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Vol. 15, No. 3, March 1975
*Printed in Belgium*

# Existence Theorems for General Control Problems of Bolza and Lagrange

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DEDICATED TO GARRETT BIRKHOFF

The existence of solutions is established for a very general class of problems in the calculus of variations and optimal control involving ordinary differential equations or contingent equations. The theorems, while relatively simple to state, cover, besides the more classical cases, problems with considerably weaker assumptions of continuity or boundedness. For example, the cost functional may only be lower semicontinuous in the control and may approach $+\infty$ as one nears certain boundary points of the control region; both endpoints in the problem may be "free". Earlier results of Cesari, Olech and the author are thereby extended.

The development is based on the theory of convex integral functionals and their conjugates. The first step is to show that, for purposes of existence theory, the problem can be reduced to a simpler model where control variables are not present as such. This model, resembling a classical problem of Bolza in the calculus of variations, but where the functions are extended-real-valued, is then investigated using, above all, the conjugacy correspondence between generalized Lagrangians and Hamiltonians.

## 1. INTRODUCTION

The problem we consider consists of minimizing the functional

$$\Psi(x, u) = \int_0^1 K(t, x(t), \dot{x}(t), u(t))\, dt + l(x(0), x(1))$$

over all absolutely continuous functions $x$: $[0, 1] \to R^n$ and Lebesgue measurable functions $u$: $[0, 1] \to R^m$, where

$$K: [0, 1] \times R^n \times R^n \times R^m \to R \cup \{+\infty\},$$
$$l: R^n \times R^n \to R \cup \{+\infty\}.$$

* Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under grant AFOSR-72-2269.







We denote this problem by $(Q)$ and call it a *control problem of Bolza.* Here the value $+\infty$ is used as a penalty to incorporate constraints into the definition of $K$ and $l$, so that constraints do not appear explicitly in the model.

This model was introduced in an earlier paper [1, Section 6]. There we discussed its relationship with optimal control problems in other formulations, as well as the question of its equivalence with the *reduced problem* $(P)$ in which one minimizes the functional

$$\Phi(x) = \int_0^1 L(t, x(t), \dot{x}(t))\, dt + l(x(0), x(1))$$

over all absolutely continuous functions $x$: $[0, 1] \to R^n$, where

$$L(t, x, v) = \inf_{u \in R^m} K(t, x, v, u).$$

The equivalence was used in deriving, by methods of convex analysis, various extensions of results of Cesari [2] and Olech [3] on the existence of solutions.

The goal of the present paper is to demonstrate the equivalence with the reduced problem in a more general way than in [1] and thereby to obtain new existence theorems. These theorems, couched in terms of the compactness of level sets of the functional $\Phi$, involve new growth conditions broadened to include properties of $l$, as well as of $L$.

The following technical assumptions are imposed. The functions $K(t, \cdot, \cdot, \cdot)$ and $l$ are lower semicontinuous. Furthermore, $K$ is $\mathscr{L} \times \mathscr{B}$-measurable, which is to say, measurable with respect to the $\sigma$-algebra generated in $[0, 1] \times R^n \times R^n \times R^m$ by products of Lebesgue sets in $[0, 1]$ and Borel sets in $R^n \times R^n \times R^m$.

The latter assumption ensures in particular that the integrand

$$t \to K(t, x(t), \dot{x}(t), u(t))$$

is $\mathscr{L}$-measurable (Lebesgue measurable) when $x$ is absolutely continuous and $u$ is $\mathscr{L}$-measurable (since then the mapping $t \to (x(t), \dot{x}(t), u(t))$ is $\mathscr{L}$-measurable). If for a given $x$ and $u$ this integrand is majorized almost everywhere by a summable, real-valued function of $t$, and if $l(x(0), x(1)) < +\infty$, then the value of $\Psi(x, u)$ is well-defined in the customary sense of the theory of integrals (possibly $-\infty$). In the remaining case, we adopt the convention that $\Psi(x, u) = +\infty$.

By definition, then, if $x$ and $u$ are such that $\Psi(x, u) < +\infty$, the conditions

$$\dot{x}(t) \in F(t, x(t), u(t)) \quad \text{for almost all } t,$$
$$(x(0), x(1)) \in C,$$

are satisfied, where

$$F(t, x, u) = \{v \in R^m \mid K(t, x, v, u) < +\infty\},$$
$$C = \{(a, b) \in R^n \times R^n \mid l(a, b) < +\infty\}.$$

These conditions may therefore be regarded as the *implicit constraints* in $(Q)$. If they can be satisfied at all, i.e., if $\inf \Psi < +\infty$, then $(Q)$ is equivalent to minimizing $\Psi$ subject to them, and in particular every optimizing pair must satisfy them. If $l$ is an indicator function, i.e., identically 0 on the set $C$, then $(Q)$ is a *control problem of Lagrange*. If on the other hand, $K$ is an indicator function, i.e., $K(t, x, v, u) = 0$ for all $v \in F(t, x, u)$, then $(Q)$ is a *control problem of Mayer*. In the general case, the sets $F(t, x, u)$ and $C$ may not be closed, if the functions tend to $+\infty$ at certain boundary points.

Although the model treated here has a fixed time interval, normalized for convenience to $[0, 1]$, problems with variable time are also covered by the results, since most such problems can be reformulated as problems over a fixed interval. This does not raise difficulties, because the conditions imposed here are very broad and flexible. A method of reformulation is given in [1, Section 3] which, contrary to classical "parametrization" of the problem, does not necessitate any topological assumptions on the way $K$ depends on $t$.

Our approach rests essentially on the theory of convex integral functionals and their conjugates. This appears to have the advantage of yielding stronger results with less notation and fewer direct assumptions. However, the ideas are closely related to those in Olech's work [3, 4], where the setting is somewhat more geometric. In particular, the growth conditions in our first two existence theorems were inspired by corresponding conditions of Olech in [3].

## 2. Equivalence with the Reduced Problem

A multifunction $\Gamma: [0, 1] \to R^N$ is will be called *$\mathscr{L}$-measurable* if its graph

$$G(\Gamma) = \{(t, z) \mid z \in \Gamma(t)\}$$



is $\mathscr{L} \times \mathscr{B}$-measurable as a subset of $[0, 1] \times R^N$. If $\Gamma$ is closed-valued, definition is equivalent to various others in the literature; see [5, 6] and the references given there. We cite several facts from this theory that will be needed.

Proposition 1. *Suppose $\Gamma: [0, 1] \to R^N$ is a closed-valued multifunction, and let*

$$T = \{t \in [0, 1] \mid \Gamma(t) \neq \varnothing\}.$$

*In order that $\Gamma$ be $\mathscr{L}$-measurable, it is necessary and sufficient that $T$ be a Lebesgue set, and that there exist a countable family $(z_i)_{i \in I}$ of $\mathscr{L}$-measurable functions $z_i: T \to R^N$ such that*

$$\Gamma(t) = \mathrm{cl}\{z_i(t) \mid i \in I\} \quad \textit{for every} \quad t \in T.$$

Corollary (Measurable Selections). *Suppose $\Gamma: [0, 1] \to R^N$ is a closed-valued, $\mathscr{L}$-measurable multifunction such that $\Gamma(t)$ is nonempty for almost every $t \in [0, 1]$. Then there exists an $\mathscr{L}$-measurable function $z: [0, 1] \to R^N$ such that*

$$z(t) \in \Gamma(t) \quad \textit{for almost every} \quad t \in [0, 1].$$

Proposition 2. *Let $f: [0, 1] \times R^N \to R \cup \{+\infty\}$, and let*

$$\Gamma(t) = \{(z, \alpha) \in R^{N+1} \mid \alpha \geq f(t, z)\}.$$

*Then $f$ is $\mathscr{L} \times \mathscr{B}$-measurable if and only if $\Gamma$ is $\mathscr{L}$-measurable.*

The set $\Gamma(t)$ is the *epigraph* of the function $f(t, \cdot)$. We note that it is closed if and only if $f(t, z)$ is lower semicontinuous in $z$.

Proposition 2 is entirely elementary, but Proposition 1 has, of course, a deep proof, due to Rokhlin, Castaing and others.

We also state at this time a result which will not be needed until paragraph 3, but which provides one of the most valuable criteria for $\mathscr{L} \times \mathscr{B}$-measurability—the Carathéodory condition.

Proposition 3. *Let $Z$ be any Borel set in $R^N$, and let $f: [0, 1] \times Z \to R$ be such that $f(t, z)$ is continuous in $z$ for fixed $t$ and $\mathscr{L}$-measurable in $t$ for fixed $z$. Then $f$ is $\mathscr{L} \times \mathscr{B}$-measurable, and in fact all the level sets*

$$\{(t, z) \in [0, 1] \times Z \mid f(t, z) \leq \alpha\}$$

*are $\mathscr{L} \times \mathscr{B}$-measurable as subsets of $[0, 1] \times R^N$.*

*Proof.* We indicate a particularly simple proof. Let $(w_i)_{i=1}^{\infty}$ be a dense sequence in $Z$. Fix $\alpha \in R$. It is clear that $f(t, z) \leqslant \alpha$ if and only if for every positive integer $j$ there exists $w_i$ such that $| w_i - z | \leqslant 1/j$ and $f(t, w_i) \leqslant \alpha + (1/j)$. Let

$$T_{ij} = \{t \in [0, 1] \mid f(t, w_i) \leqslant \alpha + (1/j)\},$$
$$Z_{ij} = \{z \in Z \mid | w_i - z | \leqslant 1/j\}.$$

Clearly $T_{ij}$ is $\mathscr{L}$-measurable in $[0, 1]$, and $Z_{ij}$ is $\mathscr{B}$-measurable in $R^N$. From what we have said, the level set in the proposition can be expressed as

$$\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} T_{ij} \times Z_{ij}.$$

Hence it is $\mathscr{L} \times \mathscr{B}$-measurable, and the proof is complete.

We shall say that the function $K$ satisfies the ***inf-boundedness condition*** if every fixed $t \in [0, 1]$, $\alpha \in R$, and every bounded set $S \subset R^n \times R^n$, the set

$$\{u \in R^m \mid \exists (x, v) \in S \quad \text{with} \quad K(t, x, v, u) \leqslant \alpha\}$$

is bounded.

Equivalence Theorem. *Suppose that $K$ satisfies the inf-boundedness condition. Then the function $L$ is $\mathscr{L} \times \mathscr{B}$-measurable, and the infimum in its definition is always attained (hence never $-\infty$). Also, $L(t, x, v)$ is lower semicontinuous in $(x, v)$. Thus in particular, the functional $\Phi$ which one minimizes in the reduced problem $(P)$ is well-defined.*

*Furthermore, for every absolutely continuous function $x$ one has*

$$\Phi(x) = \min\{\Psi(x, u) \mid u \ \mathscr{L}\text{-measurable}\}$$

*(where the minimum is attained by at least one $\mathscr{L}$-measurable function $u$). In this sense, $(P)$ is equivalent to $(Q)$.*

*Proof.* The condition implies in particular that for fixed $(t, x, v)$ the level sets

$$\{u \in R^m \mid K(t, x, v, u) \leqslant \alpha\}, \qquad \alpha \in R,$$

are all compact, since we already have them closed by our lower semicontinuity assumption. Thus the infimum in the definition of $L$ is always attained, and we have for each $t \in [0, 1]$ and $\alpha \in R$

$$\{(x, v) \mid L(t, x, v) \leqslant \alpha\} = \{(x, v) \mid \exists u, K(t, x, v, u) \leqslant \alpha\}.$$

Invoking the inf-boundedness condition more fully, as well as the lower semicontinuity of $K(t, x, v, u)$ in $(x, v, u)$, one sees the closedness of the set on the right. The level set on the left is therefore closed; thus $L(t, x, v)$ is lower semicontinuous in $(x, v)$. Consider now the epigraph multifunctions $\Gamma$ and $\Gamma_0$ defined by

$$\Gamma(t) = \{(x, v, u, \alpha) \mid K(t, x, v, u) \leqslant \alpha\},$$
$$\Gamma_0(t) = \{(x, v, \alpha) \mid L(t, x, v) \leqslant \alpha\}.$$

These are closed-valued, since $K(L, \cdot, \cdot, \cdot)$ and $(t, \cdot, \cdot, \cdot)$ are lower semicontinuous, and $\Gamma_0(t)$ is the image of $\Gamma(t)$ under the projection

$$(x, v, u, \alpha) \to (x, v, \alpha).$$

We know from Proposition 2 that $\Gamma$ is $\mathscr{L}$-measurable, because $K$ is $\mathscr{L}$-measurable. Hence by Proposition 1 the set

$$T = \{t \in [0, 1] \mid \Gamma(t) \neq \varnothing\}$$

is $\mathscr{L}$-measurable, and there exists a countable family of $\mathscr{L}$-measurable functions $(x_i, v_i, u_i, \alpha_i)$ on $T$, $i \in I$, such that

$$\Gamma(t) = \text{cl}\{(x_i(t), v_i(t), \alpha_i(t)) \mid i \in I\} \qquad \text{for every} \quad t \in T.$$

Then we have

$$\Gamma_0(t) = \text{cl}\{(x_i(t), v_i(t), \alpha_i(t)) \mid i \in I\} \qquad \text{for every} \quad t \in T,$$

implying by Proposition 1 that $\Gamma_0$ is $\mathscr{L}$-measurable. Hence $L$ is $\mathscr{L} \times \mathscr{B}$-measurable (Proposition 2), and $\Phi$ is well-defined as claimed.

Since $K(t, x, v, u) \geqslant L(t, x, v)$ by definition, we always have $\Psi(x, u) \geqslant \Phi(x)$. To complete the proof of the theorem, we suppose therefore that $x$ is an absolutely continuous function with $\Phi(x) < +\infty$, and we demonstrate the existence of an $\mathscr{L}$-measurable function $u$ such that $\Psi(x, u) = \Phi(x)$. Define

$$f(t, u) = K(t, x(t), \dot{x}(t), u),$$
$$g(t) = L(t, x(t), \dot{x}(t)),$$
$$\Gamma_1(t) = \{u \in R^m \mid f(t, u) \leqslant g(t)\}.$$

The $t$ values here belong to the Lebesgue set $T_1$ of full measure where $\dot{x}(t)$ exists; for other $t$, we define $\Gamma_1(t) = \varnothing$. We need to produce an $\mathscr{L}$-measurable function $u$ with $u(t) \in \Gamma_1(t)$ for almost all $t$, and to do so it suffices to show that the multifunction $\Gamma_1$ satisfies the assumptions of the Corollary to Proposition 1.

Certainly $\Gamma_1$ is closed-valued, because $f(t, u)$ is lower semicontinuous in $u$. It is also nonempty-valued for $t \in T_1$, because the infimum in the definition of $L$ is always attained. We observe that $f$ is $\mathscr{L} \times \mathscr{B}$-measurable, because $K$ is $\mathscr{L} \times \mathscr{B}$-measurable, and because the mapping

$$\theta: (t, u) \to (t, x(t), \dot{x}(t), u)$$

on $T_1 \times R^m$ is measurable in the sense that $\theta^{-1}(S)$ is $\mathscr{L} \times \mathscr{B}$ measurable when the set $S$ is $\mathscr{L} \times \mathscr{B}$-measurable. (This property is evident for sets $S$ of the $\mathscr{L} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B}$ product form in $[0, 1] \times R^n \times R^n \times R^m$, and hence it also holds for all sets in the $\sigma$-algebra generated by such product sets). Similarly, the function $g$ is $\mathscr{L}$-measurable, because $L$ is $\mathscr{L} \times \mathscr{B}$-measurable and the mapping

$$\theta: t \to (t, x(t), \dot{x}(t))$$

is measurable in the sense that $\theta^{-1}(S)$ is $\mathscr{L}$-measurable when $S$ is $\mathscr{L} \times \mathscr{B}$-measurable. Since $g$ is $\mathscr{L}$-measurable and $f$ is $\mathscr{L} \times \mathscr{B}$-measurable, the set

$$\{(t, u) \in T_1 \times R^m \mid f(t, u) \leqslant g(t)\}$$

is $\mathscr{L} \times \mathscr{B}$-measurable in $[0, 1] \times R^m$. But this is the graph of $\Gamma_1$. Thus $\Gamma_1$ is $\mathscr{L}$-measurable.

## 3. Semicontinuity of the Bolza Functional

The question of whether the control problem $(Q)$ has a solution is reduced by the equivalence theorem to the question of whether $(P)$ has a solution. From now on, we therefore direct our attention at the functional $\Phi$ and the properties of its level sets relevant to the study of whether $\Phi$ attains its infimum. It is assumed, as is true in the context of the equivalence theorem, that $L$ is $\mathscr{L} \times \mathscr{B}$-measurable, and $L(t, x, v)$ is lower semicontinuous in $(x, v)$ for each $t$. The results below are valid for any such function $L$; no particular $K$ need be involved.



We shall say that $L$ satisfies the *convexity condition* if $L(t, x, v)$ is convex as a function of $v$ for each $(t, x)$. This holds if $L$ is derived as above from a function $K$ with $K(t, x, v, u)$ convex in $(v, u)$.

The *Hamiltonian* associated with $L$ is the function

$$H: [0, 1] \times R^n \times R^n \to R \cup \{\pm\infty\}$$

defined by

$$H(t, x, p) = \sup_{v \in R^n} \{p \cdot v - L(t, x, v)\}.$$

We shall say that $H$ satisfies the *basic growth condition* if for each fixed $p \in R^n$ and bounded set $S \subset R^n$ there exists a summable function $\phi: [0, 1] \to R$ such that

$$H(t, x, p) \leqslant \phi(t) \qquad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad x \in S.$$

This is a generalization of the classical conditions employed in existence theory by Nagumo and Tonelli and later by Cesari. It was first used in a form equivalent to the present one in papers of Olech [3, 4].

Before deriving the main consequences of this condition, we prove a couple of results that shed more light on it.

Proposition 4. *The Hamiltonian $H$ always has the property that $H(t, x, p)$ is convex and lower semicontinuous in $p$. If $L$ satisfies the convexity condition, the formula*

$$L(t, x, v) = \sup_{p \in R^n} \{p \cdot v - H(t, x, p)\}$$

*is also valid. If $H$ satisfies the basic growth then $H$ is $\mathscr{L} \times \mathscr{B}$-measurable, and $H(t, x, p)$ is upper semicontinuous in $(x, p)$.*

*Conversely, suppose $H$ is any $\mathscr{L} \times \mathscr{B}$-measurable function such that $H(t, x, p)$ is upper semicontinuous in $x$, convex in $p$, and everywhere less than $+\infty$. Let $L$ be defined as above. Then $L$ is an $\mathscr{L} \times \mathscr{B}$-measurable function such that $L(t, x, v)$ is everywhere greater than $-\infty$, lower semicontinuous in $(x, v)$, and the convexity condition is satisfied. Moreover, then $H$ is the Hamiltonian associated with $L$.*

*Proof.* The fact that $H(t, x, p)$ is convex and lower semicontinuous in $p$ is obvious from the formula. Indeed, $H(t, x, \cdot)$ is the conjugate of the function $L(t, x, \cdot)$. If $L(t, x, \cdot)$ is convex, in addition to being lower semicontinuous and everywhere greater than $-\infty$ as we have assumed, then it is in turn the conjugate of its conjugate. In other words, the formula given for $L$ in terms of $H$ is valid. (For the theory of conjugate functions, see [7].)

To show the measurability and upper semicontinuity of $H$, assuming the basic growth condition, we rewrite the formula for $H$ as a special case of the situation in the equivalence theorem:

$$L_0(t, x, p) = \inf_{v \in R^n} K_0(t, x, p, v),$$

where $L_0 = -H$ and

$$K_0(t, x, p, v) = L(t, x, v) - p \cdot v.$$

It is clear that $K_0$ is lower semicontinuous in $(x, p, v)$ and $\mathscr{L} \times \mathscr{B}$-measurable. We want to establish that $L_0$ is lower semicontinuous in $(x, p)$ and $\mathscr{L} \times \mathscr{B}$-measurable, and for this it suffices by the equivalence theorem to show that for fixed $t \in [0, 1]$, $\alpha \in R$ and $r \geqslant 0$, the set

$$M = \{v \mid \exists(x, p) \text{ with } |x| \leqslant r, |p| \leqslant r, L(t, x, v) - p \cdot v \leqslant \alpha\}$$

is bounded ($|\cdot| =$ Euclidean norm). For later purposes, we shall state the argument somewhat more broadly than would really be necessary at the moment. Consider any $s \geqslant 0$ and let $\{p_1, \ldots, p_k\}$ be a finite set in $R^n$ such that

$$|p| \leqslant s \quad \text{implies} \quad p \in \operatorname{co}\{p_1, \ldots, p_k\}.$$

There exist by the basic growth condition summable, real-valued functions $\phi_i$ such that

$$|x| \leqslant r \quad \text{implies} \quad H(t, x, p_i) \leqslant \phi_i(t), \qquad i = 1, \ldots, m.$$

It follows then from the convexity of $H(t, x, p)$ in $p$ that

$$|x| \leqslant r \quad \text{and} \quad |p| \leqslant s \quad \text{implies} \quad H(t, x, p) \leqslant \phi(t),$$

where

$$\phi(t) = \max_{i=1,\ldots,k} \phi_i(t) < +\infty.$$

We note in passing that the function $\phi$ is again summable. For all $(x, v)$ with $|x| \leqslant r$, we have by the definition of $H$ the inequality

$$p \cdot v - L(t, x, v) \leqslant \phi(t) \quad \text{if} \quad |p| \leqslant s,$$



and consequently

$$L(t, x, v) \geqslant \sup_{|p| \leqslant s} \{p \cdot v - \phi(t)\} = s|v| - \phi(t).$$

Supposing now that $s > r$, we observe that if $v$ belongs to the set $M$ above with corresponding $(x, p)$, then

$$\alpha \geqslant L(t, x, v) - p \cdot v \geqslant s|v| - \phi(t) - |p||v| \geqslant (s - r)|v| - \phi(t).$$

Thus every $v \in M$ satisfies

$$|v| \leqslant (\alpha + \phi(t))/(s - r),$$

and the boundedness is verified.

For the converse part of the proposition, we note that the properties of $H$ imply that $H(t, x, p)$ is continuous in $p$ (cf. [7, Theorem 10.1]). The formula for $L$ in terms of $H$ can therefore be written as

$$L(t, x, v) = \sup_{i \in I}\{p_i \cdot v - H(t, x, p_i)\} > -\infty,$$

where $(p_i)_{i \in I}$ is any countable, dense family of points in $R^n$. Each of the functions

$$(t, x, v) \to p_i \cdot v - H(t, x, p_i)$$

is lower semicontinuous in $(x, v)$ and $\mathscr{L} \times \mathscr{B}$-measurable and therefore $L$, as the supremum of a countable collection of such functions, also has these properties. Moreover $L(t, x, \cdot)$ is by definition the conjugate of $H(t, x, \cdot)$ hence convex. Since $H(t, x, \cdot)$ is a convex function everywhere less than $+\infty$, it agrees with its biconjugate, the conjugate of $L(t, x, \cdot)$. Thus $H$ is the Hamiltonian corresponding to $L$.

Proposition 5. *Suppose $H$ satisfies the basic growth condition. Then $H$ actually satisfies the growth condition*

$$H(t, x, p) \leqslant h(t, |x|, |p|) \qquad \textit{for all} \quad t, x, p,$$

*where the function $h$ on* $[0, 1] \times [0, +\infty) \times [0, +\infty)$, *defined by*

$$h(t, r, s) = \max\{H(t, x, p) \mid |x| \leqslant r, |p| \leqslant s\} < +\infty,$$

*is $\mathscr{L} \times \mathscr{B}$-measurable, nondecreasing in $r$ and $s$, upper semicontinuous in $(r, s)$, convex in $s$, and*

$$\int_0^1 h(t, r, s)\, dt < +\infty \qquad \text{for all} \quad r \geqslant 0, \qquad s \geqslant 0.$$

*Proof.* The maximum in the definition of $h$ is indeed attained and not $+\infty$, since $H(t, x, p)$ is upper semicontinuous in $(x, p)$ by Proposition 4. In fact, if we write the definition of $h$ as

$$L_1(t, r, s) = \min_{x,p} K_1(t, r, s, x, p),$$

where $L_1(t, r, s) = -h(t, |r|, |s|)$ and

$$\begin{aligned} K_1(t, r, s, x, p) &= -H(t, x, p) \qquad \text{if} \quad |x| \leqslant r, \qquad |p| \leqslant s, \\ &= +\infty \qquad \text{otherwise}, \end{aligned}$$

the hypothesis of the equivalence theorem is satisfied. We may conclude therefore that $h$ is upper semicontinuous in $(r, s)$ and $\mathscr{L} \times \mathscr{B}$-measurable. We also can express $h$ by

$$h(t, r, s) = \sup\{s(p \cdot v) - L(t, x, v) \mid |x| \leqslant r, |p| \leqslant 1\},$$

and this shows the convexity in $s$, since the supremum of a collection of affine functions of $s$ is convex. It is obvious that $h$ is nondecreasing in $r$ and $s$. The proof of Proposition 4 constructs for each $r$ and $s$ a summable function $\phi$: $[0, 1] \to R$ such that $h(t, r, s) \leqslant \phi(t)$ for $0 \leqslant t \leqslant 1$, and hence the integrability assertion about $h$ is valid.

We proceed now to state the main result of this section.

Let $\mathscr{A}$ denote the Banach space consisting of all absolutely continuous functions $x$: $[0, 1] \to R^n$, the norm being

$$\| x \|_{\mathscr{A}} = | x(0)| + \int_0^1 | \dot{x}(t)|\, dt.$$

(We denote by $| \cdot |$ the Euclidean norm in $R^n$.) Let $\mathscr{C}$ denote the usual Banach space consisting of all continuous function $x$: $[0, 1] \to R^n$,

$$\| x \|_{\mathscr{C}} = \max_{0 \leqslant t \leqslant 1} | x(t)|.$$

We have $\| x \|_{\mathscr{C}} \leqslant \| x \|_{\mathscr{A}}$ for $x \in \mathscr{A} \subset \mathscr{C}$. It is known that every weakly compact subset of $\mathscr{A}$ is strongly compact as a subset of $\mathscr{C}$. (This follows from the Ascoli-Arzela criterion for strong compactness in $\mathscr{C}$ and the Dunford–Pettis criterion for weak compactness in $\mathscr{L}^1$ spaces, applied to $\dot{x}$).



SEMICONTINUITY THEOREM. *Suppose L satisfies the convexity condition and H satisfies the basic growth condition. Then for all real numbers $\alpha$ and $r$ the set*

$$\{x \in \mathscr{A} \mid \Phi(x) \leqslant \alpha, \| x \|_{\mathscr{C}} \leqslant r\}$$

*is compact in the weak topology of $\mathscr{A}$ and hence also compact as a subset of $\mathscr{C}$ in the norm topology of $\mathscr{C}$.*

*In particular, $\Phi$ is lower semicontinuous relative to the norms of $\mathscr{A}$ and $\mathscr{C}$ and lower semicontinuous sequentially relative to the weak topology of $\mathscr{A}$.*

The proof of this theorem will be based on a fundamental result about integral functionals. Here $\mathscr{L}_N^1$ denotes the usual Lebesgue space of $R^N$-valued, summable functions on $[0, 1]$, and similarly $\mathscr{L}_N^\infty$.

PROPOSITION 6 [6, 8]. *Let $F$ be an $\mathscr{L} \times \mathscr{B}$-measurable function on $[0, 1] \times R^n$ such that $f(t, z)$ is lower semicontinuous in $z$, and let*

$$g(t, w) = \sup\{w \cdot z - f(t, z) \mid z \in R^N\}.$$

*Then g is $\mathscr{L} \times \mathscr{B}$-measurable. One has the representation*

$$\int_0^1 g(t, w(t))\, dt = \sup \left\{ \int_0^1 w(t) \cdot z(t)\, dt - \int_0^1 f(t, z(t))\, dt \,\middle|\, z \in \mathscr{L}_n^\infty \right\},$$

*provided that $\int_0^1 f(t, z(t))\, dt < +\infty$ for at least one $z \in \mathscr{L}_n^\infty$. If actually*

$$\int_0^1 f(t, z)\, dt < +\infty \qquad \text{for every} \quad z \in R^n,$$

*then for every $z \in \mathscr{L}_n^\infty$ and $\beta \in R$ the set*

$$\left\{ w \in \mathscr{L}_N^1 \,\middle|\, \int_0^1 g(t, w(t))\, dt \leqslant \beta + \int_0^1 w(t) \cdot z(t)\, dt \right\}$$

*is compact in the weak topology of $\mathscr{L}_N^1$.*

*Proof of the Semicontinuity Theorem.* We first fix $\alpha \in R$ and $r > 0$ and apply Proposition 6 to the function

$$f(t, p) = h(t, r, |p|),$$

where $h$ is the function in Proposition 5. The hypothesis are satisfied, and hence the set

$$S_\beta = \left\{x \in \mathcal{A} \,\middle|\, \int_0^1 g(t, \dot{x}(t))\, dt \leqslant \beta, \| x \|_{\mathscr{C}} \leqslant r\right\}$$

is weakly compact for every $\beta$. (We can identify $\mathcal{A}$ with $R^n \times \mathscr{L}_n^1$ for considerations involving the weak topology, $x \leftrightarrow (x(0), \dot{x})$). Since

$$p \cdot v - L(t, x, v) \leqslant H(t, x, p) \leqslant f(t, p) \qquad \text{for} \quad |x| \leqslant r,$$

we have

$$L(t, x, v) \geqslant \sup_{p \in R^n} \{p \cdot v - f(t, p)\} = g(t, v) \qquad \text{for} \quad |x| \leqslant r.$$

Thus if $\| x \|_{\mathscr{C}} \leqslant r$ we have

$$\begin{aligned} \Phi(x) &= \int_0^1 L(t, x(t), \dot{x}(t))\, dt + l(x(0), x(1)) \\ &\geqslant \int_0^1 g(t, \dot{x}(t))\, dt + \gamma, \end{aligned}$$

where (using the lower semicontinuity of $l$)

$$\gamma = \min\{l(a, b) \mid |a| \leqslant r, |b| \leqslant r\} > -\infty.$$

It follows that the level set in the theorem is contained in the level set $S_\beta$ if $\beta = \alpha - \gamma$, and hence it is weakly compact as claimed, if it is weakly closed sequentially. (A subset of a weakly compact set in a Banach space is weakly closed if and only if it is weakly closed sequentially.)

We now apply Proposition 6 again, this time to the function $f_x(t, p) = H(t, x(t), p)$, where $x \in \mathcal{A}$. The hypotheses of Proposition 6 are satisfied in view of Proposition 4, the basic growth condition, and the measurability of the mapping $(t, p) \to (t, x(t), p)$. Furthermore, the convexity condition on $L$ implies that the function $g_x$ corresponding to $f_x$ is

$$g_x(t, v) = \sup_{p \in R^n} \{p \cdot v - H(t, x(t), p)\} = L(t, x(t), v).$$



It is true therefore that

$$\Phi(x) = l(x(0), x(1)) + \sup_{p \in \mathscr{L}_N^\infty} \left\{\int_0^1 p(t) \cdot \dot{x}(t)\, dt - \int_0^1 H(t, x(t), p(t))\, dt\right\}.$$

The functional

$$x \to l(x(0), x(1))$$

is weakly lower semicontinuous on $\mathcal{A}$, so that the proof now reduces to showing that each of the functionals

$$x \to \int_0^1 p(t) \cdot \dot{x}(t)\, dt - \int_0^1 H(t, x(t), p(t))\, dt$$

is weakly lower semicontinuous sequentially, or in other words, that for each $p \in \mathscr{L}_1^\infty$ we have

$$\limsup_{k \to \infty} \int_0^1 H(t, x_k(t), p(t))\, dt \leqslant \int_0^1 H(t, x_0(t), p(t))\, dt$$

if $x_k \to x_0$ weakly. But this follows from Fatou's lemma and the upper semicontinuity of $H$ in Proposition 5, since by Proposition 6 the integrands are all bounded above by the summable function

$$\phi(t) = \max\{0, h(t, r, s)\}$$

if $r$ and $s$ are taken sufficiently large.

*Remark.* It may be wondered why we have not needed to invoke something like the well-known condition $Q$ of Cesari, which is closely related to sequential weak lower semicontinuity (see Cesari [9]). The answer is that this property of the multifunction

$$x \to \text{epigraph of } L(t, x, \cdot)$$

is equivalent, under the basic growth condition on $H$ and convexity condition on $L$, to the lower semicontinuity of $L(t, x, v)$ in $(x, v)$. The role of Cesari's condition was taken in the proof by the fact that the dual representation of $L$,

$$L(t, x, v) = \sup_p \{p \cdot v - H(t, x, p)\},$$

is valid in these circumstances with $H(t, x, p)$ upper semicontinuous in $x$.

## 4. Existence of Solutions

We demonstrate next that in certain general situations the function $\Phi$ does attain its minimum over the Banach space $\mathcal{A}$. First an immediate corollary of the semicontinuity theorem is stated.

Existence Theorem 1. *Suppose that $L$ satisfies the convexity condition and $H$ satisfies the basic growth condition. If there exists a minimizing sequence $(x_k)_{k=1}^{\infty}$ for $\Phi$ such that the sequence of norms $\| x_k \|_{\mathscr{C}}$ is bounded, then there is a subsequence converging in both the norm topology of $\mathscr{C}$ and the weak topology of $\mathcal{A}$ to an $x \in \mathcal{A}$ minimizing $\Phi$.*

*In particular, $\Phi$ attains its minimum over $\mathcal{A}$ if there is an $r > 0$ such that*

$$L(t, x, v) < +\infty \qquad \textit{implies} \quad | x | \leqslant r.$$

Further existence theorems can be obtained by divising growth conditions which ensure that the level sets of the form

$$\{x \in \mathcal{A} \mid \Phi(x) \leqslant \alpha\}$$

are bounded in the norm of $\mathscr{C}$. This seems mostly to be a matter of tricks and happy discoveries. No single growth condition presents itself as "the" natural one, encompassing all the others. We concentrate below on developing a single condition which covers a great number of important cases and yet has the virtue of being fairly easy to understand and apply. This condition is an offspring of one used by Olech [3] in a different setting of problems of Lagrange; see the result of Olech which we have formulated as Theorem 4 of [1].

Let us say for simplicity that $H$ and $l$ satisfy the *stronger growth condition* if

$$H(t, x, p) \leqslant \mu(t, p) + | x | (\sigma(t) + \rho(t) | p |),$$
$$l(a, b) \geqslant l_0(a) + l_1(b),$$

where $\sigma(t)$, $\rho(t)$ and $\mu(t, p)$ are finite and summable as functions of $t$ (with $\sigma$ and $\rho$ nonnegative), $l_0$ and $l_1$ are bounded below on bounded sets, and

$$\liminf_{|a| \to +\infty} l_0(a)/| a | = +\infty,$$

$$\liminf_{|b| \to +\infty} l_1(b)/| b | > -\infty.$$



Certainly in this event $H$ also satisfies the basic growth condition. Note that the properties of $l$ hold in particular if $l$ is bounded below by a constant $\beta$ and the set

$$A = \{a \in R^n \mid \exists b \in R^n \text{ with } l(a, b) < +\infty\}$$

is bounded. (Take $l_0(a) = 0$ if $a \in A$, $l_0(a) = +\infty$ if $a \notin A$, $l_1(b) \equiv \beta$).

Proposition 7. *Suppose $H$ and $l$ satisfy the stronger growth condition. Then one also has*

$$H(t, x, p) \leqslant \theta(t, | p |) + | x | (\sigma(t) + 2\rho(t) | p |),$$
$$l(a, b) \geqslant j(| a |) - \eta | b |, \qquad \eta \geqslant 0,$$

*where $\theta$: $[0, 1] \times [0, +\infty) \to R$ and $j$: $[0, +\infty) \to R \cup \{+\}$ are certain functions such that $\theta$ is $\mathscr{L} \times \mathscr{B}$-measurable, $\theta(t, s)$ is summable in $t$, convex and nondecreasing in $s$, $j(s)$ is nondecreasing in $s$ and*

$$\lim_{s \to +\infty} j(s)/s = +\infty.$$

*Proof.* We begin by demonstrating that for each $s \geqslant 0$ there is a summable function $\phi_s(t)$ such that

$$h(t, r, s) \leqslant \phi_s(t) + r(\sigma(t) + 2s\rho(t)) \qquad \text{for all} \quad r \geqslant 0.$$

Let $\{p_1, ..., p_k\}$ be a finite subset of $R^n$ with $| p_i | \leqslant 2s$, such that

$$| p | \leqslant s \qquad \text{implies} \quad p \in \text{co}\{p_1, ..., p_k\}.$$

We have

$$H(t, x, p_i) \leqslant \mu(t, p_i) + | x | (\sigma(t) + 2s\rho(t)), \qquad i = 1, ..., k.$$

Since $H(t, x, p)$ is convex in $p$, it follows that for $| x | \leqslant r$ and $| p | \leqslant s$ we have

$$H(t, x, p) \leqslant \mu(t, p_i) + r(\sigma(t) + 2s\rho(t)), \qquad i = 1, ..., k.$$

The desired inequality is therefore true for

$$\phi_s(t) = \max_{i=1,...,k} \mu(t, p_i),$$

and this function is summable in $t$ because $\mu(t, p_t)$ is summable in $t$. We now define

$$\theta_0(t, s) = \sup_{r \geqslant 0}\{h(t, r, s) - r(\sigma(t) + 2s\rho(t))\}, \qquad s \geqslant 0,$$

where the supremum is bounded above by $\phi_s(t)$. Since $h(t, r, s)$ is upper semicontinuous and nondecreasing in $r$, the supremum is the same if restricted to rational values of $r$, and hence $\theta_0$ is the pointwise supremum of a countable family of functions of the form

$$(t, s) \to h(t, r, s) - r(\sigma(t) + 2s\rho(t)),$$

each of which is $\mathscr{L} \times \mathscr{B}$-measurable (Propositions 3 and 5). Therefore $\theta_0$ is $\mathscr{L} \times \mathscr{B}$-measurable. The functions just mentioned are also convex in $s$, and therefore so is $\theta_0$. Setting

$$\theta(t, s) = \max\{\theta_0(t, s), \theta_0(t, 0), 0\},$$

we have these properties, and also $\theta(t, s) \geqslant \theta(t, 0)$ for all $s \geqslant 0$; the latter ensures that $\theta(t, s)$ is nondecreasing in $s$. Since $0 \leqslant \theta(t, s) \leqslant \phi_s(t)$, it is clear that $\theta(t, s)$ is summable in $t$.

As for the assertions about $l$, the assumptions about $l_1$ give us the existence of $\eta \geqslant 0$, $\bar{s} \geqslant 0$ and $\gamma \geqslant 0$ such that

$$\begin{aligned} l_1(b) &\geqslant -\eta \mid b \mid \qquad &&\text{if} \quad \mid b \mid \geqslant \bar{s}, \\ l_1(b) &\geqslant -\gamma &&\text{if} \quad \mid b \mid \leqslant \bar{s}. \end{aligned}$$

We then have

$$l_1(b) \geqslant -\gamma - \eta \mid b \mid \qquad \text{for all } b.$$

A function $j$ with the desired property is then defined by

$$j(s) = -\gamma + \inf\{l_0(a) \mid \mid a \mid \leqslant s\}.$$

EXISTENCE THEOREM 2. *Assume that $L$ satisfies the convexity condition, and that $H$ and $l$ satisfy the stronger growth condition. Then all the level sets*

$$\{x \in \mathscr{A} \mid \Phi(x) \leqslant \alpha\}, \qquad \alpha \in R,$$



*are compact in the weak topology of $\mathscr{A}$ (and hence also in the norm topology of $\mathscr{C}$), so that $\Phi$ attains its minimum over $\mathscr{A}$.*

*Proof.* We take the growth condition in the form of Proposition 7 and show that the level sets are bounded in the norm of $\mathscr{C}$. A crucial fact is that Proposition 6 can be applied to the function $f(t, s) = \theta(t, \mid s \mid)$ on $[0, 1] \times R$. This is to be done near the end of the proof.

Starting out, we observe that the inequality

$$p \cdot v - L(t, x, v) \leqslant H(t, x, p) \leqslant \theta(t, \mid p \mid) + \mid x \mid (\sigma(t) + 2\rho(t) \mid p \mid)$$

implies

$$s \mid v \mid - L(t, x, v) \leqslant \theta(t, s) + \mid x \mid (\sigma(t) + 2\rho(t)s) \qquad \text{for all} \quad s \geqslant 0.$$

Therefore

$$\begin{aligned} L(t, x, v) + \sigma(t) \mid x \mid &\geqslant \sup_{s \geqslant 0}\{s[\mid v \mid - 2\rho(t) \mid x \mid] - \theta(t, s)\} \\ &= g(t, \max\{0, \mid v \mid - 2\rho(t) \mid x \mid\}), \end{aligned}$$

where

$$g(t, w) = \sup_{s \in R}\{ws - f(t, s)\} = \sup_{s \geqslant 0}\{\mid w \mid s - \theta(t, s)\}.$$

Suppose now that $x \in \mathscr{A}$ satisfies $\Phi(x) \leqslant \alpha$. We shall derive an upper bound for $\| x \|_{\mathscr{C}}$. Let

$$\omega(t) = \max\{0, \mid \dot{x}(t) \mid - 2\rho(t) \mid x(t) \mid\},$$

so that

$$L(t, x(t), \dot{x}(t)) \geqslant -\sigma(t) \mid x(t) \mid + g(t, \omega(t)).$$

(This and some of the subsequent assertions are true, of course, only in the "almost everywhere" sense.) The formula

$$\begin{aligned} (d/dt) \mid x(t) \mid &= [\dot{x}(t) \cdot x(t)]/\mid x(t) \mid \qquad \text{if} \quad \mid x(t) \mid > 0, \\ &= 0 \qquad \text{if} \quad \mid x(t) \mid = 0, \end{aligned}$$

yields for us

$$(d/dt) \mid x(t) \mid - 2\rho(t) \mid x(t) \mid \leqslant \omega(t).$$

Setting

$$r(t) = \exp\left\{-2\int_0^t \rho(\tau)\, d\tau\right\} > 0,$$

we thereby have the estimate

$$(d/dt)[r(t) \mid x(t)\mid] \leqslant \omega(t).$$

Therefore

$$r(t) \mid x(t)\mid - r(0) \mid x(0)\mid \leqslant \int_0^t \omega(\tau)\, d\tau,$$

or in other words, since $k$ is decreasing and $\omega \geqslant 0$,

$$r(1) \mid x(t)\mid \leqslant \mid x(0)\mid + \int_0^1 \omega(\tau)\, d\tau \qquad \text{for all} \quad t \in [0, 1].$$

This shows that

$$\| x \|_{\mathscr{C}} \leqslant \left[\mid x(0)\mid + \int_0^1 \omega(t)\, dt\right] \Big/ r(1).$$

We proceed to deduce bounds on $\mid x(0)\mid$ and $\int_0^1 \omega(t)\, dt$ from the fact that

$$\begin{aligned} \alpha \geqslant \Phi(x) &= \int_0^1 L(t, x(t), \dot{x}(t))\, dt + l(x(0), x(1)) \\ &\geqslant -\int_0^1 \sigma(t) \mid x(t)\mid dt + \int_0^1 g(t, \omega(t))\, dt + k(\mid x(0)\mid) - \eta \mid x(1)\mid \\ &\geqslant \int_0^1 g(t, \omega(t))\, dt + k(\mid x(0)\mid) - \| x \|_{\mathscr{C}} \left[\eta + \int_0^1 \sigma(t)\, dt\right] \\ &\geqslant \int_0^1 g(t, \omega(t))\, dt + k(\mid x(0)\mid) - \bar{s}\left[\mid x(0)\mid + \int_0^1 \omega(t)\, dt\right], \end{aligned}$$

where

$$\bar{s} = \left[\eta + \int_0^1 \sigma(t)\, dt\right] \Big/ r(1).$$

The expression $k(\mid a \mid) - \bar{s} \mid a \mid$ is bounded below by some number $\gamma$. This follows from the properties of $k$ asserted in Proposition 7. Hence

$$\int_0^1 g(t, \omega(t))\, dt \leqslant \alpha - \gamma + \bar{s} \int_0^1 \omega(t)\, dt.$$

But the set of function $\omega$ satisfying this inequality is bounded in $\mathscr{L}_1^1$,



according to Proposition 6. There is a number $D$, therefore, such that every $\omega$ satisfying this inequality also satisfies

$$\int_0^1 \omega(t)\, dt \leqslant D.$$

This $D$ can be used then in the estimate for $\| x \|_{\mathscr{C}}$ derived above.

To get a corresponding estimate for $\mid x(0)\mid$, we argue next that

$$\alpha \geqslant k(\mid x(0)\mid) - \bar{s} \mid x(0)\mid + \int_0^1 [g(t, \omega(t)) - \bar{s}\omega(t)]\, dt,$$

and consequently

$$k(\mid x(0)\mid) - \bar{s} \mid x(0)\mid \leqslant \alpha + \int_0^1 \theta(t, \bar{s})\, dt < +\infty,$$

inasmuch as

$$g(t, \omega(t)) \geqslant \bar{s}\omega(t) - \theta(t, \bar{s})$$

by definition. Making use again of the properties of $k$ in Proposition 7, we see the existence of a number $D'$ such that every $a \in R^n$ satisfying

$$k(\mid a \mid) - \bar{s} \mid a \mid \leqslant \alpha + \int_0^1 \theta(t, \bar{s})\, dt$$

also satisfies $\mid a \mid \leqslant D'$. In the estimate for $\| x \|_{\mathscr{C}}$, this now yields

$$\| x \|_{\mathscr{C}} \leqslant [D' + D]/r(1).$$

Thus every $x \in \mathscr{A}$ satisfying $\Phi(x) \leqslant \alpha$ satisfies this bound, and the theorem is thereby proved.

The preceding theorem can be applied, of course, with the roles of $x(0)$ and $x(1)$ reversed in the growth condition on $l$. However, it has definite limitations; for instance it is not relevant for any problems of Lagrange in which neither $x(0)$ nor $x(1)$ is implicitly constrained to a bounded set. Presumably such cases might be handled by some growth condition involving $L$ (or equivalently $H$) and $l$ *jointly*. To show that this is not a hopeless idea, we restate a result in this direction proved in [1].

Some notation must be introduced. Let $D_{\mathscr{C}}$ be the set of all pairs $(c, d) \in R^n \times R^n$ such that for some $\beta \in R$ sufficiently large one has

$$l(a, b) \geqslant c \cdot a - b \cdot d - \beta \qquad \text{for all} \quad (a, b).$$

For each $t \in [0, 1]$, $p \in R^n$, let $E(t, p)$ be the closure of the set of all $w \in R^n$ such that, for some $\beta$ sufficiently large, one has

$$L(t, x, v) \geqslant w \cdot x + p \cdot v - \beta \qquad \text{for all} \quad (x, v).$$

Let $D_L$ be the set of all pairs $(c, d) \in R^n \times R^n$ such that there exists $p \in \mathcal{A}$ satisfying

$$\dot{p}(t) \in E(t, p(t)) \text{ almost everywhere}, \qquad p(0) = c, \qquad p(1) = d.$$

It is easy to see that the sets $D_l$ and $D_L$ are convex. For a convex set $D$ we denote by aff $D$ the affine hull of $D$, and by ri $D$ the interior of $D$ relative to aff $D$.

Finally, let $H_0$ be the function obtained by taking the concave hull of $H(t, x, p)$ in $x$, i.e., $H_0(t, \cdot, p)$ is for each $(t, p)$ the least concave function (extended-real-valued) majorizing $H(t, \cdot, p)$.

EXISTENCE THEOREM 3. *Suppose that L satisfies the convexity condition, $H_0$ satisfies the basic growth condition, and*

$$\operatorname{ri} D_L \cap \operatorname{ri} D_l \neq \varnothing, \qquad \operatorname{aff}\ [D_L \cup D_l] = R^n \times R^n.$$

*Then the conclusions of Existence Theorem 2 are valid.*

COROLLARY. *Suppose that L satisfies the convexity condition, l majorizes at least one affine function, and the function*

$$M(t, p, w) = \sup_{x,v}\{w \cdot x + p \cdot v - L(t, x, v)\}$$

*is nowhere $+\infty$ and in fact satisfies*

$$\int_0^1 M(t, p, 0)\, dt < +\infty \qquad \textit{for all} \quad p \in R^n.$$

*Then the conclusions of Existence Theorem 2 are valid.*

*Proof.* We clearly have $D_l \neq \varnothing$, $E(t, p) = R^n$, and hence $D_L = R^N \times R^N$. $M$ is $\mathcal{L} \times \mathcal{B}$-measurable by Proposition 6, and

$$M(t, p, 0) = \sup_x H(t, x, p).$$

Thus for each $t$ and $p$ the function $x \to H(t, x, p)$ is majorized by a constant



function of $x$ whose value is summable in $t$ (namely $\max\{0, M(t, p, 0)\}$). The same is then true of $H_0$, and in particular we see that $H_0$ satisfies the basic growth condition.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.