1(\xi_1) - \xi_2 \qquad \text{for } \textit{almost every } (\xi_1, \xi_2) \in \Xi$$

is in this example distinctly less restrictive than the constraint

$$0 \leqslant x_2(\xi_1, \xi_2) \leqslant x_1(\xi_1) - \xi_2 \qquad \text{for } \textit{every } (\xi_1, \xi_2) \in \Xi.$$

COUNTEREXAMPLE 2. In this case all the assumptions are satisfied except condition (b) of the definition of "laminary", but again (1.13) is false. Let $N = 2$, $\Xi_1 = R^1 = \Xi_2$, $R^{n_1} = R^1 = R^{n_2}$. Let

$$\sigma(S) = \tfrac{1}{2}\operatorname{mes}(S \cap T) \qquad \text{for all Borel sets } S,$$

where

$$T = ([0, 1] \times [0, 1]) \cup ([-1, 0] \times [-1, 0]).$$

Then $\Xi = T$. Define f on $\Xi \times R^2$ by

$$f(\xi_1, \xi_2, x_1, x_2) = \begin{cases} 0 & \text{if} \quad \xi_2 \geqslant 0 \quad \text{and} \quad -2 + 3\xi_2 \leqslant x_1 \leqslant 2, \\ 0 & \text{if} \quad \xi_2 \leqslant 0 \quad \text{and} \quad -2 \leqslant x_1 \leqslant 2 + 3\xi_2, \\ +\infty & \text{otherwise.} \end{cases}$$

For the function $\bar{x} \in \mathcal{N}_\infty$ defined by

$$\bar{x}(\xi) = (\bar{x}_1(\xi_1), \bar{x}_2(\xi_1, \xi_2)) = \begin{cases} (\tfrac{3}{2}, 0) & \text{if} \quad \xi_1 > 0, \\ (0, 0) & \text{if} \quad \xi_1 = 0, \\ (-\tfrac{3}{2}, 0) & \text{if} \quad \xi_1 < 0, \end{cases}$$

we have $\bar{x}(\xi) + \frac{1}{2} B \subset D(\xi)$ almost everywhere, and hence in particular

$$\min\{F(x) \mid x \in \mathcal{N}_\infty\} = F(\bar{x}) = 0. \tag{1.19}$$

However, if x is any function in $\mathcal{N}_{\mathscr{D}}$ we have $x(0, 1) \in D(0, 1)$, implying $x_1(0) \geqslant 1$, while at the same time $x(0, -1) \in D(0, -1)$, implying $x_1(0) \leqslant -1$. These conditions on $x_1(0)$ are incompatible, so $\mathcal{N}_{\mathscr{D}}$ must in fact be empty;

$$\inf\{F(x) \mid x \in \mathcal{N}_{\mathscr{D}}\} = +\infty. \tag{1.20}$$

COUNTEREXAMPLE 3. This is an elementary example from the theory of continuous selections, showing that equality can fail in (1.15) if (A4) is omitted, even if all the other assumptions are satisfied and $N = 1$. Let $\Xi_1 = R^1$, $R^{n_1} = R^1$,

$$\sigma(S) = \tfrac{1}{2}\,\mathrm{mes}(S \cap [-1, 1]) \qquad \text{for all Borel sets } S,$$

so that $\Xi = [-1, 1]$. Define

$$f(\xi_1, x_1) = \begin{cases} 0 & \text{if} \quad \xi_1 > 0 \quad \text{and} \quad x_1 \in [1, 2], \\ 0 & \text{if} \quad \xi_1 = 0 \quad \text{and} \quad x_1 \in [-2, 2], \\ 0 & \text{if} \quad \xi_1 < 0 \quad \text{and} \quad x_1 \in [-2, -1], \\ +\infty & \text{otherwise.} \end{cases}$$

Consider the function $\bar{x} \in \mathcal{N}_{\mathscr{D}}$ defined by

$$\bar{x}(\xi) = \bar{x}_1(\xi_1) = \begin{cases} \tfrac{3}{2} & \text{if} \quad \xi_1 > 0, \\ 0 & \text{if} \quad \xi_1 = 0, \\ -\tfrac{3}{2} & \text{if} \quad \xi_1 < 0. \end{cases}$$

Then in fact,

$$\bar{x}(\xi) + \tfrac{1}{2} B \subset D(\xi) \qquad \text{for all } \xi \in \Xi,$$

and

$$\min\{F(x) \mid x \in \mathcal{N}_{\mathscr{D}}\} = F(\bar{x}) = 0. \tag{1.21}$$

But there does not exist any $x \in \mathcal{N}_{\mathscr{C}}$ with $F(x) < +\infty$, so that

$$\min\{F(x) \mid x \in \mathcal{N}_{\mathscr{C}}\} = +\infty. \tag{1.22}$$

Remark. An extension of Theorem 2 to recourse functions with values in a Banach space, rather than R^n, would be desirable, for example, in connection with models in optimal control, where the component $x_k(\xi_1, ..., \xi_k)$ could represent a function of time to be chosen over an interval $[t_k, t_{k+1}]$ in response to the observation of certain functions of time $\xi_1, ..., \xi_k$ over previous intervals $[t_0, t_1], ..., [t_{k-1}, t_k]$. Unfortunately, the proofs below make essential use of finite-dimensionality in many ways.

2. BACKGROUND RESULTS

Here we state and prove some facts, partly of interest in themselves, which will be needed in establishing Theorems 1 and 2.

Proposition 1 [11, p. 458]. *Let Ξ' be an arbitrary compact Hausdorff space. Let $\Gamma: \Xi' \to R^n$ be a multifunction such that for each ξ the set $\Gamma(\xi)$ is convex with* $\operatorname{int} \Gamma(\xi) \neq \varnothing$. *Then Γ is lower-semicontinuous if and only if the set* $\{(\xi, x) \mid x \in \operatorname{int} \Gamma(\xi)\}$ *is open.*

Proposition 2. *Let Ξ' be an arbitrary compact Hausdorff space. Let $\Gamma: \Xi' \mapsto R^n$ be a lower-semicontinuous multifunction such that for each ξ the set $\Gamma(\xi)$ is convex with* $\operatorname{int} \Gamma(\xi) \neq \varnothing$. *Then for all $\epsilon > 0$ sufficiently small the multifunction*

$$\Gamma^\epsilon: \xi \to \{x \in R^n \mid x + \epsilon B \subset \Gamma(\xi), \mid x \mid \leqslant 1/\epsilon\} \tag{2.1}$$

has the same properties. If Γ is actually continuous, so is Γ^ϵ.

Proof. Clearly $\Gamma^\epsilon(\xi)$ is convex. Let

$$\Gamma_*^\epsilon(\xi) = \{x \in R^n \mid x + \epsilon B \subset \operatorname{int} \Gamma(\xi), \mid x \mid < 1/\epsilon\}. \tag{2.2}$$

For each ξ, we can find $x \in R^n$ and $\epsilon > 0$ such that $x \in \Gamma_*^\epsilon(\xi)$. Then, in view of the lower-semicontinuity of Γ and Lemma 1, there is a neighborhood $U \times V$ of (ξ, x) such that $x' \in \Gamma_*^\epsilon(\xi')$ for all $\xi' \in U$ and $x' \in V$. Thus the sets

$$G_\epsilon = \{(\xi, x) \mid x \in \Gamma_*^\epsilon(\xi)\}$$

are open in $\Xi' \times R^n$, and their projections on Ξ' (also open, and increasing as $\epsilon \downarrow 0$) cover Ξ'. Since Ξ' is compact, one of the projections must be all of Ξ'. In other words, for some $\epsilon > 0$ sufficiently small, we have $\Gamma_*^\epsilon(\xi) \neq \varnothing$ for all $\xi \in \Xi'$. But $\Gamma_*^\epsilon(\xi) = \operatorname{int} \Gamma^\epsilon(\xi)$ if $\Gamma_*^\epsilon(\xi) \neq \varnothing$. Therefore, for ϵ sufficiently small, the set $\{(\xi, x) \mid x \in \operatorname{int} \Gamma^\epsilon(\xi)\}$ is the open set G_ϵ, implying via Proposition 1 that Γ^ϵ is lowersemicontinuous.

If Γ is continuous, the graph set $G(\Gamma)$ is closed. Then $G(\Gamma^\epsilon)$ is obviously closed as well so that Γ^ϵ also is continuous.

Proposition 3 (Continuous Selections). *Let Ξ' be an arbitrary compact Hausdorff space, and let $\Gamma: \Xi' \mapsto R^n$ be a lower-semicontinuous, convex-valued multifunction such that* $\operatorname{int} \Gamma(\xi) \neq \varnothing$ *for all $\xi \in \Xi'$. Let $\tilde{\Xi}$ be a closed subset of Ξ', and let $\gamma: \tilde{\Xi} \to R^n$ be a continuous function such that $\gamma(\xi) \in \operatorname{int} \Gamma(\xi)$ for all $\xi \in \Xi'$. Then γ can be extended to a continuous function on all of Ξ' such that, for some $\epsilon > 0$,*

$$\gamma(\xi) + \epsilon B \subset \Gamma(\xi) \qquad \textit{for all } \xi \in \Xi'. \tag{2.3}$$

Proof. Since $\tilde{\Xi}$ is compact and γ is continuous, the graph $\{(\xi, \gamma(\xi)) \mid \xi \in \tilde{\Xi}\}$ is a compact subset of $\{(\xi, x) \mid x \in \operatorname{int} \Gamma(\xi)\}$. Moreover the latter set is open by Proposition 1. There does exist, therefore, an $\epsilon > 0$ such that

$$\gamma(\xi) + 2\epsilon B \subset \Gamma(\xi) \qquad \text{for all } \xi \in \tilde{\Xi}.$$

Thus, applying Proposition 2, we obtain the existence of ϵ such that $\Gamma^{2\epsilon}$ is a lower-semicontinuous, nonempty-convex-valued multifunction such that $\gamma(\xi) \in \Gamma^{2\epsilon}(\xi)$ for $\xi \in \bar{\Xi}$. The multifunction $\xi \mapsto \text{cl}\, \Gamma^{2\epsilon}(\xi)$ then has these same properties and is compact-valued. Michael's theorem on continuous selections [7] assures the existence of a continuous extension of γ satisfying $\gamma(\xi) \in \text{cl}\, \Gamma^{2\epsilon}(\xi)$ for all ξ. Since

$$\text{cl}\, \Gamma^{2\epsilon}(\xi) \subset \Gamma^{\epsilon}(\xi) \qquad \text{for all } \xi,$$

this γ has the property (2.3) that we wanted.

PROPOSITION 4 (Summability Property). *Let $\bar{D}: \xi \mapsto \text{cl}\, D(\xi)$, and define the multifunction $\bar{D}^{\epsilon}$ as in* (2.1), *i.e.,*

$$\bar{D}^{\epsilon}(\xi) = \{x \in R^n \mid x + \epsilon B \subset \text{cl}\, D(\xi), \mid x \mid \leqslant 1/\epsilon\}.$$

Under assumptions (A1)–(A6), *for arbitrarily small $\epsilon > 0$, there is a summable function $\alpha: \Xi \mapsto R^1$ such that*

$$f(\xi, x) \leqslant \alpha(\xi) \qquad \textit{whenever } x \in \bar{D}^{\epsilon}(\xi). \tag{2.4}$$

Proof. The hypothesis of Proposition 2 is fulfilled with $\Gamma = \bar{D}$ continuous. Thus for ϵ sufficiently small, the multifunction $\bar{D}^{\epsilon}$ is continuous and compact-convex-valued, with int $\bar{D}^{\epsilon}(\xi) \neq \varnothing$. Fix any such ϵ. Let $\bar{\xi} \in \Xi$ be arbitrary. We shall demonstrate the existence of an open neighborhood U of $\bar{\xi}$ relative to Ξ and a summable function $\alpha_U: U \mapsto R^1$ such that

$$f(\xi, x) \leqslant \alpha_U(\xi) \qquad \text{whenever } x \in \bar{D}^{\epsilon}(\xi) \text{ and } \xi \in U. \tag{2.5}$$

Since Ξ is compact, it can be covered by a finite collection of such open sets U with associated functions α_U, so the conclusion of the proposition will be immediate.

Obviously,

$$\bar{D}^{\epsilon}(\xi) \subset \text{int}\, \bar{D}(\xi) = \text{int}\, D(\xi) \qquad \text{for every } \xi \in \Xi. \tag{2.6}$$

In light of this inclusion, there exist points $a_i \in \text{int}\, D(\bar{\xi})$ such that the polytope $P = \text{co}\{a_1, ..., a_m\}$ has int $P \supset \bar{D}^{\epsilon}(\bar{\xi})$ [9, Theor. 20.4]. The continuity of $\bar{D}^{\epsilon}$ then yields the existence of an open neighborhood U_0 of $\bar{\xi}$ such that int $P \supset \bar{D}^{\epsilon}(\xi)$ for all $\xi \in U_0$. On the other hand, there exist by Proposition 1 open neighborhoods U_i of $\bar{\xi}$ and V_i of a_i, $i = 1, ..., m$, such that

$$V_i \subset \text{int}\, \bar{D}(\xi) = \text{int}\, D(\xi) \qquad \text{for all } \xi \in U_i. \tag{2.7}$$

Let $U = U_0 \cap U_1 \cap \cdots \cap U_m$. If $\xi \in U$, we have $\bar{D}^\epsilon(\xi) \subset P$, implying by the convexity of $f(\xi, \cdot)$, that (2.5) holds for

$$\alpha_U(\xi) = \max_{i=1,\ldots,m} f(\xi, a_i).$$

Moreover, $f(\xi, a_i)$ is summable in $\xi \in U_i$ by (A6) and (2.7), since $a_i \in V_i$. Hence, α_U is summable in $\xi \in U$ as desired.

PROPOSITION 5. *If σ is laminary (with compact support), then so is the "projection" σ^k of σ on $\Xi_1 \times \cdots \times \Xi_k$ for $1 \leqslant k < N$; the (regular Borel) probability measure σ^k is defined by*

$$\sigma^k(T) = \sigma(S), \qquad \textit{where} \qquad S = [T \times \Xi_{k+1} \times \cdots \times \Xi_N] \cap \Xi. \tag{2.8}$$

Proof. The support of σ^k is the (compact) projection Ξ^k of Ξ. If T is any (Borel) measurable set in $\Xi_1 \times \cdots \times \Xi_k$, then T is the projection S^k of the measurable set S defined in (2.8). Moreover, for $1 \leqslant l < k$, the set

$$A_l^T(\xi_1, \ldots, \xi_l) = \{(\xi_{l+1}, \ldots, \xi_k) \mid (\xi_1, \ldots, \xi_l, \xi_{l+1}, \ldots, \xi_k) \in T\}$$

is the projection on $\Xi_{l+1} \times \cdots \times \Xi_k$ of the set

$$A_l^S(\xi_1, \ldots, \xi_l) = \{(\xi_{l+1}, \ldots, \xi_N) \mid (\xi_1, \ldots, \xi_l, \xi_{l+1}, \ldots, \xi_N) \in S\}.$$

Taking $T = \Xi^k$ (hence $S = \Xi$), we see that σ^k has property (b) of the definition of "laminary," because the lower semicontinuity of a multifunction is trivially preserved when the multifunction is composed with a continuous function (in this case a projection mapping). More generally, if T is any measurable set in Ξ^k with $\sigma^k(T \backslash \Xi^k) = 0$, then the set S in (2.8) has $(\Xi \backslash S) = 0$, and $T^l = S^l$ for $1 \leqslant l < k$. Since σ is laminary, we know therefore that

$$\operatorname{cl} A_l^S(\xi_1, \ldots, \xi_l) = A_l^\Xi(\xi_1, \ldots, \xi_l) \qquad \text{for almost every } (\xi_1, \ldots, \xi_l) \in T^l.$$

Projecting the sets onto $\Xi_{l+1} \times \cdots \times \Xi_k$, we obtain the relation

$$\operatorname{cl} A_l^T(\xi_1, \ldots, \xi_l) = A_l^{\Xi^k}(\xi_1, \ldots, \xi_l) \qquad \text{for almost every } (\xi_1, \ldots, \xi_l) \in T^l.$$

Thus σ^k also has property (a) of the definition of "laminary."

PROPOSITION 6. *Assume that σ is laminary, and let S' be any measurable subset of Ξ such that $\sigma(\Xi \backslash S') = 0$. Then there exists a measurable subset S of S' such that $\sigma(\Xi \backslash S) = 0$ and for every k, $0 \leqslant k < N$, the projection S^k of S on $\Xi_1 \times \cdots \times \Xi_k$ is measurable and satisfies*

$$\operatorname{cl} A_k^S(\xi_1, \ldots, \xi_k) = A_k^\Xi(\xi_1, \ldots, \xi_k) \qquad \textit{for every } (\xi_1, \ldots, \xi_k) \in S^k. \tag{2.9}$$

Proof. We shall demonstrate the assertion in the slightly stronger form where S is actually sigma-compact (the union of a countable family of compact sets). This assertion is true trivially if $N = 1$, since the regularity of σ implies the existence for every integer $m > 0$ of a compact set $K_m \subset S'$ with

$$\sigma(K_m) > \sigma(S') - (1/m).$$

Let us suppose now that $N > 1$ and make the induction hypothesis that the assertion is true for all cases of $N - 1$ components. Regarding $\Xi_1 \times \cdots \times \Xi_N$ as the product of the $N - 1$ spaces $(\Xi_1 \times \Xi_2), \Xi_3, ..., \Xi_N$, we deduce the existence of a sigma-compact S in S' such that the desired properties hold for every k with $2 \leqslant k < N$. The projection S^1 of S on Ξ^1 is then measurable, because the projection of a sigma-compact set is sigma-compact. Since σ is laminary, we have

$$\operatorname{cl} \Lambda_1{}^S(\xi_1) = \Lambda_1{}^{\Xi}(\xi_1) \qquad \text{for almost every } \xi_1 \in S^1. \tag{2.10}$$

Therefore, the equality in (2.10) holds for all $\xi_1 \in T$, where T is some sigma-compact subset of S^1 with $\sigma^1(S^1 \setminus T) = 0$. Replacing S by its (sigma-compact) intersection with $T \times \Xi_2 \times \cdots \times \Xi_N$, we obtain a new S such that the desired properties hold for $1 \leqslant k < N$.

PROPOSITION 7. *Let $\bar{x}: \Xi \to R^n$ be an essentially nonanticipative, measurable function satisfying*

$$\bar{x}(\xi) \in \Delta(\xi) \qquad \textit{for almost every } \xi \in \Xi, \tag{2.11}$$

where $\Delta: \Xi \to R^n$ is a multifunction whose graph is closed and whose values $\Delta(\xi)$ are all contained within some fixed ball. Assume that σ is laminary. Then there is a nonanticipative, measurable function $\bar{\bar{x}}: \Xi \to R^n$ which agrees almost everywhere with $\bar{x}$ and satisfies

$$\bar{\bar{x}}(\xi) \in \Delta(\xi) \qquad \textit{for every } \xi \in \Xi. \tag{2.12}$$

Proof. Let S' be the subset of Ξ for which the relation in (2.11) holds, and let S be a subset of S' with the properties guaranteed by Proposition 6. In particular, then, S is dense in Ξ and

$$\bar{x}(\xi) \in \Delta(\xi) \qquad \text{for every } \xi \in S. \tag{2.13}$$

It follows from our assumptions on Δ that $\Delta(\xi) \neq \varnothing$ for all $\xi \in \Xi$.

Define $\Delta_k: \Xi^k \to R^{n_1} \times \cdots \times R^{n_k}$ recursively by $\Delta_N = \Delta$ and

$$\begin{aligned} \Delta_{k-1}(\xi_1, ..., \xi_{k-1}) = \{(x_1, ..., x_{k-1}) \mid \forall \xi_k \text{ with } (\xi_1, ..., \xi_{k-1}, \xi_k) \in \Xi^k, \\ \exists x_k \text{ with } (x_1, ..., x_{k-1}, x_k) \in \Delta_k(\xi_1, ..., \xi_{k-1}, \xi_k)\}. \end{aligned} \tag{2.14}$$

We claim that for every k:

(i) Δ_k is a multifunction whose graph set is closed and whose values are all contained in some fixed ball; and

(ii) $(\bar{x}_1(\xi_1),..., \bar{x}_k(\xi_1 ,..., \xi_k)) \in \Delta_k(\xi_1 ,..., \xi_k)$ for every $(\xi_1 ,..., \xi_k) \in S^k$. These properties hold, as we have seen, for $k = N$. We suppose next that they hold for a given k and prove that they then hold also for $k - 1$.

The boundedness property in (i) obviously carries over to Δ_{k-1}. For the closedness property, let

$$\{(\xi_1{}^i,..., \xi_{k-1}^i , x_1{}^i,..., x_{k-1}^i)\}_{i\in I} \tag{2.15}$$

be a generalized sequence in the graph of Δ^{k-1} converging to an element

$$(\bar{\xi}_1 ,..., \bar{\xi}_{k-1} , \tilde{x}_1 ,..., \tilde{x}_{k-1}). \tag{2.16}$$

Let $\bar{\xi}_k$ be such that $(\bar{\xi}_1 ,..., \bar{\xi}_{k-1} , \bar{\xi}_k) \in \Xi^k$. The multifunction

$$(\xi_1 ,..., \xi_{k-1}) \to \{\xi_k \mid (\xi_1 ,..., \xi_{k-1} , \xi_k) \in \Xi^k\} \tag{2.17}$$

is lower semicontinuous on Ξ^k by Proposition 5 (the laminary property (b) of σ^k). Hence there exist elements $\xi_k{}^i$ converging to $\bar{\xi}_k$ such that

$$(\xi_1{}^i,..., \xi_{k-1}^i , \xi_k{}^i) \in \Xi^k \qquad \text{for all } i \in I. \tag{2.18}$$

Since the elements (2.15) lie in the graph of Δ^{k-1}, there exist elements $x_k{}^i$ such that

$$(x_1{}^i,..., x_{k-1}^i , x_k{}^i) \in \Delta_k (\xi_1{}^i,..., \xi_{k-1}^i , \xi_k{}^i) \qquad \text{for all } i \in I. \tag{2.19}$$

Passing to a (generalized) subsequence if necessary, we can suppose (from the boundedness property of Δ_k) that $x_k{}^i$ converges to a certain $\tilde{x}_k$. Then,

$$(\tilde{x}_1 ,..., \tilde{x}_{k-1} , \tilde{x}_k) \in \Delta_k(\bar{\xi}_1 ,..., \bar{\xi}_{k-1} , \bar{\xi}_k) \tag{2.20}$$

by (2.19) and the closedness of the graph of Δ_k. We have thus shown that for every $\bar{\xi}_k$ with $(\bar{\xi}_1 ,..., \bar{\xi}_{k-1} , \bar{\xi}_k) \in \Xi^k$ there is an element $\tilde{x}_k$ such that (2.20) holds. This means that the element (2.16) belongs to the graph of Δ^{k-1}, and the closedness of this graph is thereby established.

For property (ii) of Δ^{k-1}, we make use of the fact that

$$\begin{aligned} &\operatorname{cl}\{\xi_k \mid (\xi_1 ,..., \xi_{k-1} , \xi_k) \in S^k\} \\ &\quad = \{\xi_k \mid (\xi_1 ,..., \xi_{k-1} , \xi_k) \in \Xi^k\} \qquad \text{for every } (\xi_1 ,..., \xi_{k-1}) \in S^{k-1}. \end{aligned} \tag{2.21}$$

This is true since S was chosen to have the properties in Proposition 6, and hence satisfies, in particular,

$$\begin{aligned} &\mathrm{cl}\{(\xi_k, \ldots, \xi_N) \mid (\xi_1, \ldots, \xi_{k-1}, \xi_k, \ldots, \xi_N) \in S\} \\ &\quad = \{(\xi_k, \ldots, \xi_N) \mid (\xi_1, \ldots, \xi_{k-1}, \xi_k, \ldots, \xi_N) \in \Xi\} \\ &\quad\quad \text{for every } (\xi_1, \ldots, \xi_{k-1}) \in S^{k-1}. \end{aligned} \tag{2.22}$$

Let $(\xi_1, \ldots, \xi_{k-1}) \in S^{k-1}$, and let ξ_k be such that $(\xi_1, \ldots, \xi_{k-1}, \xi_k) \in \Xi^k$. To establish (ii) for Δ_{k-1}, we want to show the existence of x_k such that

$$(\bar{x}_1(\xi_1), \ldots, \bar{x}_{k-1}(\xi_1, \ldots, \xi_{k-1}), x_k) \in \Delta_k(\xi_1, \ldots, \xi_{k-1}, \xi_k). \tag{2.23}$$

In view of (2.21), it is possible to choose a generalized sequence of the form $\{(\xi_1, \ldots, \xi_{k-1}, \xi_k^i)_{i\in I}$ in S^k converging to $(\xi_1, \ldots, \xi_{k-1}, \xi_k)$. From property (ii) of Δ_k, we have

$$(\bar{x}_1(\xi_1), \ldots, \bar{x}_{k-1}(\xi_1, \ldots, \xi_{k-1}), \bar{x}_k(\xi_1, \ldots, \xi_{k-1}, \xi_k^i) \in \Delta_k(\xi_1, \ldots, \xi_{k-1}, \xi_k^i). \tag{2.24}$$

The boundedness property of Δ_k in (i) allows us to suppose, passing to a (generalized) subsequence if necessary, that $\bar{x}_k(\xi_1, \ldots, \xi_{k-1}, \xi_k^i)$ converges in $i \in I$ to some x_k. Then (2.23) holds by the closedness property of Δ_k in (i).

The proof that (i) and (ii) hold for all k is thereby finished. Observe that these properties imply

$$\Delta_k(\xi_1, \ldots, \xi_k) \neq \varnothing \qquad \text{for all } (\xi_1, \ldots, \xi_k) \in \Xi^k, \tag{2.25}$$

since S^k (being the projection of the set S of full measure in Ξ) is dense in Ξ^k. The closedness of the graph of Δ_k also implies that for every compact set K in $R^{n_1} \times \cdots \times R^{n_k}$ the set

$$\{(\xi_1, \ldots, \xi_k) \in \Xi^k \mid \Delta_k(\xi_1, \ldots, \xi_k) \cap K \neq \varnothing\}$$

is closed, hence measurable. Thus Δ_k is a measurable multifunction in the sense of Castaing (see [8, p. 5]).

We are now ready for the construction of the function $\bar{\bar{x}}$ in the proposition, which will be effected component by component, starting with $k = 1$. This construction is based on the fundamental selection theorem for measurable multifunctions (due to Rokhlin and others, see [8, Corollary 1.1]). *Every nonempty-closed-valued measurable multifunction with values (for example) in a euclidean space has a measurable selection.*

For $k = 1$, we consider the multifunction $\Delta_1' : \Xi^1 \to R^{n_1}$ defined by

$$\Delta_1'(\xi_1) = \begin{cases} \{\bar{x}_1(\xi_1)\} & \text{if} \quad \xi_1 \in S^1, \\ \Delta_1(\xi_1) & \text{if} \quad \xi_1 \in \Xi^1 \setminus S^1. \end{cases} \tag{2.27}$$

Since S^1 is a measurable set and $\bar{x}_1$ is a measurable function, Δ_1' is like Δ_1 measurable, as well as nonempty-closed-valued. Hence by the selection theorem, there exists a measurable function $\bar{\bar{x}}_1$ satisfying

$$\bar{\bar{x}}_1(\xi_1) \in \Delta_1'(\xi_1) \qquad \text{for all } \xi_1 \in \Xi^1.$$

In other words (using property (ii) of Δ_1),

$$\bar{\bar{x}}_1(\xi_1) \in \Delta_1(\xi_1) \qquad \text{for all } \xi_1 \in \Xi^1,$$
$$\bar{\bar{x}}_1(\xi_1) = \bar{x}_1(\xi_1) \qquad \text{for all } \xi_1 \in S^1.$$

Assume now inductively that we have constructed $\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_{k-1}$ on $\Xi^1, \ldots, \Xi^{k-1}$ with

$$(\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1})) \in \Delta_{k-1}(\xi_1, \ldots, \xi_{k-1}) \qquad \text{for all } (\xi_1, \ldots, \xi_{k-1}) \in \Xi_k^{-1}, \tag{2.28}$$

$$(\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1})) = (\bar{x}_1(\xi_1), \ldots, x_{k-1}(\xi_1, \ldots, \xi_{k-1})) \qquad \text{for all } (\xi_1, \ldots, \xi_{k-1}) \in S^{k-1}. \tag{2.29}$$

Define

$$\tilde{\Delta}_k(\xi_1, \ldots, \xi_k) = \{x_k \mid (\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1}), x_k) \in \Delta_k(\xi_1, \ldots, \xi_k)\}. \tag{2.30}$$

From (2.28) and the definition (2.14) of Δ_{k-1}, we know that

$$\tilde{\Delta}_k(\xi_1, \ldots, \xi_k) \neq \varnothing \qquad \text{for all } (\xi_1, \ldots, \xi_k) \in \Xi^k, \tag{2.31}$$

while (2.29) and property (ii) of Δ_k imply

$$\bar{x}_k(\xi_1, \ldots, \xi_k) \in \tilde{\Delta}_k(\xi_1, \ldots, \xi_k) \qquad \text{for all } (\xi_1, \ldots, \xi_k) \in S^k. \tag{2.32}$$

It is clear from property (i) of Δ_k that $\tilde{\Delta}_k$ is closed-valued. We claim further that $\tilde{\Delta}_k$ is measurable. To see this, consider the multifunction T_k defined on Ξ^k by

$$T_k(\xi_1, \ldots, \xi_k) = \{(x_1, \ldots, x_k) \mid x_l = \bar{\bar{x}}_l(\xi_1, \ldots, \xi_l) \text{ for } l = 1, \ldots, k-1\}.$$

This is measurable, because the functions $\bar{\bar{x}}_l$ are measurable and for every compact set $K \subset R^{n_1} \times \cdots \times R^{n_k}$ one has

$$\{(\xi_1, \ldots, \xi_k) \in \Xi^k \mid T_k(\xi_1, \ldots, \xi_k) \cap K \neq \varnothing\}$$
$$= \{(\xi_1, \ldots, \xi_k) \in \Xi^k \mid (\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1})) \in K'\},$$

where K' is the (closed) projection of K on $R^{n_1} \times \cdots \times R^{n_{k-1}}$. Since T_k and

Δ_k are both closed-valued and measurable, so is the intersection multifunction

$$T_k \cap \Delta_k : (\xi_1, \dots, \xi_k) \to T_k(\xi_1, \dots, \xi_k) \cap \Delta_k(\xi_1, \dots, \xi_k)$$

(see [8, Corollary 1.3]). For any closed set C in R^{n_k}, we have

$$\begin{aligned} &\{(\xi_1, \dots, \xi_k) \in \Xi^k \mid \tilde{\Delta}_k(\xi_1, \dots, \xi_k) \cap C \neq \varnothing\} \\ &\quad = \{(\xi_1, \dots, \xi_k) \in \Xi^k \mid [(T_k \cap \Delta_k)(\xi_1, \dots, \xi_k)] \cap P_k^{-1}(C) \neq \varnothing\}, \end{aligned} \tag{2.33}$$

where P_k is the projection from $R^{n_1} \times \cdots \times R^{n_k}$ onto R^{n_k}. Since $T_k \cap \Delta_k$ is measurable and $P_k^{-1}(C)$ is closed, the second set in (2.33) is measurable; thus $\tilde{\Delta}_k$ is a measurable multifunction as claimed.

Define the multifunction $\Delta_k' : \Xi^k \to R^{n_k}$ by

$$\Delta_k'(\xi_1, \dots, \xi_k) = \begin{cases} \{\bar{x}_k(\xi_1, \dots, \xi_k)\} & \text{if} \quad (\xi_1, \dots, \xi_k) \in S^k, \\ \tilde{\Delta}_k(\xi_1, \dots, \xi_k) & \text{if} \quad (\xi_1, \dots, \xi_k) \in \Xi^k \setminus S^k. \end{cases} \tag{2.34}$$

Then Δ_k' is again closed-valued and measurable, and from (2.31) and (2.32) we have

$$\varnothing \neq \Delta_k'(\xi_1, \dots, \xi_k) \subset \tilde{\Delta}_k(\xi_1, \dots, \xi_k) \qquad \text{for all } (\xi_1, \dots, \xi_k) \in \Xi^k. \tag{2.35}$$

The selection theorem is therefore applicable to Δ_k', and there exists a measurable function $\bar{\bar{x}}_k : \Xi^k \to R^{n_k}$ satisfying

$$\bar{\bar{x}}_k(\xi_1, \dots, \xi_k) \in \Delta_k'(\xi_1, \dots, \xi_k) \qquad \text{for all } (\xi_1, \dots, \xi_k) \in \Xi^k.$$

This relation means, according to the definitions (2.34) and (2.30) of Δ_k' and $\tilde{\Delta}_k$, that

$$(\bar{\bar{x}}_1(\xi_1), \dots, \bar{\bar{x}}_k(\xi_1, \dots, \xi_k)) \in \Delta_k(\xi_1, \dots, \xi_k) \qquad \text{for all } (\xi_1, \dots, \xi_k) \in \Xi^k, \tag{2.36}$$

$$(\bar{\bar{x}}_1(\xi_1), \dots, \bar{\bar{x}}_k(\xi_1, \dots, \xi_k)) = (\bar{x}_1(\xi_1), \dots, \bar{x}_k(\xi_1, \dots, \xi_k)) \qquad \text{for all } (\xi_1, \dots, \xi_k) \in \Xi^k. \tag{2.37}$$

Thus we have extended the relations (2.28), (2.29) from $k - 1$ to k.

The construction proceeds until we reach $k = N$ ($\Xi^N = \Xi$, $S^N = S$, $\Delta_N = \Delta$). At that point, the conclusion of Proposition 7 is fulfilled.

3. Deduction of the Main Results

Proof of Theorem 1. The remaining argument consists of putting together some known facts from the theory of convex integral functionals. Assumptions (A2), (A3) and (A5) imply by [10, Lemma 2] that f is a normal convex integrand, that is, in addition to (A2) there is a sequence of measurable functions $x^j: \Xi \to R^n$ such that $f(\xi, x^j(\xi))$ is measurable in ξ for each j, and the set $D(\xi) \cap \{x^j(\xi) \mid j = 1, 2,...\}$ is dense in $D(\xi)$ for each ξ. With (A1), (A4) and (A5), we then have from [11, Theor. 5] that $\mathscr{W}$ contains at least one continuous function x, and for every such x the function $\xi \mapsto f(\xi, x(\xi))$ is summable over Ξ. Let $\bar{x}$ denote a particular continuous function x, and for every such x denote a particular continuous function in $\mathscr{W}$. Then there is an $\epsilon > 0$ such that $f(\xi, \bar{x}(\xi) + \epsilon y)$ is summable in ξ for each $y \in B$. The hypothesis of [11, Theor. 2] is satisfied accordingly; this yields the conclusion that F is a well-defined, lower-semicontinuous, convex functional in the sense described. (The weak lower-semicontinuity follows because F is (by [11, Theor. 1]) the conjugate of a convex functional on $\mathscr{L}^1$.)

We still must verify (1.7). We have already noted that $\mathscr{W} \neq \varnothing$. It is evident that $\mathscr{W}$ is open and

$$\operatorname{int}\{x \in \mathscr{L}^\infty \mid F(x) < +\infty\} \subset \mathscr{W}.$$

Thus to complete the proof it suffices to show that $F(x) < +\infty$ for every $x \in \mathscr{W}$. But this fact is a corollary of Proposition 4.

Proof of Theorem 2. The first part of the theorem is immediate from Proposition 7 applied to $\Delta(\xi) = \operatorname{cl} D(\xi) = \bar{D}(\xi)$. For the rest, we observe at the outset that since $\mathscr{N}_\infty \cap \mathscr{W} \neq \varnothing$, one has

$$\inf\{F(x) \mid x \in \mathscr{N}_\infty\} = \inf\{F(x) \mid x \in \mathscr{N}_\infty \cap \mathscr{W}\}. \tag{3.1}$$

Indeed, if x is any element of $\mathscr{N}_\infty$ such that $F(x) < +\infty$, and if $\bar{x} \in \mathscr{N}_\infty \cap \mathscr{W}$, then for $0 < \lambda \leqslant 1$, we have $(1 - \lambda)\, x + \lambda\bar{x}$ in $\mathscr{N}_\infty \cap \mathscr{W}$ by (1.7) and

$$F((1 - \lambda)\, x + \lambda\bar{x}) \leqslant (1 - \lambda) F(x) + \lambda F(\bar{x}).$$

Thus, considering what happens as $\lambda \downarrow 0$, we see that for every $\epsilon > 0$ there is an $x' \in \mathscr{N}_\infty \cap \mathscr{W}$ with $F(x') < F(x) + \epsilon$, and this establishes (3.1).

Now fix any $\bar{x} \in \mathscr{N}_\infty \cap \mathscr{W}$ and $\delta > 0$. We shall demonstrate the existence of $\tilde{x} \in \mathscr{N}_{\mathscr{C}}$ such that

$$F(\tilde{x}) < F(\bar{x}) + 2\delta, \tag{3.2}$$

and this will prove the theorem.

Choose $\epsilon > 0$ small enough that the properties in Propositions 2 and 4 hold for $\Gamma = \bar{D}$, and in addition $\bar{x}(\xi) + \epsilon B \subset \bar{D}^\epsilon(\xi)$ for almost every $\xi \in \Xi$. Altering $\bar{x}$ on a set of measure zero if necessary, we can suppose that actually

$$\bar{x}(\xi) + \epsilon B \subset \bar{D}^\epsilon(\xi) \qquad \text{for every } \xi \in \Xi. \tag{3.3}$$

This follows from the continuity of D^ϵ by applying Proposition 7 to the multifunction

$$\Delta(\xi) = \{x \in R^N \mid x + \epsilon B \subset D^\epsilon(\xi)\} \qquad \text{for } \xi \in \Xi.$$

For the summable function $\alpha(\xi)$ in Proposition 4 and the function $f(\xi, \bar{x}(\xi))$ (which is summable in ξ by Theorem 1), there exists $\eta > 0$ such that whenever $S \subset \Xi$ is a (Borel) measurable set with measure $\sigma(S) < \eta$, one has

$$\int_S \alpha(\xi)\, d\sigma < \delta \qquad \text{and} \qquad \int_S f(\xi, \bar{x}(\xi))\, d\sigma > -\delta. \tag{3.4}$$

According to Lusin's Theorem, there exists for the measurable function $\bar{x}$ compact set $\tilde{\Xi} \subset \Xi$ such that $\bar{x}$ is continuous relative to $\tilde{\Xi}$ and $\sigma(S) < \eta$, where $S = \Xi \setminus \tilde{\Xi}$. Then,

$$\bar{x}(\xi) \in \operatorname{int} \bar{D}^\epsilon(\xi) \qquad \text{for every } \xi \in \tilde{\Xi} \tag{3.5}$$

by (3.3), because $\bar{D}^\epsilon$ is continuous.

Suppose we can construct $\tilde{x} \in \mathcal{N}_{\mathscr{C}}$ satisfying

$$\tilde{x}(\xi) \in D^\epsilon(\xi) \quad \text{for all } \xi \in \Xi, \qquad \text{with} \qquad x(\xi) = \bar{x}(\xi) \quad \text{for } \xi \in \tilde{\Xi}. \tag{3.6}$$

This will yield, via Proposition 4,

$$\begin{aligned} F(\tilde{x}) &= \int_{\tilde{\Xi}} f(\xi, x(\xi))\, d\sigma + \int_S f(\xi, \tilde{x}(\xi))\, d\sigma \\ &\leqslant F(\bar{x}) - \int_S f(\xi, \bar{x}(\xi))\, d\sigma + \int_S \alpha(\xi)\, d\sigma \\ &\leqslant F(\bar{x}) + \delta + \delta, \end{aligned}$$

as desired in (3.2). The rest of our proof is devoted to this construction, which amounts to a complicated application of Michael's theorem on continuous selections in the form of Proposition 3. The latter result could be invoked directly to get the existence of a continuous function x satisfying (3.6), but this x might fail to be nonanticipative. This is where the complications arise; their resolution requires a multistage approach based on lemmas stated below.

There is no loss of generality if we suppose each of the spaces Ξ_i is compact. (Replace Ξ_i by the projection of Ξ on Ξ_i.) Define the multifunction $\Gamma_N: \Xi_1 \times \cdots \times \Xi_N \mapsto R^n$ by

$$\Gamma_N(\xi) = \begin{cases} \bar{D}^\epsilon(\xi) & \text{if} \quad \xi \in \Xi, \\ (1/\epsilon)\, B & \text{if} \quad \xi \notin \Xi, \end{cases} \tag{3.7}$$

and then define

$$\Gamma_k: \Xi_1 \times \cdots \times \Xi_k \to R^{n_1} \times \cdots \times R^{n_k}$$

recursively for $k = N - 1, \ldots, 1$, by

$$\begin{aligned} \Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1}) \\ = \{(x_1, \ldots, x_{k-1}) \mid \forall \xi_k \in \Xi_k, \exists x_k \in R^{n_k}, \\ (x_1, \ldots, x_{k-1}, x_k) \in \Gamma_k(\xi_1, \ldots, \xi_{k-1}, \xi_k)\}. \end{aligned} \tag{3.8}$$

LEMMA 1. *There exists a measurable, nonanticipative function*

$$\bar{\bar{x}}: \Xi_1 \times \cdots \times \Xi_N \mapsto R^n = R^{n_1} \times \cdots \times R^{n_N}$$

coinciding with $\bar{x}$ *on* $\hat{\Xi}$, *such that for* $k = 1, \ldots, N$ *one has*

$$(\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_k(\xi_1, \ldots, \xi_k)) + \epsilon B \subset \Gamma_k(\xi_1, \ldots, \xi_k) \qquad \textit{for all } (\xi_1, \ldots, \xi_k). \tag{3.9}$$

(Here B denotes the closed unit euclidean ball of the appropriate dimension.)

Proof. Define $\bar{\bar{x}}_k: \Xi_1 \times \cdots \times \Xi_k \to R^{n_k}$ by

$$\bar{\bar{x}}_k(\xi_1, \ldots, \xi_k) = \begin{cases} \bar{x}_k(\xi_1, \ldots, \xi_k) & \text{if} \quad (\xi_1, \ldots, \xi_k) \in \Xi^k, \\ 0 & \text{if} \quad (\xi_1, \ldots, \xi_k) \notin \Xi^k. \end{cases} \tag{3.10}$$

The function $\bar{\bar{x}}$ given by

$$\bar{\bar{x}}(\xi) = (\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_N(\xi_1, \ldots, \xi_N))$$

is then measurable and nonanticipative on $\Xi_1 \times \cdots \times \Xi_N$, and

$$\bar{\bar{x}}(\xi) = \bar{x}(\xi) \qquad \text{for all } \xi \in \Xi. \tag{3.11}$$

This implies by (3.3) of course that

$$\bar{\bar{x}}(\xi) + \epsilon B \subset D^\epsilon(\xi) \qquad \text{for every } \xi \in \Xi. \tag{3.12}$$

On the other hand, since $D^\epsilon(\xi) \subset (1/\epsilon) B$ by definition, (3.3) implies also that $|\bar{x}(\xi)| \leqslant (1/\epsilon) - \epsilon$ for all $\xi \in \Xi$. But $\bar{\bar{x}}(\xi)$ is for each ξ a vector of the form

$$(x_1(\xi_1), \ldots, \bar{x}_k(\xi_1, \ldots, \xi_k), 0, \ldots, 0),$$

for some k. Thus $|\bar{\bar{x}}(\xi)| \leqslant (1/\epsilon) - \epsilon$ for all ξ, or equivalently,

$$\bar{\bar{x}}(\xi) + \epsilon B \subset (1/\epsilon) B \qquad \text{for every } \xi \in \Xi_1 \times \cdots \times \Xi_N. \tag{3.13}$$

Of course, (3.12) and (3.13) assert that (3.9) holds as desired for $k = N$.

We now assume inductively that (3.9) holds for a given k and verify that it then holds for $k-1$. If for given $(\xi_1, \ldots, \xi_{k-1})$ we have

$$(x_1, \ldots, x_{k-1}) \in (\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1})) + \epsilon B,$$

then for all $\xi_k \in \Xi_k$,

$$|(x_1, \ldots, x_{k-1}, \bar{\bar{x}}_k(\xi_1, \ldots, \xi_k)) - (\bar{\bar{x}}_1(\xi_1), \ldots, \bar{\bar{x}}_{k-1}(\xi_1, \ldots, \xi_{k-1}), \bar{\bar{x}}_k(\xi_1, \ldots, \xi_k))| \leqslant \epsilon.$$

Then from the induction hypothesis,

$$(x_1, \ldots, x_{k-1}, \bar{x}_k(\xi_1, \ldots, \xi_k)) \in \Gamma_k(\xi_1, \ldots, \xi_k) \qquad \text{for all } \xi_k,$$

so that $(x_1, \ldots, x_{k-1}) \in \Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1})$ by definition. Therefore,

$$(\bar{x}_1(\xi), \ldots, \bar{x}_{k-1}(\xi_1, \ldots, \xi_{k-1})) + \epsilon B \subset \Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1}),$$

and (3.9) is thus verified for $k-1$.

Lemma 2. *The multifunctions Γ_k are all lower-semicontinuous and compact-convex-valued.*

Proof. The assertion is evident from formula (3.7) in the case of $k = N$, because $\bar{D}^\epsilon$ is known to have these properties on Ξ, and $\bar{D}^\epsilon(\xi) \subset (1/\epsilon) B$ by definition. Again we proceed by induction, assuming the assertion is valid for a given k and verifying that it then holds also for $k-1$. Let

$$\Gamma'_{k-1}(\xi_1, \ldots, \xi_k) = \{(x_1, \ldots, x_{k-1}) \mid \exists x_k, (x_1, \ldots, x_{k-1}, x_k) \in \Gamma_k(\xi_1, \ldots, \xi_k)\}, \tag{3.14}$$

so that

$$\Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1}) = \cap \{\Gamma'_{k-1}(\xi_1, \ldots, \xi_{k-1}, \xi_k) \mid \xi_k \in \Xi_k\}. \tag{3.15}$$

It is elementary that Γ'_{k-1} inherits from Γ_k the property of being a lower-semicontinuous, compact-convex-valued multifunction. Therefore Γ_{k-1} is also compact-convex-valued by (3.15).

To establish the lower semicontinuity of Γ_{k-1}, we make use of Proposition 1 and the fact (Lemma 1) that

$$\operatorname{int} \Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1}) \neq \varnothing \qquad \text{for all } \xi. \tag{3.16}$$

Suppose that

$$(\bar{x}_1, \ldots, \bar{x}_{k-1}) \in \operatorname{int} \Gamma_{k-1}(\bar{\xi}_1, \ldots, \bar{\xi}_{k-1}). \tag{3.17}$$

It will suffice by Proposition 1 to demonstrate the existence of $\lambda > 0$ and a neighborhood U of $(\xi_1, \ldots, \xi_{k-1})$ such that

$$(\bar{x}_1, \ldots, \bar{x}_{k-1}) + \lambda B \subset \Gamma_{k-1}(\xi_1, \ldots, \xi_{k-1}) \qquad \text{for all } (\xi_1, \ldots, \xi_{k-1}) \in U. \tag{3.18}$$

We have

$$(\bar{x}_1, \dots, \bar{x}_{k-1}) \in \operatorname{int} \Gamma'_{k-1}(\bar{\xi}_1, \dots, \bar{\xi}_{k-1}, \xi_k) \qquad \text{for all } \xi_k \in \Xi_k \tag{3.19}$$

by (3.15) and (3.17). Since Γ_{k-1} is lower semicontinuous, there exist for each $\bar{\xi}_k \in \Xi_k$ neighborhoods U of $(\bar{\xi}_1, \dots, \bar{\xi}_{k-1})$ and U' of $\bar{\xi}_k$ and $\lambda > 0$ such that

$$\begin{aligned}(\bar{x}_1, \dots, \bar{x}_{k-1}) + \lambda B \subset \Gamma'_{k-1}(\xi_1, \dots, \xi_{k-1}, \xi_k)& \\ \text{for all } (\xi_1, \dots, \xi_{k-1}) \in U \quad \text{and} \quad \xi_k \in U'.&\end{aligned} \tag{3.20}$$

Using the compactness of Ξ_k, we can deduce from this the existence of $\lambda > 0$ and a neighborhood U of $(\bar{\xi}_1, \dots, \bar{\xi}_{k-1})$ such that

$$\begin{aligned}(\bar{x}_1, \dots, \bar{x}_{k-1}) + \lambda B \subset \Gamma'_{k-1}(\xi_1, \dots, \xi_{k-1}, \xi_k)& \\ \text{for all } (\xi_1, \dots, \xi_{k-1}) \in U \quad \text{and} \quad \xi_k \in \Xi_k.&\end{aligned} \tag{3.21}$$

But then (3.15) yields the desired conclusion (3.18).

LEMMA 3. *Let $1 \leqslant k < N$. Suppose we have constructed*

$$(\tilde{x}_1(\xi_1), \dots, \tilde{x}_k(\xi_1, \dots, \xi_k)),$$

depending continuously on ξ, agreeing with $(\bar{x}_1(\xi_1), \dots, \bar{x}_k(\xi_1, \dots, \xi_k))$ on $\bar{\Xi}$, and satisfying

$$(\tilde{x}_1(\xi_1), \dots, \tilde{x}_k(\xi_1, \dots, \xi_k)) \in \operatorname{int} \Gamma_k(\xi_1, \dots, \xi_k) \qquad \textit{for all } \xi. \tag{3.22}$$

Define

$$\begin{aligned}&\tilde{\Gamma}_{k+1}(\xi_1, \dots, \xi_k, \xi_{k+1}) \\ &\quad = \{x_{k+1} \mid (\tilde{x}_1(\xi_1), \dots, \tilde{x}_k(\xi_1, \dots, \xi_k), x_{k+1}) \in \Gamma_{k+1}(\xi_1, \dots, \xi_k, \xi_{k+1})\}.\end{aligned} \tag{3.23}$$

Then $\tilde{\Gamma}_{k+1}$ is a lower-semicontinuous multifunction with compact-convex values, and

$$\begin{aligned}&\operatorname{int} \tilde{\Gamma}_{k+1}(\xi_1, \dots, \xi_{k+1}) \\ &\quad = \{x_{k+1} \mid (\tilde{x}_1(\xi_1), \dots, \tilde{x}_k(\xi_1, \dots, \xi_k), x_{k+1}) \in \operatorname{int} \Gamma_{k+1}(\xi_1, \dots, \xi_{k+1})\} \\ &\quad \neq \varnothing \qquad \textit{for all } \xi.\end{aligned} \tag{3.24}$$

Moreover,

$$\bar{x}_{k+1}(\xi_1, \dots, \xi_{k+1}) \in \operatorname{int} \tilde{\Gamma}_{k+1}(\xi_1, \dots, \xi_{k+1}) \qquad \textit{for all } \xi \in \bar{\Xi}. \tag{3.25}$$

Proof. Since Γ_{k+1} is compact-convex-valued (Lemma 2), $\tilde{\Gamma}_{k+1}$ is clearly compact-convex-valued. Also, (3.22) implies by definition (3.8) that, for every $\xi_{k+1} \in \Xi_{k+1}$,

$$(\tilde{x}_1(\xi_1),..., \tilde{x}_k(\xi_1,..., \xi_k)) \in \text{int}\{(x_1,..., x_k) \mid \exists x_{k+1}, \\ \times (x_1,..., x_k, x_{k+1}) \in \Gamma_{k+1}(\xi_1,..., \xi_k, \xi_{k+1})\}, \tag{3.26}$$

where $\text{int}\, \Gamma_{k+1}(\xi_1,..., \xi_k, \xi_{k+1}) \neq \varnothing$ by Lemma 1. It follows then from convexity that (3.24) holds. The lower-semicontinuity of Γ_{k+1} (Lemma 2) in characterization of Proposition 1 implies via (3.24) and the continuity of $(\tilde{x}_1(\xi_1),..., \tilde{x}_k(\xi_1,..., \xi_k))$ that the set

$$\{(\xi_1,..., \xi_{k+1}, x_{k+1}) \mid x_{k+1} \in \text{int}\, \tilde{\Gamma}_{k+1}(\xi_1,..., \xi_{k+1})\}$$

is open, and hence $\tilde{\Gamma}_{k+1}$ is lower-semicontinuous. Finally, since for $\xi \in \bar{\Xi}$ we have

$$(\tilde{x}_1(\xi_1),..., \tilde{x}_k(\zeta_1,..., \xi_k), \bar{x}_{k+1}(\xi_1,..., \xi_{k+1})) \\ = (\bar{x}_1(\xi_1),..., \bar{x}_k(\xi_1,..., \xi_k), \bar{x}_{k+1}(\xi_1,..., \xi_{k+1})) \in \text{int}\, \Gamma_{k+1}(\xi_1,..., \xi_{k+1})$$

by our hypothesis with $\bar{x}$ as in Lemma 1, it also follows from (3.24) that (3.25) holds

Completion of the Proof of Theorem 2 Our task is to construct $\tilde{x} \in \mathcal{N}_{\mathscr{C}}$ satisfying (3.6) This can be effected by constructing continuous component functions

$$\tilde{x}_k: \Xi_1 \times \cdots \times \Xi_k \mapsto R^{n_k} \tag{3.27}$$

in such a way that

$$\tilde{x}(\xi) = (\tilde{x}_1(\xi_1),..., \tilde{x}_N(\xi_1,..., \xi_N))$$

satisfies

$$\tilde{x}(\xi) \in \Gamma_N(\xi) \qquad \text{for all } \xi \in \Xi_1 \times \cdots \times \Xi_N, \tag{3.28}$$

and

$$\tilde{x}_k(\xi_1,..., \xi_k) = \bar{x}_k(\xi_1,..., \xi_k) \qquad \text{for all } (\xi_1,..., \xi_k) \in \bar{\Xi}_k, \tag{3.29}$$

where $\bar{\Xi}^k$ is the (compact) projection of $\bar{\Xi}$ on $\Xi_1 \times \cdots \times \Xi_k$.

Starting with $k = 1$, we recall that the function $\bar{x}_1$ is continuous on $\bar{\Xi}^1$ and satisfies

$$\bar{x}_1(\xi_1) \in \text{int}\, \Gamma_1(\xi_1) \qquad \text{for all } \xi_1 \in \bar{\Xi}^1 \tag{3.30}$$

(Lemma 1). Moreover, Γ_1 is a lower-semicontinuous, convex-valued multifunction (Lemma 2) having $\text{int}\, \Gamma_1(\xi_1) \neq \varnothing$ for every $\xi_1 \in \Xi_1$ (Lemma 1).

We can therefore apply Proposition 3 and get a continuous extension $\tilde{x}_1$ of $\bar{x}_1$ with

$$\tilde{x}_1(\xi_1) \in \operatorname{int} \Gamma_1(\xi_1) \qquad \text{for all } \xi_1 \in \Xi_1 . \tag{3.31}$$

Proceeding now by induction, let us suppose $\tilde{x}_1 ,..., \tilde{x}_k$ have been constructed with the properties in Lemma 3. Defining $\tilde{\Gamma}_{k+1}$ as in Lemma 3, we have a lower-semicontinuous multifunction with properties (3.24) and (3.25). Proposition 3 can again be applied, and we thereby obtain a continuous function $\tilde{x}_{k+1}$ on $\Xi_1 \times \cdots \times \Xi_{k+1}$ agreeing with $\bar{x}_{k+1}$ on $\tilde{\Xi}_{k+1}$ and having

$$\tilde{x}_{k+1}(\xi_1 ,..., \xi_{k+1}) \in \operatorname{int} \tilde{\Gamma}_{k+1}(\xi_1 ,..., \xi_{k+1}) \qquad \text{for all } \xi. \tag{3.32}$$

But then,

$$(\tilde{x}_1(\xi_1),..., \tilde{x}_{k+1}(\xi_1 ,..., \xi_{k+1})) \in \operatorname{int} \Gamma_{k+1}(\xi_1 ,..., \xi_{k+1}) \qquad \text{for all } \xi \tag{3.33}$$

by (3.24), so the construction can be carried one stage further using the same argument. At the last stage, we have $(\tilde{x}_1 ,..., \tilde{x}_N) = \tilde{x}$ satisfying (3.28) and (3.29) as required.

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