A DUAL SOLUTION PROCEDURE FOR QUADRATIC

STOCHASTIC PROGRAMS WITH SIMPLE RECOURSE

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Abstract

We exhibit *a* dual of a stochastic program with simple recourse -- with random parameters in the technoloty matrix and the right-hand sides,and with quadratic recourse costs -- that is essentially a deterministic quadratic program except for some simple stochastic upper bounds. We then describe a solution procedure for problems of this type based on a finite element representation of the dual variables.

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We consider the following class of quadratic stochastic programs with simple recourse:

(0.1) find $x \in R^n$ such that

$$0 \le x_j \le r_j, \qquad j = 1,\dots,n$$

$$\sum_{j=i}^{n} a_{ij}x_j \le b_i \quad i = 1,\dots,m$$

and

$$\sum_{j=1}^{n}(c_jx_j - \frac{d_j}{2r_j}x_j^2) - E\left\{\sum_{h=1}^{\ell} q_h(\omega)e_h\theta(e_h^{-1}v_h(\omega)\right\}$$

is maximized, where

(0.2) $$v_h(\omega) = \sum_{j=1}^{n} t_{hj}(\omega)x_j - p_h(\omega).$$

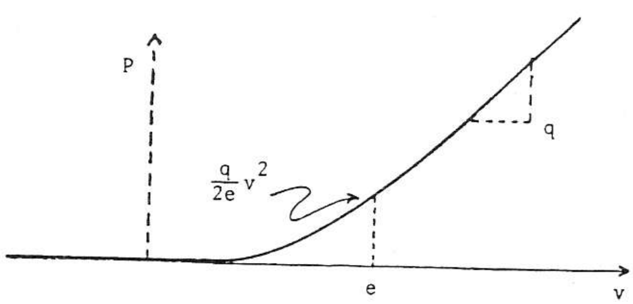
The function θ is defined by

$$\theta(\tau) = \begin{cases} 0 & \text{if } \tau \le 0, \\ \tau^2/2 & \text{if } 0 \le \tau \le 1, \\ \tau - 1/2 & \text{if } \tau \ge 1; \end{cases}$$

so that the recourse cost function

$$\rho_h(v_h) = q_he_h\,\theta\,(e_h^{-1}v_h)$$

has the form



0.3 Figure: recourse cost function

In the limit as e_h goes to 0, the function ρ_h tends to the piecewise linear function ρ_h^ℓ with

$$\rho_h^\ell(v_h) = 0 \quad \text{if } v_h \le 0,$$
$$= q_h v_h \text{ if } v_h \ge 0.$$

which brings us to the case of stochastic programs with simple recourse and linear recourse costs [1]. Note that there is no loss of generality in having ρ_h^ℓ and ρ_h with slope 0 when $v_h \le 0$. If the original problem is not of this form, a simple transformation involving an adjustment of the $(c_j, j=1,\ldots,n)$ and the $(q_h, h=1,\ldots,\ell)$ will reduce the original problem to the canonical form (0.1).

The coefficients

$$q_h(\cdot), \quad h=1,\ldots,\ell$$
$$t_{hj}(\cdot), \quad j=1,\ldots,n; \quad h=1,\ldots,\ell$$
$$p_h(\cdot), \quad h=1,\ldots,\ell$$

are random variables with known distribution function. We assume that these random variables have second moments so that the $v_h(\cdot)$ defined through (0.2) also have finite second moments. Consequently the expectation that appears in the objective of (0.1) is well-defined. We shall assume that (0.1) is solvable, i.e., that exists a vector x^* that solves (0.1); in particular this implies that the linear system

$$0 \le x_j \le r_j, \quad j=1,\ldots,n; \quad \sum_{j=1}^n a_{ij} x_j \le b_i, \quad i=1,\ldots m,$$

is feasible. The coefficients r_j, d_j for $j=1,\ldots,n$, and e_h for $h=1,\ldots,\ell$ as well as the random variables $q_h(\cdot)$ are strictly positive. In particular this guarantees the concavity of the object.

We regard model (0.1) as the quadratic version of the simple recourse problem [2] involving random coefficients in the technology matrix, the cost and the right hand sides.

In the next section we show that the following problem (0.4) is dual to the quadratic stochastic programs with simple recourse:

(0.4) find $y \in R^m$ and $z(\cdot): \Omega \to R^\ell$ such that

$$y_i \geq 0 \qquad i=1,\ldots,m$$

$$0 \leq z_h(\omega) \leq q_h(\omega) \text{ a.s. } \quad h=1,\ldots,\ell$$

and

$$\sum_{i=1}^m b_i y_i + E\sum_{h=1}^{\ell} \left\{p_h(\omega) z_h(\omega) + \frac{e_h}{2q_h(\omega)} z_h^2(\omega)\right\}$$

$$+\sum_{j=1}^n r_j d_j \theta(d_j^{-1} w_j)$$

is minimized, where $j=1,\ldots,m$,

$$w_j = c_j - \sum_{i=1}^m a_{ij} y_i - E\left(\sum_{h=1}^{\ell} z_h(\omega) t_{hj}(\omega)\right). \tag{0.5}$$

Although this problem is related to the dual problem that would be obtained by a straight forward application of the results of [3] these are significant differences. It is the specific structure of *this* dual problem which is exploited in the algorithmic procedure described in Section 2.

Our work was originally motivated by a problem coming from the division of IIASA (International Institute for Applied Systems Analysis) dealing with Resources and Environment; given the hydrodynamic flow, highly affected by atmospheric conditions, between subbasins of a given shallow lake, one needs to design (size) and locate tertiary treatment plants that will filter the inflow so as to minimize (in a least square sense) the deviation between the observed concentration of certain pollutants and given desirable levels. Here both $p(\cdot)$ and $T(\cdot)$ were random but q was fixed (nonstochastic).

1. DUALITY AND ITS DERIVATION

The primal problem (0.1) and dual problem (0.4) are linked together as the two halves of a certain minimax problem. Let

(1.1) $X = \{x=(x_1,\dots,x_n) \in R^n \mid 0 \le x_j \le r_j\}$,

$Y = \{ y=(y_1,\dots,y_m) \in R^m \mid 0 \le y_j\}$,

$Z = \{ z(\cdot)=\big(z_1(\cdot),\dots,z_\ell(\cdot)\big): \Omega \to R^\ell \mid 0 \le z_h(\omega) \le q_h(\omega) \text{ a.s.}\}$

(where the functions $z_h(\cdot)$ are assumed to be measurable and are in fact square integrable, because the functions $q_h(\cdot)$ are). Define the function L on XxYxZ by

$$L\big(x,y,z(\cdot)\big)=\sum_{j=1}^{n}\left(c_j x_j - \frac{d_j}{2r_j}x_j^2\right) + E\left\{\sum_{h=1}^{\ell}\left(p_h(\omega)z_h(\omega)+\frac{e_h}{2q_h(\omega)}z_h^2(\omega)\right)\right\} \tag{1.2}$$

$$+\sum_{i=1}^{m}y_i b_i - \sum_{j=1}^{n}\left(\sum_{i=1}^{m}y_i a_{ij} + E\left\{\sum_{h=1}^{\ell}z_h(\omega)t_{hj}(\omega)\right\}\right)x_j .$$

This function is obviously quadratic concave in x for fixed $\big(y,z(\cdot)\big)$ and quadratic convex in $\big(y,z(\cdot)\big)$ for fixed x. Two optimization problems are naturally associated with it, namely

(1.3) maximize $f(x)$ over all $x \in X$, where

$$f(x)=\inf_{(y,z(\cdot))\in Y\times Z} L\big(x,y,z(\cdot)\big)$$

and

minimize $g\big(y,z(\cdot)\big)$ over all $\big(y,z(\cdot)\big)\in Y\times Z$, where

$$g\big(y,z(\cdot)\big)=\sup_{x\in X}L\big(x,y,z(\cdot)\big).$$

As is well known in optimization theory, no matter what the choice of the sets X,Y and Z and the formula for L, the saddlepoint condition

$$L\big(x,\bar y,\bar z(\cdot)\big) \le L\big(\bar x,\bar y,\bar z(\cdot)\big) \le L\big(\bar x,y,z(\cdot)\big) \quad \text{for all } x \in X,\ \big(y,z(\cdot)\big)\in Y\times Z \tag{1.5}$$

is satisfied by elements $\bar x \in X$ and $\big(\bar y,\bar z(\cdot)\big) \in Y\times Z$ if and only if $\bar x$ gives the maximum in problem (1.3), $\big(\bar y,\bar z(\cdot)\big)$ gives the minimum in problem (1.4) and the optimal values in these two problems are equal.

In fact (1.3) and (1.4) can be identified with our primal and dual problems (0.1) and (0.4), so the assertions just made are true of the latter. This is shown by direct calculation: one has from the formulas in (1.3) and (1.4) and the definitions (1.1) and (1.2) that

$$f(x) = \begin{cases} \sum_{j=1}^{n} \left(c_j x_j - \frac{d_j}{2r_j} x_j^2\right) - E\left\{\sum_{h=1}^{n} q_h(\omega) e_h \theta\left(e_h^{-1} v_h(\omega)\right)\right\} \\ \qquad\qquad \text{if } \sum_{j=1}^{n} a_{ij} x_j \le b_i \text{ for } i=1,\dots,m \\ -\infty \text{ otherwise} \end{cases}$$

where $v_h(\omega)$ is given by (0.2), and

$$g(y,z(\cdot)) = \sum_{i=1}^{m} b_i y_i + E\left\{\sum_{h=1}^{\ell} \left(\rho_h(\omega) z_h(\omega) + \frac{e_h}{2q_h(\omega)} z_h(\omega)^2\right\}\right.$$

$$+ \sum_{j=1}^{n} r_j d_j \theta\left(d_j^{-1} w_j\right),$$

where w_j is given by (0.5). The calculation makes use of the fact that the conjugate of the function θ is

$$\theta^*(t) = \sup_{\tau \in R} \{t\tau - \theta(\tau)\} = \begin{cases} t^2/2 \text{ if } 0 \le t \le 1, \\ \infty \text{ otherwise} \end{cases}$$

DUALITY THEOREM. *Suppose that the primal problem* (0.1) *is feasible, i.e., that there exists* $x \in R^n$ *satisfing*

(1.6) $\quad 0 \le x_j \le r_j$ *for* $j=1,\dots,n$, *and* $\sum_{j=1}^{n} a_{ij} x_j \le b_j$ *for* $i=1,\dots,m$.

Then the primal problem (0.1) *has an optimal solution* $\bar{x}$, *the dual problem* (0.4) *has an optimal solution* $(\bar{y},\bar{z}(\cdot))$, *and the optimal values in the two problems are equal. Moreover,* $\bar{x}$ *and* $(\bar{y},\bar{z}(\cdot))$ *are optimal if and only if the saddlepoint condition* (1.5) *is fulfilled.*

PROOF. These assertions will follow from the general observations above, once it is shown that there do exist $\bar{x} \in X$ and $(\bar{y},\bar{z}(\cdot)) \in Y\text{x}Z$ satisfying the saddle-point condition. To show this we consider an auxiliary minimax problem in terms of the function

$$L_o(x,z(\cdot)) = \sum_{j=1}^{n}\left(c_jx_j - \frac{d_j}{2r_j}x_j\right) + E\left\{\sum_{h=1}^{\ell}\left(p_h(\omega)z_h(\omega) + \frac{e_h}{2q_h(\omega)}z_h^2(\omega)\right)\right\}$$

$$+ \sum_{j=1}^{n} E\left\{\sum_{h=1}^{\ell} z_h(\omega)t_{hj}(\omega)\right\}x_j$$

on $X_o xZ$, where X_o consists of the vectors x which satisfy (1.6). (Note that L_o differs from L only in the absence of all y terms.) Again $L_o(x,z(\cdot))$ is concave in x and convex in $z(\cdot)$ and it is continuous in x and $z(\cdot)$ relative to the usual topology on $X \subset R^n$ and the norm topology that Z receives as a subset of a Hilbert space of square integrable functions. Any convex function which is continuous in the norm topology on a Hilbert space is also lower semicontinuous in the weak topology, and in the latter topology the convex set Z is compact. Of course the convex set X_o is also compact. Thus we are dealing with a function on a product of two nonempty compact convex sets, which *is in* particular upper semicontinuous and concave in the first argument and lower semicontinuous and convex in the second. According to the minimax theorem of Ky Fan, see [4], such a function is sure to have a saddlepoint.

Denote such a saddlepoint by $(\bar{x},\bar{z}(\cdot))$: one has $\bar{x} \in X$, $\bar{z}(\cdot) \in Z$ and

(1.7) $$L_o(x,\bar{z}(\cdot)) \le L_o(\bar{x},\bar{z}(\cdot)) \le L_o(\bar{x},z(\cdot)) \quad \text{for all } x \in X_o,\ z(\cdot) \in Z.$$

Since the quadratic concave function $x \mapsto L_o(x,\bar{z}(\cdot))$ attains its maximum at $\bar{x}$ relative to the set X_o, i.e., relative to the linear constraints (1.6), there exists a Lagrange multiplier vector $\bar{y} \in Y$ such that

(1.8) $$L_o(x,\bar{z}(\cdot)) + \sum_{i=1}^{m}\bar{y}_i\left(b_i - \sum_{j=1}^{n}a_{ij}x_j\right)$$

$$\le L_o(\bar{x},\bar{z}(\cdot)) + \sum_{i=1}^{m}\bar{y}_i\left(b_i - \sum_{j=1}^{n}a_{ij}\bar{x}_j\right)$$

$$\le L_o(\bar{x},\bar{z}(\cdot)) + \sum_{i=1}^{m}y_i\left(b_i - \sum_{j=1}^{n}a_{ij}\bar{x}_j\right)$$

for all $x \in X$ and $y \in Y$.

Inasmuch as

$$L_o\big(x,\bar z(\cdot)\big) + \sum_{i=1}^{m} y_i\big(b_i - \sum_{j=1}^{n} a_{ij}x_j\big) = L\big(x,y,\bar z(\cdot)\big)$$

by definition, the combination of (1.7) and (1.8) is equivalent to the desired saddlepoint condition (1.5) thus $(\bar x,\bar y,\bar z(\cdot))$ is a saddlepoint of L on XxYxZ. □

COROLLARY. *Suppose* $(\bar y,\bar z(\cdot))$ *is an optimal solution to the dual problem* (0.4). *Then the unique optimal solution* $\bar x$ *to the primal problem* (0.1) *is given by*

$$\bar x_j = \operatorname{argmax}\left\{w_j x_j - \frac{d_j}{2r_j}x_j^2\right\} \tag{1.9}$$

$$= \begin{cases} 0 \text{ if } w_j < 0 \\ r_j w_j/d_j \text{ if } 0 \le w_j \le d_j \\ r_j \text{ if } w_j > d_j, \end{cases}$$

where w_j *is given by* (0.5).

The corollary follows from the saddlepoint condition: $L(x,\bar y,\bar z(\cdot))$ must achieve its maximum over X at $\bar x$, and this expression is strictly concave,

$$L\big(x,\bar y,\bar z(\cdot)\big) = \sum_{j=1}^{n}\big(w_j x_j - \frac{d_j}{2r_j}x_j^2\big).$$

2. A SOLUTION PROCEDURE FOR THE DUAL PROBLEM

We are concerned with problem (0.4), repeated here for convenient reference,

(2.1) find $y \in R_+^m$ and $z(\cdot): \Omega \to R^\ell$ measurable such that

$0 \le z_h(\omega) \le q_h(\omega)$ a.s. $h=1,\dots,\ell$

and $\Phi(y,z)$ is minimized,

where

$$\Phi(y,z) = \sum_{i=1}^{m} b_i y_i + \sum_{h=1}^{\ell} E\left\{p_h(\omega)z_h(\omega) + \frac{e_h}{2q_h(\omega)}z_h^2(\omega)\right\} + \sum_{j=1}^{n} r_j d_j \theta\big(d_j^{-1}w_j\big) \tag{2.2}$$

with, for $j=1,\dots,n$

(2.3) $$w_j = c_j - \sum_{i=1}^{m} a_{ij} y_i - E\left\{\sum_{h=1}^{\ell} z_h(\omega) t_{hj}(\omega)\right\}.$$

Here Ω denotes the support (the smallest closed set of measure 1) of the random variables. It has been shown [5] that the solution to (2.1) remains unaffected if the condition

$$0 \le z_h(\omega) \le q_h(\omega) \quad \text{a.s.}$$

is replaced by the condition

(2.4) $$0 \le z_h(\omega) \le q_h(\omega) \quad \text{for all } \omega \in \Omega.$$

It is this last version of these constraints that we shall use.

The main idea of the algorithm is to substitute for (2.1) a finite dimensional approximation based on a finite element representation of (2.1) for z. We restrict $z_h(\cdot)$ to the linear span of a finite collection of functions, i.e.,

$$z_h(\cdot) = \sum_{k=1}^{\nu} \lambda_{hk} \zeta_{hk}(\cdot)$$

where the $\zeta_{hk}(\cdot)$ are given and the $\lambda_{hk} \in R$. With this representation for z, problem (2.1) becomes:

(2.5) find $y \in R_+^m$ and $\lambda_{hk} \in R$ for $k=1,\dots,\nu$, $h=1,\dots,\ell$, such that

$$0 \le \sum_{k=1}^{\nu} \lambda_{hk} \zeta_{hk}(\omega) \le q_h(\omega) \quad \text{for all } \omega \in \Omega,\ h=1,\dots,\ell$$

$$w_j = c_j - \sum_{i=1}^{m} a_{ij} y_i - \sum_{k=1}^{\nu} \lambda_{hk} E\{\zeta_{hk}(\omega) t_{hj}(\omega)\} \quad \text{for } j=1,\dots,n,$$

and $\Phi^{\nu}(y,\lambda)$ is minimized

where

$$\begin{aligned}\Phi^{\nu}(y,\lambda) = & \sum_{i=1}^{m} b_i y_i \\ & + \sum_{h=1}^{\ell} \sum_{k=1}^{\nu} \lambda_{hk} E\{\zeta_{hk}(\omega) p_h(\omega)\} \\ & + \sum_{h=1}^{\ell} \Big(\sum_{k=1}^{\nu} \sum_{k'=1}^{\nu} \lambda_{hk} \lambda_{hk'} E\left\{\frac{e_h}{2q_h(\omega)} \zeta_{hk}(\omega) \zeta_{hk'}(\omega)\right\}\Big) \\ & + \sum_{j=1}^{n} r_j d_j \theta(d_j^{-1} w_j).\end{aligned}$$

Let us denote the integrals that appear in (2.5) by

$$\bar{t}_{hkj} = E\{\zeta_{hk}(\omega) t_{hj}(\omega)\},$$

$$\bar{p}_{hk} = E\{\zeta_{hk}(\omega) p_h(\omega)\},$$

and

$$\bar{e}_{hkk'} = E\left\{\frac{e_h}{q_h(\omega)}\,\zeta_{hk}(\omega)\zeta_{hk'}(\omega)\right\},$$

we then get the following form for (2.5):

(2.6) find $y \in R^m_+$ and $\lambda_{hk} \in R$ for $k=1,\ldots,\nu$, $h=1,\ldots,\ell$ such that for

$$w_j = c_j - \sum_{i=1}^m a_{ij} y_i - \sum_{h=1}^{\ell} \sum_{k=1}^{\nu} \lambda_{hk} \bar{t}_{hkj} \quad \text{for } j=1,\ldots,n,$$

$\Phi^\nu(y,\lambda)$ is minimized

and

(2.7) $$0 \le \sum_{k=1}^{\nu} \lambda_{hk} \zeta_{hk}(\omega) \le q_h(\omega) \quad \text{for all } \omega \in \Omega,\ h=1,\ldots,\ell.$$

The function Φ^ν takes on the form

(2.8) $$\Phi^\nu(y,\lambda) = \sum_{h=1}^{\ell} \left(\sum_{k=1}^{\nu} \bar{p}_{hk}\lambda_{hk} + \tfrac{1}{2}\sum_{k=1}^{\nu} \sum_{k'=1}^{\nu} \bar{e}_{hkk'}\lambda_{hk}\lambda_{hk'}\right)$$

$$\sum_{i=1}^m b_i y_i + \sum_{j=1}^n r_j d_j\, \theta(d_j^{-1} w_j)$$

Except for the stochastic constraints (2.7) this is a deterministic quadratic program for which efficient subroutine are available; for example MINOS [6]; recall that θ is a piece-wise quadratic and linear function. Thus the only serious obstacle is the fact that the simple upper-bounding constraints (2.7) are stochastic. We overcome this difficulty by constructing the representations of the functions $z_h(\cdot)$ so that they automatically satisfy these constraints.

Suppose that functions ζ_{hk} are themselves bounded below by 0 and above by q_h, then the constraints(2.7) will be satisfied if rather than taking linear combinations of the functions ζ_{hk} we limit ourselves to *convex* combinations. Assuming that we proceed in this fashion, problem (2.6) becomes:

(2.9) find $y \in R_+^m$ and $\lambda_{hk} \in R_+$ for $k=1,\ldots,\nu$; $h=1,\ldots \ell$ such that

$$w_j = c_j - \sum_{i=1}^{m} a_{ij} y_i - \sum_{h=1}^{\ell} \sum_{k=1}^{\nu} \lambda_{hk} \bar{t}_{hkj} \quad \text{for } j=1,\ldots,n,$$

$$1 = \sum_{k=1}^{\nu} \lambda_{hk}, \quad h=1,\ldots,\ell,$$

and $\Phi^{\nu}(y,\lambda)$ is minimized.

The choice of the functions ζ_{hk} is adaptive. We view problem (2.9) as the ν-th iteration of an approximation process, in the sense that the convex combination of the functions ζ_{hk} only yields a finite element representation of the functions z_h. The choice of $\zeta_{h,\nu}$ is such that it guarantees a decrease in the value of $\Phi(y,z)$ when the solution to the ν-th quadratic program is used to represent z, i.e.,

$$z_h^{\nu}(\cdot) = \sum_{k=1}^{\nu} \hat{\lambda}_{hk} \zeta_{hk} (\cdot),$$

instead of the coefficients that would be generated through ealier versions of (2.9); here $\hat{\lambda}_{hk}$ are the optimal solutions of (2.9). Let

$$x_j^{\nu}, \quad j=1,\ldots,n,$$

be the (dual) multipliers associated with the equations

$$w_j = c_j - \sum_{i=1}^{m} a_{ij} y_i - \sum_{h=1}^{\ell} \sum_{k=1}^{\nu} \lambda_{hk} \bar{t}_{hkj} \quad j=1,\ldots,n,$$

at the optimum. For $h=1,\ldots,\ell$, we define

$$\text{(2.10)} \quad \zeta_h^{\nu+1}(\omega) = q_h(\omega)\, \theta' \left(e_h^{-1} \left[\sum_{j=1}^{n} t_{hj}(\omega) x_j^{\nu} - p_h(\omega)\right]\right)$$

where θ' is the derivative of θ, i.e.,

$$\theta'(\tau) = \begin{cases} 0 & \text{if } \tau \le 0, \\ \tau & \text{if } 0 \le \tau \le 1, \\ 1 & \text{if } \tau \ge 1. \end{cases}$$

In view of (2.10), we always have that

$$0 \le \zeta_h^{\nu+1}(\cdot) \le q_h(\cdot).$$

The functions $\zeta^{\nu+1} = (\zeta_1^{\nu+1},..,\zeta_\ell^{\nu+1})$ are such that

$$(2.11) \quad \zeta^{\nu+1} \in \text{argmin}\ [\Phi(y^\nu,\zeta)\,|\,0 \le \zeta_h \le q_h(\cdot),\quad h=1,\dots,\ell].$$

To see this simply note that

$$\frac{\partial}{\partial z_h}\Phi = p_h + \frac{e_h}{g_h} z_h - \sum_{j=1}^n r_j\,\theta'\left(d_j^{-1}w_j\right)\frac{\partial}{\partial z_h} w_j$$

from which it follows that

$$\frac{\partial}{\partial z_h}\Phi = e_h q_h^{-1} z_h + p_h - \sum_{j=1}^n t_{hj} x_j^\nu$$

since $\frac{\partial}{\partial z_h} w_j = t_{hj}$ and from (1.9) and the definition of θ' we get

$$x_j = r_j\theta'\left(d_j^{-1}w_j\right) = \begin{cases} 0 & \text{if } w_j < 0, \\ r_j d^{-1} w_j & \text{if } 0 \le w_j \le d_j, \\ r_j & \text{if } d_j < w_j. \end{cases}$$

This then yields (2.10) since we obtain $\zeta_h^{\nu+1}$ from the equation

$$\frac{\partial}{\partial z_h}\Phi = 0$$

if it turns out that the resulting value is between 0 and q_h.

The choice of $\zeta^{\nu+1}$ guarantees that unless we already have found the optimal solution, the new representation

$$z_h^{\nu+1}(\cdot) = \sum_{h=1}^{\nu+1} \hat{\lambda}_{hk}\,\zeta_{hk}(\cdot)$$

will yield an improved solution, here the $\hat{\lambda}_{hk}$ being the coefficients obtained by solving (2.9), setting $\nu = \nu+1$ in (2.9).

The algorithm thus proceeds as follows:

Step 0. Choose any function ζ^1 such that

$0 \le \zeta_h^1(\omega) \le q_h(\omega) \quad h=1,\ldots,\ell.$

$\left(\text{Say } \zeta_h^1(\omega) \equiv 0\right).$

Set $\nu = 1$.

Step 1. Solve (2.9), recording $\left(x_j^\nu,\ j=1,\ldots,n\right)$ the dual variables associated to the constraints defining w_j. Let $\hat{\lambda}$ denote the optimal values of the λ-variables.

Step 2. Define $\zeta^{\nu+1}$ through (2.10).

If $\zeta_h^{\nu+1} = z_h^\nu = \sum_k \hat{\lambda}_{hk}\,\zeta_{hk}$, terminate: the $\left(x_j^\nu,\ j=1,\ldots,n\right)$ solve problem (0.1).

Otherwise return to Step 1 with $\nu = \nu+1$.

Observe that having $\zeta^{\nu+1} = z^\nu$ implies that no function of type ζ can be found that could give a representation for z generating a decrease in Φ. The fact that the $\left(x_j^\nu,\ j=1,\ldots,n\right)$ are then optimal solutions of the original problem (0.1) follows from the Duality Theorem of Section 1.

We conclude by making a few comments about implementation. First note that to store the function ζ^ν it really suffices to store the finite dimensional vector $\left(x_j^\nu,\ j=1,\ldots,n\right)$; the definition of ζ^ν, through (2.10), corresponds to a simple probabilistic subset (event) of Ω completely determined by x^ν. This is also all that is necessary to compute the quanities $\bar{t}_{hkj}$, $\bar{p}_{hk}$ and $\bar{e}_{hkk'}$ which are obtained by numerical integration. Finally, one should not really rely on the stopping criterion given in Step 2, but on bounds that can be obtained from the optimal value of (2.9) similar to those used in the Frank-Wolf algorithm [7].

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