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On the Interchange of Subdifferentiation and Conditional Expectation for Convex Functionals

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We show that the operators $E^{\mathscr{G}}$(conditional expectation given a σ-field $\mathscr{G}$) and ∂ (subdifferentiation), when applied to a normal convex integrand f, commute if the effective domain multifunction $\omega \to \{x \in R^n | f(\omega, x) < +\infty\}$ is $\mathscr{G}$-measurable.

We deal with interchange of conditional expectation and subdifferentiation in the context of stochastic convex analysis. The purpose is to give a condition that allows the commuting of these two operators when applied to convex integral functionals.

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $\mathscr{G}$ a σ-field contained in $\mathscr{A}$, and f an *$\mathscr{A}$-normal convex integrand* defined on $\Omega \times R^n$ with values in $R \cup \{\infty\}$. The latter means that the map

$$\omega \mapsto \operatorname{epi} f(\omega, \cdot) = \{(x, \alpha) \in R^{n+1} | \alpha \geqq f(\omega, x)\}$$

is a closed-convex-valued $\mathscr{A}$-measurable multifunction. See [2] and [9] for more on normal integrands and their properties. In particular recall that for any $\mathscr{A}$-measurable function $x: \Omega \to R^n$, the function

$$\omega \mapsto f(\omega, x(\omega))$$

is a $\mathscr{A}$-measurable and the *integral functional* associated with f is defined by

$$I_f(x) = \int f(\omega, x(\omega)) P(d\omega).$$

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To bypass some trivialities we impose the following summability conditions:

(1) there exists a $\mathcal{G}$-measurable $x:\Omega\to R^n$ such that $I_f(x)$ is finite,

(2) there exists $v\in\mathcal{L}_n^1(\mathcal{G})=\mathcal{L}^1(\Omega,\mathcal{G},P;R^n)$ such that $I_{f*}(v)$ is finite,

where f^* is the ($\mathcal{A}$-normal) conjugate convex integrand, i.e.

$$f^*(\omega,x)=\sup_{x\in R^n}[v\cdot x-f(\omega,x)]$$

Finally, we assume that $\mathcal{A}$-- and hence also $\mathcal{G}$-- is countably generated, and that there exists a *regular* conditional probability (given $\mathcal{G}$), $P^{\mathcal{G}}:\mathcal{A}\times\Omega\to[0,1]$. Whenever we refer to the conditional expectation given $\mathcal{G}$, we always mean the version obtained by integrating with respect to $P^{\mathcal{G}}$. Consequently all conditional expectations will be regular.

In particular the conditional expectation $E^{\mathcal{G}}f$ of f is the $\mathcal{G}$-normal integrand defined by

$$(E^{\mathcal{G}}f)(\omega,x)=\int f(\zeta,x)P^{\mathcal{G}}(d\zeta|\omega)$$

Also given $\Gamma:\Omega\rightrightarrows R^n$, a closed-valued $\mathcal{A}$-measurable multifunction, its conditional expectation given $\mathcal{G}$ is a closed-valued $\mathcal{G}$-measurable multifunction obtained via a projection-type operation from a set

$$\mathcal{L}_\Gamma^1=\{u\in\mathcal{L}^1(\Omega,\mathcal{A},P;R^n)|u(\omega)\in\Gamma(\omega)\text{ a.s.}\}\subset\mathcal{L}_n^1(\mathcal{A})$$

onto $\mathcal{L}_n^1(\mathcal{G})\Gamma=\mathcal{L}^1(\Omega,\mathcal{G},P;R^n)$. Valadier has shown that a regular version $E^{\mathcal{G}}\Gamma:\Omega\to R^n$ is given by the expression

$$E^{\mathcal{G}}\Gamma(\omega)=cl\{\int u(\zeta)P^{\mathcal{G}}(d\zeta|\omega)|u\in\mathcal{L}_n^1(\mathcal{A}),u(\omega)\in\Gamma(\omega)\text{ a.s.}\}.$$

We refer to [14] and the references given therein for the properties of $E^{\mathcal{G}}f$; in particular to the article of Dynkin and Estigneev [3], which specifically deals with regular conditional expectations of measurable multifunctions.

We consider I_f and $I_{E^{\mathcal{G}}f}$ as (integral) functionals on $\mathcal{L}_n^\infty(\mathcal{A})$ and $\mathcal{L}_n^\infty(\mathcal{G})$ respectively. The natural pairings of $\mathcal{L}^\infty$ and $\mathcal{L}^1$ and $(\mathcal{L}^\infty)^*$ yield for each functional two different subgradient multifunctions. We shall use ∂I_f and $\partial I_{E^{\mathcal{G}}f}$ for designating $\mathcal{L}^1$-subgradients and $\partial^* I_f$ and $\partial^* I_{E^{\mathcal{G}}f}$ for $(\mathcal{L}^\infty)^*$-subgradients. Rockafellar [8, Corollary 1B] shows that when the summability conditions (1) and (2) are satisfied, one has the following representation for $(\mathcal{L}^\infty)^*$-subgradients:

$$\partial^* I_f(x)=\{v+v_s | v\in I_f(x), v_s\in \mathscr{S}_n(\mathscr{A}) \text{ with } v_s[x-x']\leqq 0\,\forall x'\in \operatorname{dom} I_f\} \tag{3}$$

where $\mathscr{S}_n(\mathscr{A})$ is the space of *singular* continuous linear functionals on $\mathscr{L}_n^\infty(\mathscr{A})$, and

$$\operatorname{dom} I_f=\{x\in \mathscr{L}_n^\infty(\mathscr{A}) | I_f(x)<+\infty\}$$

is the effective domain of I_f. (For the decomposition of $(\mathscr{L}_n^\infty)^*$ consult [2, Chapter VIII]). Furthermore the $\mathscr{L}^1$-subgradient set is given by

$$\partial I_f(x)=\{v\in \mathscr{L}_n^1(\mathscr{A}) | v(\omega)\in \partial f(\omega, x(\omega)) \text{ a.s.}\}. \tag{4}$$

The summability conditions (1) and (2) on f imply similar properties for $E^{\mathscr{G}}f$, so the formulas above also apply to $I_{E^{\mathscr{G}}f}$. Thus for $x\in \mathscr{L}_n^\infty(\mathscr{G})$ we get

$$\partial^* I_{E^{\mathscr{G}}f}(x)=\{u+u_s | u\in \partial I_{E^{\mathscr{G}}f}(x), u_s\in \mathscr{S}_n(\mathscr{G}) \text{ with } u_s[x-x']\geqq 0, \forall x'\in \operatorname{dom} I_{E^{\mathscr{G}}f}\} \tag{5}$$

and

$$\partial I_{E^{\mathscr{G}}f}(x)=\{u\in \mathscr{L}_n^1(\mathscr{G}) | u(\omega)\in \partial E^{\mathscr{G}}f(\omega, x(w)) \text{a.s.}\}. \tag{6}$$

We are interested in the relationship between ∂I_f and $\partial I_{E^{\mathscr{G}}f}$. Relying on the formulas just given, Castaing and Valadier [2, Theorem VIII.37] show that if in place of the summability conditions (1) and (2), one makes the stronger assumption:

there exists $x^\circ\in \mathscr{L}_n^\infty(\mathscr{G})$ at which I_f is finite and norm continuous, (7)

then for every $x\in \mathscr{L}_n^\infty(\mathscr{G})$ one gets:

$$\partial I_{E^{\mathscr{G}}f}(x)=E^{\mathscr{G}}(\partial I_f(x))+rc[\partial I_{E^{\mathscr{G}}f}(x)], \tag{8}$$

where rc denotes the recession (or asymptotic) cone [2, 7]. If $x\in \operatorname{int} \operatorname{dom} I_{E^{\mathscr{G}}f}$, $\partial I_{E^{\mathscr{G}}f}(x)$ is weakly compact and then $rc[\partial I_{E^{\mathscr{G}}f}(x)]=\{0\}$, in which case

$$\partial I_{E^{\mathscr{G}}f}(x)=E^{\mathscr{G}}\partial I_f(x). \tag{9}$$

This was already observed by Bismut [1, Theorem 4]. For the subspace of

$\mathcal{L}_n^\infty$ of constant functions, Hiriart-Urruty [4] obtain a similar result for the ε-subdifferentials of convex functions. For finite-valued Lipschitz integrands, Thibauld [12, Proposition 4.7] obtained recently a related result involving Clarke generalized subgradients.

Here we shall go one step further and provide a condition under which the rc term can be dropped from the identity (8) without requiring that $x \in \operatorname{int} \operatorname{dom} I_f$. Very simple examples show that the rc term is sometimes inescapable in (8). For instance, suppose $\mathscr{G} = \{\phi, \Omega\}$ (so $E^{\mathscr{G}} = E$) and consider $f(\omega, \cdot) = \psi_{(-\infty, \xi(\omega)]}$, the indicator of the unbounded interval $(-\infty, \xi(\omega)]$, where ξ is a random variable uniformly distributed on $[0,1]$. In this case $\psi_{(-\infty,0]} = Ef = E^{\mathscr{G}} f = I_{E^{\mathscr{G}} f}$, so that $\partial I_{E^{\mathscr{G}} f}(0) = R_+$ but $E^{\mathscr{G}}(\partial I_f(0)) = E\{0\} = \{0\}$. Thus (8) would fail without the rc term. Another example appears in [13, p. 63] where it is the condition of the Theorem: "If $(x) < +\infty$ for every $x \in \mathscr{L}_n^\infty(\mathscr{G})$ such that $x(\omega) \in \operatorname{dom} f(\omega, \cdot)$", that fails to be satisfied.

THEOREM *Suppose f is an $\mathscr{A}$-normal convex integrand such that the closure of its effective domain multifunction*

$$\omega \mapsto D(\omega) := cl \operatorname{dom} f(\omega, \cdot) = cl\{x \in R^n \,|\, f(\omega, x) < +\infty\} \tag{10}$$

is $\mathscr{G}$-measurable. Assume that $I_f(x) < +\infty$ for every $x \in \mathscr{L}_n^\infty(\mathscr{G})$ such that $x(\omega) \in \operatorname{dom} f(\omega, \cdot)$ a.s., and that there exists $x^0 \in \mathscr{L}_n^\infty(\mathscr{G})$ at which I_f is finite and norm continuous. Then for every $x \in \mathscr{L}_n^\infty(\mathscr{G})$ one has

$$\partial E^{\mathscr{G}} f(\cdot, (x(\cdot)) = E^{\mathscr{G}} \partial f(\cdot, x(\cdot)) \text{ a.s.}, \tag{11}$$

or in other words, the closed-valued $\mathscr{G}$-measurable multifunctions

$$\omega \mapsto \partial E^{\mathscr{G}} f(\omega, x(\omega))$$

and

$$\omega \mapsto E^{\mathscr{G}}[\partial f(\cdot, x(\cdot))](\omega)$$

are almost surely equal.

Proof. From (8) it follows that

$$\partial I_{E^{\mathscr{G}} f}(x) \supset E^{\mathscr{G}}(\partial I_f(x)).$$

In view of (6) and (4) this holds if and only if

$$\partial E^{\mathscr{G}} f(\cdot, x(\cdot)) \supset E^{\mathscr{G}} \partial f(\cdot, x(\cdot)) \; a.s.$$

It thus suffices to prove the reverse inclusion. Let us suppose that $u \in \partial E^{\mathscr{G}} f(\cdot, x(\cdot))$. For every $y \in R^n$, define

$$g(\omega, y) = f(\omega, y) - u(\omega) \cdot y.$$

This is an $\mathscr{A}$-normal convex integrand which inherits all the properties assumed for f in the Theorem (recall that $u \in \mathscr{L}_n^1(\mathscr{G})$). Moreover $0 \in \partial E^{\mathscr{G}} g(\cdot, x(\cdot))$. We shall show that $0 \in E^{\mathscr{G}} \partial g(\cdot, x(\cdot))$, which in turn will imply that $u \in E^{\mathscr{G}} \partial f(\cdot, x(\cdot))$ and thereby complete the proof of the Theorem.

Since almost surely $0 \in \partial E^{\mathscr{G}} g(\omega, x(\omega))$, we know that $0 \in \partial I_{E^{\mathscr{G}}g}(x) \subset \partial^* I_{E^{\mathscr{G}}g}(x)$. Hence x minimizes $I_{E^{\mathscr{G}}g}$ on $\mathscr{L}_n^\infty(\mathscr{G})$. Let inj denote the natural injection of $\mathscr{L}_n^\infty(\mathscr{G})$ into $\mathscr{L}_n^\infty(\mathscr{A})$ with

$$\mathscr{W} = \text{inj}\,[\mathscr{L}_n^\infty(\mathscr{G})].$$

Now note that inj $x = x$ also minimizes $I_{E^{\mathscr{G}}g}$ on $\mathscr{W} \subset \mathscr{L}_n^\infty(\mathscr{A})$, or equivalently I_g on $\mathscr{W}$, since the two integral functionals coincide on $\mathscr{W}$ (by the definition of conditional expectation.) Thus

$$0 \in \partial^*(I_g + \psi_{\mathscr{W}})(x),$$

where $\psi_{\mathscr{W}}$ is the indicator function of $\mathscr{W}$, or equivalently:

$$0 \in \partial^* \mathrm{I}_g(x) + \partial^* \psi_{\mathscr{W}}(x),$$

since g is (norm) continuous at some $x^0 = \text{inj}\, x^0 \in \mathscr{W}$. By (3), this means that there exist $v \in \mathscr{L}_n^1(\mathscr{A})$, $v_s \in \mathscr{S}_n(\mathscr{A})$, such that

$$v(\omega) \in \partial g(\omega, x(\omega)) \text{ a.s.,} \tag{12}$$

$$v_s[x - x'] \geqq 0 \qquad \text{for all } x' \in \text{dom}\, I_g, \tag{13}$$

and $-(v + v_s)$ is orthogonal to $\mathscr{W}$, i.e.

$$(v + v_s)\,[x'] = 0 \qquad \text{for all } x' \in \mathscr{W}. \tag{14}$$

This last relation can also be expressed as

$$(v + v_s)\,[\text{inj}\, y] = 0 \qquad \text{for all } y \in \mathscr{L}_n^\infty(\mathscr{G}),$$

or still for all $y \in \mathscr{L}_n^\infty(\mathscr{G})$

$$\text{inj}^*(v + v_s)\,[y] = 0,$$

where $\text{inj}^* : (\mathscr{L}_n^\infty(\mathscr{A}))^* \to (\mathscr{L}_n^\infty(\mathscr{G}))^*$ is the adjoint of inj. Thus the continuous linear functional $\text{inj}^*(v + v_s)$ must be identically 0 on $\mathscr{L}_n^\infty(\mathscr{G})$, i.e. on $\mathscr{L}_n^\infty(\mathscr{G})$

one has

$$\text{inj}^* v_s = -\text{inj}^* v = -E^{\mathscr{G}} v. \tag{15}$$

The last equality follows from the observation that $E^{\mathscr{G}} = \text{inj}^*$ when inj* is restricted to $\mathscr{L}_n^1(\mathscr{A})$, cf. [2, p.265] for example.

We shall complete the proof by showing that the assumptions (12), (13) and (15) imply that

$$(v - E^{\mathscr{G}} v)(\omega) \in \partial g(\omega, x(\omega)) \text{ a.s.} \tag{16}$$

This will certainly do, since it trivially yields the sought-for relation

$$0 = E^{\mathscr{G}}(v - E^{\mathscr{G}} v) \in E^{\mathscr{G}} \partial g(\cdot, x(\cdot)).$$

To obtain (16), it will be sufficient to show that

$$E\{(-E^{\mathscr{G}} v) \cdot [x - y]\} \geqq 0 \tag{17}$$

for all $y \in \text{dom}\, I_g \subset \mathscr{L}_n^\infty(\mathscr{A})$. To see this, recall that the relations (17) and $v \in \partial I_g(x)$ (cf. (12)) imply that $v - E^{\mathscr{G}} v \in \partial I_g(x)$, from which (16) follows via the representation of $\mathscr{L}^1$-subgradients given by (4). In fact, because the effective domain multifunction, or more precisely its closure $\omega \mapsto D(\omega)$, is $\mathscr{G}$-measurable, it is sufficient to show that (17) holds for every $y \in \text{dom}\, I_g \cap \mathscr{W}$. Suppose to the contrary that (17) holds for every $y \in \text{dom}\, I_g \cap \mathscr{W}$ -- or equivalently because of the $\leqq$ inequality that (17) holds for every $y \in cl$ dom $I_g \cap \mathscr{W}$ -- but there exists $\hat{y} \in \mathscr{L}_n^\infty(\mathscr{A})$ such that $I_g(\hat{y}) < +\infty$ and for which (17) fails, i.e. we have

$$E\{(-E^{\mathscr{G}} v) \cdot [x - \hat{y}]\} < 0.$$

Because $-E^{\mathscr{G}} v$ and x are $\mathscr{G}$-measurable, this inequality implies that

$$E\{(-E^{\mathscr{G}} v) \cdot [-E^{\mathscr{G}} \hat{y}]\} < 0. \tag{18}$$

Moreover, since $I_g(\hat{y}) < +\infty$, it follows that almost surely

$$\hat{y}(\omega) \in \text{dom}\, g(\omega, \cdot) \subset D(\omega).$$

Taking conditional expectation on both sides, we see that

$$(E^{\mathscr{G}} \hat{y})(\omega) \in E^{\mathscr{G}} D(\omega) = D(\omega), \tag{19}$$

because D is a closed-value $\mathscr{G}$-measurable multifunction. Naturally $E^{\mathscr{G}}\hat{y}\in\mathscr{W}$. Because I_g is by assumption finite on $\{z\in\mathscr{L}_n^\infty(\mathscr{G})| z(\omega)\in \text{dom } g(\omega,\cdot)\text{a.s.}\}$, and $D(\omega)=cl \text{ dom } g(\omega,\cdot)$, it follows from (19) that $E^{\mathscr{G}}\hat{y}\in cl \text{ dom } I_g$. Hence (17) cannot hold for every $y\in\text{dom } I_g\cap\mathscr{W}$ since $E^{\mathscr{G}}\hat{y}$ belongs to $(cl \text{ dom } I_g)\cap\mathscr{W}$ and satisfies (18).

There remains only to show that (17) holds for every $y\in\mathscr{L}_n^\infty(\mathscr{G})$ such that inj $y=y\in\text{dom } I_g$. But now from (13) we have that for each such y

$$v_s[x-y]=v_s[\text{inj } x-\text{inj } y]\geqq 0,$$

or again equivalently: for each $y\in\text{dom } I_g\cap\mathscr{L}_n^\infty(\mathscr{G})$,

$$(\text{inj}^* v_s)[x-y]\geqq 0.$$

But this is precisely (17), since we know from (15) that on $\mathscr{L}_n^\infty(\mathscr{G})$, $\text{inj}^*v_s= -E^{\mathscr{G}}v$. □

COROLLARY *Suppose f is a $\mathscr{A}$-normal convex integrand such that $F(x)< +\infty$ whenever $x\in\text{dom} f(\omega,\cdot)$ a.s., where*

$$F(x)=E\{f(\omega,x)\}.$$

Suppose moreover that there exists $x^0\in R^n$ at which F is finite and continuous, and that the multifunction

$$\omega\mapsto D(\omega)=cl\,\text{dom} f(\omega,\cdot)$$

is almost surely constant. Then for all $x\in R^n$,

$$E[\partial f(\cdot,x)]=\partial F(x), \tag{20}$$

where the expectation of the closed-valued measurable multifunction Γ is defined by

$$E\Gamma=cl\{\textstyle\int v(\omega)P(d\omega)\big|v\in\mathscr{L}_n^1(\mathscr{A}), v(\omega)\in\Gamma(\omega)\text{ a.s.}\}$$

Proof Just apply the Theorem with $G=\{\phi,\Omega\}$, and identify the class of constant functions--the $\mathscr{G}$-measurable functions--with R^n. □

This Corollary was first derived by Ioffe and Tikhomirov [5] and later generalized by Levin [6]. Note that our definition of the expectation of a closed-valued measurable multifunction is at variance with the definition now in vogue for the integral of a measurable multifunction, which does not involve the closure operation. (Otherwise the definition of the integral of a multifunction would be inconsistent with that of its conditional expectation, in particular with respect to $\mathscr{G}=\{\phi,\Omega\}$, and also when $\Gamma\to E\Gamma$ is viewed as an integral on a space of closed sets it could generate an element that it is not an element of that space.)

Application

Consider the *stochastic optimization problem*:
find

$$\inf E[f(\omega,x_1(\omega),x_2(\omega))] \text{ over all } x_1\in\mathscr{L}^\infty_{n_1}(\mathscr{G}), x_2\in\mathscr{L}^\infty_{n_2}(\mathscr{A}), \tag{21}$$

where $\mathscr{A}$ *and* $\mathscr{G}$ are as before, and f is an $\mathscr{A}$-normal convex integrand which satisfies the norm-continuity condition:
there exists

$$(x_1^0,x_2^0)\in\mathscr{L}^\infty_{n_1}(\mathscr{G})\times\mathscr{L}^\infty_{n_2}(\mathscr{A})$$

at which I_f is finite and norm continuous. Suppose also that the effective domain multifunction

$$\omega\to\operatorname{dom}f(\omega,\cdot,\cdot)=\{(x_1,x_2)\in R^{n_1}\times R^{n_2}\mid f(\omega,x_1,x_2)<+\infty\}$$

is uniformly bounded and that there exists a summable function $h\in\mathscr{L}^1(\mathscr{A})$ such that $(x_1,x_2)\in\operatorname{dom}f(\omega,\cdot\cdot)$ implies that $|f(\omega,x_1,x_2)|\leqq h(\omega)$. Finally suppose that the multifunction

$$\omega\mapsto D_1(\omega)=cl\{x_1\in R^{n_1}\mid \exists x_2\in R^{n_2} \text{ such that } f(\omega,x_1,x_2)<+\infty\}$$

is $\mathscr{G}$-measurable. For a justification and discussion of these assumptions cf. [11, Section 2]. From Theorem 1 of [11], it follows that the problem: find

$$\inf E[g(\omega,x_1(\omega))] \text{ over all } x_1\in\mathscr{L}^\infty_{n_1}(\mathscr{G}), \tag{23}$$

where

$$g(\omega,x_1)=E^{\mathscr{G}}[\inf_{x_2\in R^{n_2}}f(\cdot,x_1,x_2)\}(\omega),$$

is equivalent to (21) in the sense that if $(\bar{x}_1,\bar{x}_2)$ solves (21), then $\bar{x}_1$ solves (23), and similarly any solution x_1 of (23) can be "extended" to a solution (x_1,x_2) of (21). Both problems also have the same optimal value.

The hypotheses imply that

$$(\omega,x_1)\mapsto\inf_{x_2}f(\omega,x_1,x_2)$$

is an $\mathscr{A}$-normal convex integrand, since the multifunction $\omega \mapsto \text{epi}(\inf_{x_2} f(\omega, x_1, x_2))$ is closed-convex-valued and $\mathscr{A}$-measurable. Its effective domain multifunction, or more precisely

$$\omega \mapsto D_1(\omega) := cl\,\text{dom}\,q(\omega, \cdot),$$

is $\mathscr{G}$-measurable. Combining (11) with the representation for the subgradients of infimal functions [15, VIII.4], we have that for every $x_1 \in \mathscr{L}^{\infty}_{n_1}(\mathscr{G})$

$$\partial q(\cdot, x_1(\cdot)) = E^{\mathscr{G}}\{v(\omega) | (v(\omega), 0) \in_{\text{a.s.}} \partial f(\omega, x_1(\omega), x_2)$$

$$\text{for some } x_2 \in R^{n_2}\}(\cdot),$$

from which Theorem 2, the main result of [11], follows directly.

Remark If the underlying probability measure P has finite support, then $(\mathscr{L}^{\infty}_n)^* = \mathscr{L}^1_n$, and (11) and (20) are satisfied without any other restriction.

On the other hand, if P is nonatomic, and the effective domain multifunction (or its closure) is not $\mathscr{G}$-measurable, then the identities (11) and (20) do not apply. More precisely, suppose that there exists a subset C of R^n such that the $\mathscr{A}$-measurable set

$$\{\omega | \text{dom} f(\omega, \cdot) \cap C \neq \phi\}$$

has (strictly) positive mass and is not $\mathscr{G}$-measurable. Then the term $rc[\partial I_{E^{\mathscr{G}}f}(x)]$ can never be dropped from the representation of $\partial I_{E^{\mathscr{G}}f}$ given by (8), as can be seen from an adaptation of the arguments in Section 4 of [10]. In those cases the inclusion $E^{\mathscr{G}} \partial f \subset \partial E^{\mathscr{G}} f$ will be strict for at least some $x \in \mathscr{L}^{\infty}_n(\mathscr{G})$.

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