GAME THEORY AND MATHEMATICAL ECONOMICS
O. Moeschlin, D. Pallaschke (editors)

OPTIMALITY CONDITIONS FOR CONVEX CONTROL PROBLEMS WITH NONNEGATIVE STATES AND THE POSSIBILITY OF JUMPS

R. Tyrrell Rockafellar

Department of Mathematics
University of Washington
Seattle, Washington
U.S.A.

Optimal control problems of convex type are considered in which the primal and dual state constraints define cones, such as nonnegative orthants, and moreover, jumps in the states might conceivably occur. Necessary and sufficient conditions for optimality are derived by convex analysis.

INTRODUCTION

In optimal control problems of convex type, such as often occur in economic models, it is typical for the states to be constrained to some convex set, e.g., a nonnegative orthant. Classical theory suggests that necessary conditions for optimality should involve dual (adjoint) state variables which might have jumps. In fact, a lack of strong growth conditions in some problems raises the possibility that jumps should be admitted for the primal state variables too. Standard necessary conditions do not cover such situations and indeed may run into other difficulties as well. For example, in an economic problem with states restricted to an orthant the expression to be optimized may contain a function which is defined only on the orthant in question and cannot be extended smoothly beyond it.

In this paper we use convex analysis to treat an abstract problem with the possibility of all these features. Necessary and sufficient conditions are derived in terms of subgradient versions of the Euler-Lagrange equation and transversality relation. We have previously developed such conditions for convex variational problems of similar kinds, but with special structure not allowing for jumps in both primal and dual variables [1], or not adapted to general boundary terms and constraints [2]. Here, by contrast, we have primal and dual jumps and general boundary terms. However, we make other restrictions of a lesser sort, mainly in order to simplify the technical discussion and bring out more clearly some of the properties of greatest interest in economic applications: we only treat autonomous systems (data not time-dependent), and we assume that the primal and dual state constraints concern membership in a convex *cone*.

For the way the model problem adopted here can be used to represent problems in other formulations, including those with explicit control variables, see the account in [3].

Let L and l be lower semicontinuous, convex functions from $R^n \times R^n$ to $R \cup \{+\infty\}$. For a fixed interval $[t_0,t_1]$ we wish to minimize

$$(1.1)\qquad I(q) = l(q(t_0),q(t_1)) + \int_{t_0}^{t_1} L(q(t),\dot{q}(t))dt$$

over some space of functions $q : [t_0,t_1] \to R^n$, $\dot{q} = dq/dt$ (at least in an almost everywhere sense). In terms of the effective domains

$$(1.2)\qquad D = \{(q_0,q_1)\,|\,l(q_0,q_1) < \infty\}, \qquad E = \{(x,y)\,|\,L(x,y) < \infty\},$$

which are the convex subsets of $R^n \times R^n$ where l and L are finite, the inequality $I(q) < \infty$ implies that

$$(1.3)\qquad (q(t_0),q(t_1)) \in D, \qquad (q(t),\dot{q}(t)) \subset E \quad \text{a.e.},$$

and hence in particular for the convex set

$$(1.4)\qquad Q = \text{cl}\{x \in R^n \,|\, \exists\, y \in R^n \text{ with } (x,y) \in E\}$$

that

$$(1.5)\qquad q(t) \in Q \quad \text{a.e.}$$

Thus, (1.3) and (1.5) are implicit constraints in the minimization of I, whatever space of functions q we happen to choose. Condition (1.5) is called the *primal state constraint.*

A natural candidate for the space over which to minimize is

$$(1.6)\qquad A = \{q : [t_0,t_1] \to R^n \,|\, q \text{ is absolutely continuous}\}.$$

On A, I is a well-defined convex functional with values in $R \cup \{\infty\}$ (see [3]). However, unless L satisfies in particular a certain growth condition that may be too stringent for some applications, there is serious doubt about I actually attaining a minimum, on A: it is possible that a minimizing sequence for I tends to a discontinuous function q not in A. The growth condition in question is fulfilled if and only if the convex set

$$(1.7)\qquad P = \text{cl}\{z \in R^n \,|\, \exists\, w \in R^n \text{ with } \inf_{x,y}[L(x,y) - w\cdot x - z\cdot y] > -\infty\}$$

is all of R^n. As a matter of fact, in the theory developed in [4] to cover problems where I is minimized over A, the condition

$$(1.8)\qquad p(t) \in P(t) \quad \text{a.e.}$$

appears as the *dual state constraint* that must be satisfied by the functions $p : [t_0,t_1] \to R^n$ which enter into a certain dual problem related to the characterization of optimality in the original problem. In economic applications, $p(t)$ often can be interpreted as a price vector, and the constraint (1.8) may reflect intrinsic requirements like nonnegativity. In the presence of such requirements, therefore, the growth condition on L which is appropriate to a minimization problem over A fails, and we need to look instead at an extended problem where the function q may be discontinuous.

STATEMENT OF THE MAIN RESULTS

In formulating and justifying the appropriate extension of the problem of minimizing I over A, we shall limit attention here to the following situation.

ASSUMPTION 1. Not only are the functions l *and* L *lower semicontinuous and convex, but the closed convex sets* Q *and* P *in* (1.4) *and* (1.7) *are cones with nonempty interior, and*

$$(2.1)\qquad D \subset Q \times Q.$$

ASSUMPTION 2. There is at least one function $q \in A$ *such that*

(2.2) $\quad q(t) \in \text{int}\, Q(t) \quad$ for all $t \in [t_0,t_1]$,

(2.3) $\quad \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\,dt < \infty$.

ASSUMPTION 3. The relative interior of the convex set D *meets the relative interior of the convex set*

(2.4) $\quad C = \{(q_0,q_1) \mid \exists\, q \in A$ *satisfying* (2.3) *with* $q(t_0) = q_0, q(t_1) = q_1\}$.

As in [2], we shall work with the space

(2.5) $\quad B = \{q : [t_0,t_1] \to R^n \mid q$ is of bounded variation$\}$

under the convention that two functions q_1 and q_2 in B are identical if $q_1(t+) = q_2(t+)$ and $q_1(t-) = q_2(t-)$ for all $t \in [t_0,t_1]$; at the endpoints of $[t_0,t_1]$ we interpret

(2.6) $\quad q(t_0-) = q(t_0)$ and $q(t_1+) = q(t_1)$ for $q \in B$.

Recall that a function $q \in B$ does have a right limit $q(t+)$ at every $t \in [t_0,t_1)$ and a left limit $q(t-)$ at every $t \in (t_0,t_1]$, and the jump $\Delta q(t) = q(t+) - q(t-)$ can be nonzero for at most countably many values of t. The derivative $\dot{q}$ still exists almost everywhere, but it is *not* necessarily true that

(2.7) $\quad q(\tau_1+) = q(\tau\, -) + \int_{\tau_0}^{\tau_1} \dot{q}(t)\,dt$ for all $[\tau_0,\tau_1] \subset [t_0,t_1]$,

and indeed the latter property holds if and only if q belongs to the subspace A of B.

In general, the R^n-valued Lebesque-Stieltjes measure of dq on $[t_0,t_1]$ may have a singular part, which in turn can be decomposed into an atomic measure (corresponding to jumps in q) but possibly also a nonatomic component. The measures dq can be identified with the continuous linear functionals on the Banach space

(2.8) $\quad C = \{r : [t_0,t_1] \to R^n \mid r$ is continuous$\}$,

and the correspondence $q \leftrightarrow (q(t_0), dq)$ therefore furnishes an isomorphism between B and $R^n \times C^*$, where C^* is the dual of C. In this way a Banach space structure is induced on B, but we shall be more concerned with the *weak*topology* on B, by which we mean the topology induced by the weak*topology on C^* under the isomorphism just mentioned.

DEFINITION. A function $q \in B$ *will be called* P-singularly monotone *if for every interval* $[\tau_0,\tau_1] \in [t_0,t_1]$ *one has*

(2.9) $\quad q(\tau_1+) - q(\tau_0-) - \int_{\tau_0}^{\tau_1} q(t)\,dt \in P^\circ$.

Likewise, a function $p \in B$ *will be called* Q-*singularly monotone if for every* *interval* $[\tau_0,\tau_1] \subset [t_0,t_1]$ *one has*

$$(2.10) \qquad p(\tau_1+) - p(\tau_0+) < \int_{\tau_0}^{\tau_1} p(t)dt \in Q^\circ \quad .$$

Here, of course, Q° is the polar of the cone Q.

If $P = R^n$, the P-singular monotonicity of q reduces to the requirement that $q \in A$, while if $P = R^n_+$ it means that

$$(2.11) \qquad q(\tau_1+) \leq q(\tau_0-) + \int_{\tau_0}^{\tau_1} L(q(t),\dot{q}(t))dt \text{ for all } [\tau_0,\tau_1] \subset [t_0,t_1] \quad .$$

In the general case it says that the singular part of the measure dq is P°-valued. An important consequence, seen by taking $\tau_0 = \tau_1 = t$ in (2.9), is that

$$(2.12) \qquad \Delta q(t) \in P^\circ \text{ for all } t \subset [t_0,t_1] \quad .$$

Since every element of A is in particular a P-singularly monotone element of $\mathcal{B}$, regardless of the choice of P, the following functional $\bar{I}$ is an extension of I from A to $\mathcal{B}$:

$$(2.13) \qquad \bar{I}(q) = \begin{cases} I(q) \text{ if } q \text{ is P-singularly monotone with } (q(t_0),q(t_1)) \in Q \times Q , \\ +\infty \quad \text{otherwise} . \end{cases}$$

The problem of minimizing $\bar{I}$ over $\mathcal{B}$ turns out to be the natural "closure" of the problem of minimizing I over A.

THEOREM 1. *Under Assumptions* 1,2, *and* 3, *one has*

$$(2.14) \qquad \inf_{q \in \mathcal{B}} \bar{I}(q) = \inf_{q \in A} I(q) \quad .$$

Moreover, any $q \in \mathcal{B}$ *which is the limit in the weak* topology of* $\mathcal{B}$ *of some minimizing sequence for* I *on* A *is a minimizing element for* $\bar{I}$ *on* $\mathcal{B}$. *In particular, any* $q \in A$ *which minimizes* I *on* A *also minimizes* $\bar{I}$ *on* $\mathcal{B}$.

The extended problem of minimizing $\bar{I}$ over $\mathcal{B}$ has the implicit constraint $\bar{I}(q) < \infty$. This forces q to satisfy the conditions in (2.13) as well as the ones noted earlier for I, namely the elementary implications of having

$$l(q(t_0),q(t_1)) < \infty \text{ and } \int_{t_0}^{t_1} L(q(t),\dot{q}(t))dt < \infty \quad .$$

The extended problem therefore can be stated equivalently in greater detail as

minimize $l(q(t_0),q(t_1)) + \int_{t_0}^{t_1} L(q(t),\dot{q}(t))dt$
over all $q \in \mathcal{B}$ such that

$(q(t_0),q(t_1)) \in D , \qquad (q(t),\dot{q}(t)) \in E \quad$ a.e.
$q(t+) \in Q$ and $q(t-) \in Q$ for all $t \in [t_0,t_1]$,

and q is P-singularly monotone.

The form of the state constraint here is derived from the earlier form (1.5) using the closedness of Q, the endpoint constraint $(q(t_0),q(t_1))\in D$, and assumption (2.2). (Limits alone would only give $q(t_0+)$ and $q(t_0-)$ in Q and tell nothing about the true endpoints of q.)

Theorem 1 will be proved in the next section along with the following characterization of optimality in terms of subgradients [5,§23] of the convex functions l and L.

THEOREM 2. *Under Assumptions* 1, 2, *and* 3, *for* $q\in B$ *to solve the extended problem of minimizing* $\bar{I}$ *over* B *it is both necessary and sufficient that there exist* $p\in B$ *such that*

(a) $(p(t_0),-p(t_1))\in\partial l(q(t_0),q(t_1))$,

(b) $(\dot{p}(t),p(t))\in\partial L(q(t),\dot{q}(t))$ *for almost every* $t\in[t_0,t_1]$,

(c) $q(t\pm)\in Q$ *and* $p(t\pm)\in P$ *for all* $t\in[t_0,t_1]$,

(d) q is P-*singularly monotone and* p *is* Q-*singularly monotone*,

(e) $q\cdot p$ *is absolutely continuous on* $[t_0,t_1]$.

In (e), of course, we must appeal to the conventions used in defining the space B: we really mean that $q\cdot p$ can be *identified* with a function $r:[t_0,t_1]\to R$ of bounded variation in the sense of having $r(t) = q(t+)\cdot p(t+) = q(t-)\cdot p(t-)$. Most of condition (c) is redundant and is included mainly for emphasis; all but the assertion that

(2.15) $\quad (p(t_0),p(t_1))\in P\times P$

is actually a consequence of (a) and (b).

Special cases of the "transversality" condition (a) and "Euler-Lagrange" condition (b) in Theorem 2 have been described in [3]. (See [6] for generalizations of such subgradient conditions to nonconvex problems over A). Condition (d) implies that the measure dp is Q°-valued and in particular

(2.16) $\quad \Delta p(t)\in Q^\circ \quad$ for all $t\in[t_0,t_1]$.

This and (2.12), which likewise is a consequence of (e), tell us in conjunction with (c) and the relation

(2.17) $\quad q(t+)\cdot p(t+) = q(t-)\cdot p(t-) \quad$ for all $t\in[t_0,t_1]$,

which holds under (e), that

(2.18) $\quad q(t\pm)\cdot\Delta p(t) = 0$ and $p(t\pm)\cdot\Delta q(t) = 0$ for all $t\in[t_0,t_1]$.

PROOFS

The arguments will be based mainly on our previous results in [2] for convex problems of Lagrange. These results concern the minimization of the functional

$$(3.1) \qquad I_L(q) = \int_{t_0}^{t_1} L(q(t),\dot{q}(t))dt$$

on A subject to endpoint constraints of the elementary form $q(t_0) = q_0$ and $q(t_1) = q_1$. They likewise involve an extended problem over B where the functional is

$$(3.2) \qquad \overline{I}_L(q) = \begin{cases} I_L(q) \text{ if } q \text{ is P-singularly monotone,} \\ +\infty \text{ otherwise.} \end{cases}$$

This expression is simpler than the general one in [2]; it corresponds to the present case where P is a cone rather than an arbitrary convex set. Other simplifications over the situation in [2] occur because L is merely a convex function on $R^n \times R^n$ and does not vary with t, so that the sets Q and P do not vary with t either. Our assumption that Q and P have nonempty interior ensures that the basic assumptions (S_1), (S_2) and (S_3) in [2] are fulfilled. (Q and P are the *closures* of the convex sets denoted in [2] by X and P.)

Duality is an essential feature of the results in [2]. Since L is a lower semicontinuous, convex function from $R^n \times R^n$ to $R \cup \{+\infty\}$ which is not identically $+\infty$ (due to Assumption 2), the function

$$(3.3) \qquad M(z,w) = \sup_{x,y} \{z\cdot y + w\cdot x - L(x,y)\} = L^*(w,z) \quad ,$$

where L^* is the conjugate [5,§12] of L, has these properties too and satisfies

$$(3.4) \qquad L(x,y) = \sup_{z,w} \{z\cdot y + w\cdot x - M(z,w)\} = M^*(y,x) \quad .$$

Duality properties have to do with the functional

$$(3.5) \qquad I_M(p) = \int_{t_0}^{t_1} M(p(t),\dot{p}(t))dt$$

on A and its corresponding extension on B:

$$(3.6) \qquad \overline{I}_M(p) = \begin{cases} I_M(p) \text{ if } p \text{ is Q-singularly monotone,} \\ +\infty \text{ otherwise.} \end{cases}$$

Like $\overline{I}_L$, $\overline{I}_M$ is a well-defined convex functional from B to $R \cup \{+\infty\}$. In fact both functionals are *lower semicontinuous in the weak* topology on* B (as defined in §2). The reasoning behind this is given in [2, pp. 186-190, starting with (6.24)], where it is shown that (in present notation and simpler circumstances) the expression

$$(3.7) \qquad \sup_{\substack{r\in C \\ q\in A}} \left\{ \int_{t_0}^{t_1} r(t)\cdot dp(t) + \int_{t_0}^{t_1} \dot{q}(t)\cdot p(t)dt - \int_{t_0}^{t_1} L(r(t),\dot{q}(t))dt \right\}$$

equals $\bar{I}_M(p)$ when it is not $+\infty$ (and trivially then too even when it is $+\infty$, because of (3.3) and the monotonicity condition in (3.6)); this says that $\bar{I}_M$ is the pointwise supremum of a collection of weak*-continuous linear functionals on $\mathcal{B}$ (one for each $x \in C$ and $q \in A$ such that the last integral in (3.7) is finite), and hence $\bar{I}_M$ is weak* lower semicontinuous. The same holds then for $\bar{I}_L$ by symmetry.

The basic results we must invoke from [2] involve the following convex functions on $R^n \times R^n$:

$$(3.8) \qquad f_L^A(q_0,q_1) = \inf\{I_L(q) \mid q \in A, q(t_0) = q_0, q(t_1) = q_1\} \quad ,$$

$$(3.9) \qquad f_L^B(q_0,q_1) = \inf\{\bar{I}_L(q) \mid q \in \mathcal{B}, q(t_0) = q_0, q(t_1) = q_1\} \quad ,$$

$$(3.10) \qquad f_M^B(p_0,p_1) = \inf\{\bar{I}_M(p) \mid p \in \mathcal{B}, p(t_0) = p_0, p(t_1) = p_1\} \quad .$$

Observe that the set C in (2.4) is dom f_L^A, where "dom" denotes the effective domain of a convex function (the set of points where it is not $+\infty$). Denoting by "ri" the relative interior of a convex set [5,§6], we may quote the key facts that will be needed.

THEOREM 3 [2,p.180]. *Under Assumptions* 1 *and* 2, *one has*

$$(3.11) \qquad \mathrm{ri}([Q \times Q] \cap \mathrm{dom}\, f_L^B) = \mathrm{ri\ dom}\, f_L^A = \mathrm{ri}\, C \quad ,$$

and on this set f_L^B *agrees with* f_L^A. *Furthermore, for each* $(p_0,p_1) \in R^n \times R^n$ *the infimum in* (3.10) *is attained by at least one* $p \in \mathcal{B}$, *and one has*

$$(3.12) \qquad \sup_{q_0,q_1} \{q_1 \cdot p_1 - q_0 \cdot p_0 - f_L^B(q_0,q_1)\} = \begin{cases} f_M^B(p_0,p_1) & \text{if } (p_0,p_1) \in P \times P \quad , \\ \infty & \text{otherwise.} \end{cases}$$

The first part of this result, along with the weak* lower semicontinuity of $\bar{I}_L$, is what will give us Theorem 1. Obviously

$$(3.13) \qquad \inf_{q \in A} I(q) = \inf_{q_0,q_1} \{l(q_0,q_1) + f_L^A(q_0,q_1)\} \quad ,$$

$$(3.14) \qquad \inf_{q \in \mathcal{B}} \bar{I}(q) = \inf_{q_0,q_1} \{l(q_0,q_1) + f_L^B(q_0,q_1)\} \quad .$$

Since the infimum of a convex function f on $R^n \times R^n$ is the same as the infimum of f over ri dom f [5,§7], the infimum on the right in (3. 3) is unchanged when restricted to

$$(3.15) \qquad \mathrm{ri\ dom}(l + f_L^A) = \mathrm{ri}(\mathrm{dom}\, l \cap \mathrm{dom}\, f_L^A) = \mathrm{ri}(D \cap C) \quad ,$$

while the infimum on the right in (3. 4) is unchanged when restricted to

$$\text{(3.16)}\qquad \operatorname{ri}\operatorname{dom}(l+f_L^{\mathcal{B}}) = \operatorname{ri}(\operatorname{dom} l \cap \operatorname{dom} f_L^{\mathcal{B}}) = \operatorname{ri}(D \cap [Q\times Q] \cap \operatorname{dom} f_L^{\mathcal{B}}) \quad .$$

Here we have written $D\cap[Q\times Q]$ in place of D as justified by condition (2.1) in Assumption 1.

We know from Assumptions 2 and 3 that

$$\text{(3.17)}\qquad [\operatorname{int} Q\times \operatorname{int} Q]\cap C \neq \phi \quad\text{and}\quad \operatorname{ri} D\cap \operatorname{ri} C \neq \phi \quad ,$$

and since $C = \operatorname{dom} f_L^{A} = \operatorname{dom} f_L^{\mathcal{B}}$ the first implies

$$\phi \neq [\operatorname{int} Q\times \operatorname{int} Q]\cap \operatorname{ri}\operatorname{dom} f_L^{\mathcal{B}} = \operatorname{int}[Q\times Q]\cap \operatorname{ri}\operatorname{dom} f_L^{\mathcal{B}} \quad .$$

The calculus of relative interiors [5,§6] then yields via (3.11) that

$$\text{(3.18)}\qquad \begin{aligned}\phi \neq \operatorname{ri}(D\cap C) &= \operatorname{ri} D\cap \operatorname{ri} C\\ &= \operatorname{ri} D\cap \operatorname{ri}([Q\times Q]\cap \operatorname{dom} f_L^{\mathcal{B}})\\ &= \operatorname{ri} D\cap \operatorname{int}[Q\times Q]\cap \operatorname{ri}\operatorname{dom} f_L^{\mathcal{B}}\\ &= \operatorname{ri}(D\cap[Q\times Q]\cap \operatorname{dom} f_L^{\mathcal{B}}) \quad .\end{aligned}$$

Combining this with (3.15) and (3.16), we see that the infima on the right sides of (3.13) and (3.14) are both unchanged when restricted to $\operatorname{ri} D\cap \operatorname{ri} C$. But $f_L^{\mathcal{B}}$ agrees with f_L^{A} on ri C by Theorem 3, so this allows us to conclude the two infima are equal. Thus (2.14) is true, as claimed in Theorem 1.

The other assertion of Theorem 1 is now easy to verify. If $q\in\mathcal{B}$ is the weak* limit of a minimizing sequence $\{q^j\}$ of I on A we have $\bar{I}(q^j) = I(q^j)$ and hence by the lower semicontinuity of $\bar{I}$ that

$$\inf_A I = \lim_{j\to\infty} I(q^j) = \lim_{j\to\infty} \bar{I}(q^j) \geq \bar{I}(q) \geq \inf_{\mathcal{B}}\bar{I} \quad .$$

Since the extremities of this chain are equal, as just demonstrated, we have $\bar{I}(q) = \inf_{\mathcal{B}}\bar{I}$. The proof of Theorem 1 is thereby finished.

Turning to the proof of Theorem 2, we note first that q minimizes $\bar{I}$ over $\mathcal{B}$ if and only if for a certain pair $(\bar{q}_0,\bar{q}_1)$ in $R^n\times R^n$ we have that

$$\text{(3.19)}\qquad (\bar{q}_0,\bar{q}_1) \text{ minimizes } l + f_L^{\mathcal{B}} \text{ over } R^n\times R^n \quad ,$$

$$\text{(3.20)}\qquad q \text{ provides the minimum in (3.9) for } (\bar{q}_0,\bar{q}_1) \quad .$$

We shall analyze these optimality properties separately, using the fact that when they hold we must have

$$\text{(3.21)}\qquad l(\bar{q}_0,\bar{q}_1) + f_L^{\mathcal{B}}(\bar{q}_0,\bar{q}_1) = \bar{I}(q) = l(\bar{q}_0,\bar{q}_1) + I_L(q) \text{ (finite)} \quad .$$

The analysis of (3.19) starts with the observation that it is equivalent in terms of subgradients [5,§23] to having

(3.22) $$(0,0) \in \partial(l + f_L^{\mathcal{B}})(\bar{q}_0,\bar{q}_1) \quad .$$

By (3.18) and the relation $D \in [Q \times Q] = D = \text{dom}\, l$ (cf. Assumption 1), we also know that

$$\emptyset \neq \text{ri}\, D \cap \text{int}[Q \times Q] = \text{ri}(D \cap [Q \times Q]) = \text{ri dom}\, l$$

and consequently, again via (3.18), that

$$\text{ri dom}\, l \cap \text{ri dom}\, f_L^{\mathcal{B}} \neq \emptyset$$

This is a sufficient condition for

$$\partial(l + f_L^{\mathcal{B}})(\bar{q}_0,\bar{q}_1) = \partial l(\bar{q}_0,\bar{q}_1) + \partial f_L^{\mathcal{B}}(\bar{q}_0,\bar{q}_1)$$

[5,§23], in which case (3.22) corresponds to the existence of $(\bar{p}_0,\bar{p}_1) \in R^n \times R^n$ such that

(3.23) $$(\bar{p}_0,-\bar{p}_1) \in \partial l(\bar{q}_0,\bar{q}_1) \text{ (implying } (\bar{q}_0,\bar{q}_1) \in Q \times Q),$$

(3.24) $$(-\bar{p}_0,\bar{p}_1) \in \partial f_L^{\mathcal{B}}(\bar{q}_0,\bar{q}_1) \quad .$$

Recall next that (3.24) means

$$(\bar{q}_0,\bar{q}_1) \in \underset{q_0,q_1}{\text{argmax}} \{ q_1 \cdot \bar{p}_1 - q_0 \cdot \bar{p}_0 - f_L^{\mathcal{B}}(q_0,q_1) \} \quad ,$$

which is equivalent by equation (3.12) in Theorem 3 to having

(3.25) $$(\bar{p}_0,\bar{p}_1) \in P \times P \text{ and } f_M^{\mathcal{B}}(\bar{p}_0,\bar{p}_1) = \bar{q}_1 \cdot \bar{p}_1 - \bar{q}_0 \cdot \bar{p}_0 - f_L^{\mathcal{B}}(\bar{q}_0,\bar{q}_1) \quad .$$

Altogether, then, (3.19) holds if and only if there exists $(\bar{p}_0,\bar{p}_1)$ for which (3.23) and (3.25) are both fulfilled.

Condition (3.20), on the other hand, is equivalent to

(3.26) $$(q(t_0),q(t_1)) = (\bar{q}_0,\bar{q}_1) \text{ and } \mathcal{I}_L(q) = f_L^{\mathcal{B}}(\bar{q}_0,\bar{q}_1) \quad .$$

Similarly, the infimum defining $f_M^{\mathcal{B}}(\bar{p}_0,\bar{p}_1)$ is attained by $p \in \mathcal{B}$ if and only if

(3.27) $$(p(t_0),p(t_1)) = (\bar{p}_0,\bar{p}_1) \text{ and } \mathcal{I}_M(p) = f_M^{\mathcal{B}}(\bar{p}_0,\bar{p}_1) \quad ,$$

and according to Theorem 3, therefore, there is at least one $p \in \mathcal{B}$ satisfying the latter regardless of the choice of $(\bar{p}_0,\bar{p}_1)$. It follows that (3.19) and (3.20) together are characterized by having (3.26), and for some $p \in \mathcal{B}$ satisfying (3.27), conditions (3.23) and (3.24) as well. Thus q minimizes $\mathcal{I}$ over $\mathcal{B}$ if and only if it fulfills along with some $p \in \mathcal{B}$ the conditions

$$(3.28) \qquad (p(t_0), -p(t_1)) \in \partial l(q(t_0), q(t_1))$$

$$(3.29) \qquad (q(t_0), q(t_1)) \in Q \times Q \text{ and } (p(t_0), p(t_1)) \in P \times P \quad ,$$

$$(3.30) \qquad \mathbb{I}_L(q) = f_L^{\mathcal{B}}(q(t_0), q(t_1)) \text{ and } \mathbb{I}_M(p) = f_M^{\mathcal{B}}(p(t_0), p(t_1)) \quad ,$$

$$(3.31) \qquad f_M^{\mathcal{B}}(p(t_0), p(t_1)) = q(t_1)\cdot p(t_1) - q(t_0)\cdot p(t_0) - f_L^{\mathcal{B}}(q(t_0), q(t_1)) \quad .$$

Here (3.28) is the same as condition (a) of Theorem 2, while the other three conditions are together equivalent by [2,Theorem 2] to having (b), (c), and (d) hold and the following (real-valued) measures vanish:

$$(3.32) \qquad \begin{aligned} p_+\cdot(dq - \dot{q}dt) &= p_-\cdot(dq - \dot{q}dt) = 0 \quad , \\ q_+\cdot(dp - \dot{p}dt) &= q_-\cdot(dp - \dot{p}dt) = 0 \quad , \end{aligned}$$

where $p_+(t) = p(t_+)$, and so forth. Since the function p_+ is P-valued while the singular measure $dq - qdt$ is P -valued by (d), the measure $p_+\cdot(dq - qdt)$ is in any case nonpositive, and the same is true also for the other three measures listed in (3.32). Since the identity

$$d(q\cdot p) = p_+\cdot dq + q_-\cdot dp = p_-\cdot dq + q_+\cdot dp$$

is valid for any $q \in \mathcal{B}$ and $p \in \mathcal{B}$ (cf. [2,p.161]), it follows that (3.32) is equivalent (in the presence of (c) and (d) of Theorem 2) to

$$d(q\cdot p) - [p\cdot\dot{q} + q\dot{p}]dt = 0 \quad ,$$

which is (e) of Theorem 2. This finishes the proof of Theorem 2.

REFERENCES

[1] Rockafellar R.T. , State constraints in convex control problems of Bolza, SIAM J. Control 10 (1972), 691-715 .

[2] Rockafellar R.T. , Dual problems of Lagrange for arcs of bounded variation, in: Calculus of Variations and Control (Academic Press, 1976, 155-192).

[3] Rockafellar R.T. , Duality in optimal control, in: W.A. Coppel (ed.) Mathematical Control Theory,(Springer-Verlag) Lecture Notes in Math. 680 (1978), 219-257 .

[4] Rockafellar R.T. , Existence and duality theorems for convex problems of Bolza, Trans. Amer. Math. Soc. 159(1971), 1-40 .

[5] Rockafellar R.T. , Convex Analysis. (Princeton University Press, 1970) .

[6] Clarke F.H. , The Euler-Lagrange differential inclusion, J. Diff. Eqs. 19 (1975), 80-90 .

This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, United States Air Force, under grant no. 77-3204.