VARIATIONAL SYSTEMS, AN INTRODUCTION

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1. VARIATIONAL PRINCIPLES AND CONSTRAINTS.

Fundamental in many applications of mathematics is the idea of modeling a situation by first describing a set S of possible "states" that need to be considered and then introducing additional criteria that single out from S some particular state x. For example, S could represent all the configurations that might be taken on by a certain physical system, and x could be an "equilibrium" state, perhaps expressing a balance of forces or giving an extremal value to some energy function. Economic models often follow a similar pattern, except that instead of an energy function it may be a cost or utility function, say, whose minimum or maximum puts the spotlight on a particular x in S. Such models too can concern an x which is an equilibrium resulting from interactive maximization or minimization of various functions by numerous individual agents.

Modern applications in statistics, engineering, and operations research have especially focused attention on situations where a physical or economic system can be affected or controlled by outside decisions, and these decisions should be taken in the "best" possible manner. The notion of an *optimization problem* has proved very useful. In abstract terms, such a problem consists of a set S whose elements, called the *feasible solutions* to the problem, represent the alternatives that are open to a decision maker. Examples of S include the set of acceptable estimators for a statistical parameter, the set of feasible designs in a structural engineering problem, the possible control policies for an inventory process, and so on. The aim is to minimize over S a certain function f, the *objective function*. The elements x of S where the minimum is attained are called the *optimal solutions* to the problem. Of course minimization could be replaced by maximization.

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In all such cases where an x is singled out from an underlying set S on the basis of some kind of minimization or maximization, it is common to speak of x as being characterized by a *variational principle*. This terminology also carries over to many situations where x does not necessarily give a true extremum but merely satisfies conditions that generalize, or form part of, various conditions known to be associated with an extremum over S.

The question of variational principles and their role in science and technology is closely connected, therefore, with understanding and characterizing extremals of a function f over a set S. This in turn depends on the nature of f and S, and here it is that a great amount of mathematical innovation has become necessary in recent decades. The older view of variational principles was too limited. Traditional methods are simply not adequate to treat the kind of functions f and sets S that that nowadays are deemed important in such a context. We speak here not just of special techniques but of the entire outlook of classical analysis.

To begin with, some words about the sets S that may be encountered will make this clearer. In this introduction, we shall be concerned in the main with situations that can be described by a finite number of real variables, or in other words, which display "finitely many degrees of freedom". Denoting the variables by $x_1,\ldots,x_n$, we can identify the possible "states" which correspond to a situation at hand with elements $(x_1,\ldots,x_n) = x$ of the space R^n. Thus the state set S can be thought of simply as a certain subset of R^n. The exact definition of S in a particular case depends of course on various circumstances, but it typically involves a number of functional relationships among the variables $x_1,\ldots x_n$. It may also involve restrictions on the values that may be taken on by these variables. In economic models, for instance, it is common to have variables that are intrinsically nonnegative ; in structural design problems, bounded variables are the rule.

A great many situations are covered by the following kind of description :

(1.1) $S :=$ set of all $x = (x_1,\ldots,x_n) \in R^n$ such that

$$x \in X \quad \text{and} \quad f_i(x) \begin{cases} \leq 0 & \text{for } i = 1,\ldots, s, \\ = 0 & \text{for } i = s+1,\ldots,m, \end{cases}$$

where X is some given subset of R^n (usually rather simple in character, perhaps the entire space R^n) and each f_i is a real valued function on R^n. The conditions $x \in X$, $f_i(x) \leq 0$ or $f_i(x) = 0$ are called *constraints* on the state x. The inclusion of the abstract condition $x \in X$ allows an open-ended flexibility in the description of the constraints.

What most distinguishes the applications for which classical analysis was developed from the modern ones, as far as sets S of type (1.1) are concerned, are the

inequality constraints, possibly very many of them, and the frequent lack of "smoothness" of the functions f_i and set X. In elementary models for physical systems, it is frequently the case that S is completely characterized by several *equations* involving the variables $x_1,\ldots,x_n$:

$$(1.2) \quad S = \{x \in X \mid f_i(x_1,\ldots,x_n) = 0 \quad \text{for} \quad i = 1,\ldots,m\},$$

where X is an *open* set in R^n and the functions f_i are smooth, i.e. continuously differentiable. Furthermore, the equations are independent in the sense that in a neighborhood of any point of S they can be solved for some m variables as smooth functions of the other $n-m$ variables, although just which ones might depend on the point in question. Then S is a "smooth" curve, surface, or hypersurface in R^n of dimension $n-m$, the kind of object which finds its abstraction in the important mathematical concept of a differentiable manifold. We refer to such an S as a *smooth manifold*.

When inequality constraints are encountered in classical analysis, they are usually of an elementary sort and few in number. An example of a set S that can be described in terms of such constraints is a closed annulus : a region in R^2 lying between two concentric circles and including the circles themselves. This corresponds to two quadratic inequalities. Another example is a solid cube in R^3 or its boundary. Such a cube can be determined by a system of six linear inequalities. Note that when S is such a cube, its boundary is not a smooth manifold, but its structure is simple enough not to pose much trouble. The open faces and edges of S are smooth manifolds that can be investigated individually. In general, one might say that the kind of sets S seen in traditional applications are, if not smooth manifolds themselves at least the union of a modest number of smooth manifolds that are nicely juxtaposed to each other and easily listed in an explicit manner.

In contrast, many contemporary problems in economics, chemical equilibrium, physical variational principles, and other areas, concern sets S of the form (1.1) where the number of inequality constraints is in the hundreds or thousands, far larger than the number of variables x_i, which nevertheless can be huge too. Then the notions and technical tools appropriate for smooth manifolds no longer suffice. At any given point x of S some of the inequality constraints can be *active* (satisfied as equations), while others can be *inactive* (satisfied with strict inequality). Quite apart from the large numbers involved, there is usually no easy way to determine which combinations of active and inactive constraints actually do occur; cf. Figure 1. Furthermore, the consideration of such combinations does not necessarily lead to a decomposition of S into smooth manifolds, not to speak of one having a simple, direct description. Even the equality constraints appearing in (1.1) can cause diffi-

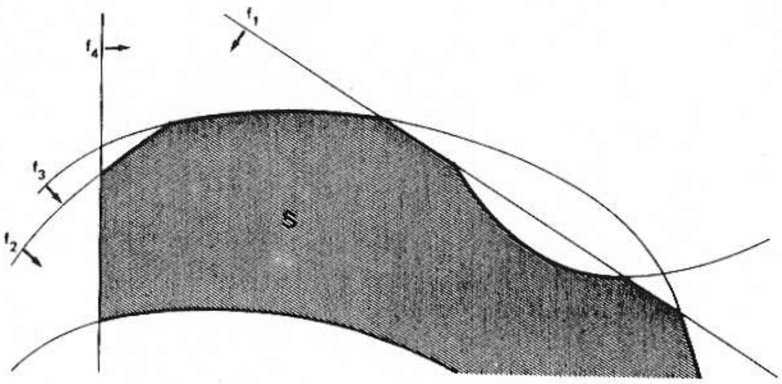

Figure 1. S the set of acceptable states.

culties by not being "independent" at critical points of S , and the set X may have complicated boundaries that need to be taken into account.

The study of evolutionary systems in the context of viability theory, cf. Aubin, 1984, obliges us to confront, in a dynamical setting, all the questions that were raised in connection with the mathematical structure of the set of acceptable states as defined by (1.1). The motivation comes from biological, ecological and macro-economics models that fit the following general evolutionary format : a closed subset S of R^n identifies the acceptable states of the system, the dynamics of the system are described by the relations

$$(1.3)\quad \dot{x}(t) \in \Gamma(t) \quad \text{and} \quad x(t_0) = x_0,$$

where $\dot{x}$ denotes the derivatives of the state x with respect to t (the time parameter), and $\Gamma(t)$ is the set of feasible dynamics at time t . In the study of the evolution of the state $x(t)$ as a function of t , we must make provisions for the behaviour of the system at its frontier of viability, i.e. when $x(t)$ belongs to the boundary of S . Because precisely these critical periods are the ones of interest in the modeling process, we cannot resort to the "smooth" case studied in classical dynamics, i.e. when the system is to evolve in an open domain or on a smooth manifold with open boundaries.

Another difficulty is that the differentiability assumptions or differential dependence of the solution on the parameters of the problem which seem (or at least used to seem) so natural in classical physics lose their luster in other subjects. Mathematical models derived from biology, economic theory or the theory of extremals in statistics, for example, often have a *convex* set X and inequalities involving *convex* functions f_i . These particular mathematical properties are of interest because they have an axiomatic significance in economic models or extremal statistics

which smoothness properties do not. This turns out to be no impass for analysis, if certain generalizations of differential calculus are pursued.

The importance of being able to work with nonsmooth functions comes from more reasons than just this. In some way, inequality constraints in themselves force the considerations of nonsmoothness. We have already observed this in the example of a solid cube in R^3 having a nonsmooth boundary. More generally, any constraint system of the form $g_k(x) \leq 0$, for $k = 1,\ldots,q$, can be lumped together as a single inequality $g(x) \leq 0$ where

$$(1.4)\qquad g(x) = \max_{k=1,\ldots,q} g_k(x)$$

The price to be paid, of course, is that g will not inherit the differentiability properties of the functions g_k , see Figure 2.

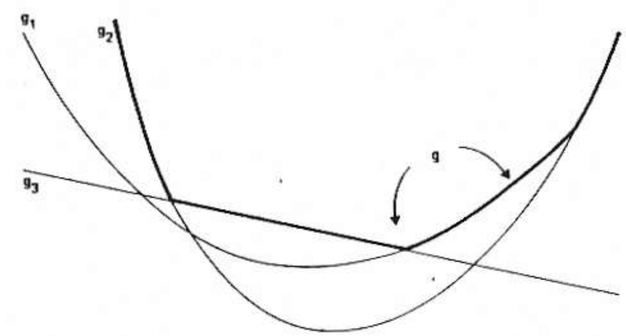

Figure 2. The max-function g.

Nevertheless the idea of lumping constraints together this way has its value, and we must be prepared to cope with it. For example any convex function $g : R^n \to R$ can be represented by a limiting version of (1.4) in which infinitely many (linear) functions are allowed.

The classical approach to a nonsmooth function g as in (1.4) would be to treat it as a *piecewise smooth*, or in other words to decompose the domain of g into finitely many smooth manifolds relative to which g is continuously differentiable. But this may be impossible without imposing painful and practically unverifiable conditions on the functions g_k and how they interact with each other.

Nonsmoothness enters the study of variational problems through the analysis of constraint systems, as we have been discussing, but also through the objective function and various consequences of optimization itself. In a problem of the form :

(1.5) find $x \in S \subset R^n$ such that $f_0(x)$ is minimized,

where f_0 is a real-valued function on R^n , there is no reason to limit the attention to the case f_0 smooth, and indeed there are many applications where f_0 is not smooth. Examples may be found even in classical approximation theory :

(1.6) find $x \in R^n$ that minimizes $f_0(x) = \max_{0 \leq t \leq 1} |h(t) - j(t,x)|$,

where h is given continuous function on $[0,1]$ which is to be approximated (in the Chebyshev sense) by one of a given family of functions $j(\cdot,x)$. In other applications f_0 can be a max-function as in (1.4).

An extremely valuable concept which opens up further sources of nonsmoothness is that of a *perturbed* or *parameterized* optimization problem. To take a relatively simple case, let us imagine a minimization problem (in variables $x_1,\ldots,x_n$) which depends on finitely many other variables $u_1,\ldots,u_d$. More specifically, suppose that for each $u = (u_1,\ldots,u_d)$ in a certain set $U \subset R^d$ we want to consider the problem :

(1.7) find $x \in S(u)$ that minimizes $F_0(x,u)$,

where

(1.8) $S(u) = \{x \in X \subset R^n \mid F_i(x,u) \leq 0,\ i = 1,\ldots,s;\ F_i(x,u) = 0, i = s+1,\ldots,m\}$.

For each $u \in U$, let

(1.9) $p(u) := \inf_{x \in S(u)} F_0(x,u)$

denote the infimal value of problem (1.7), i.e. the lowest value attained by $F_0(x,u)$ as x ranges over $S(u)$, assuming for the moment that this does exist. It is a fact of life that the *infimal function* p is unlikely to be differentiable in the classical sense, however nice the functions $F_0, F_1,\ldots, F_m$ and the set X may be; often it is not even continuous. Yet in many contexts we would like very much to have some understanding of the way $p(u)$ changes with u , and there are strong motivations for studying *rates* of change. Of course this needs to be expressed in terms of generalized derivatives of some sort. In fact there are some important applications, both theoretical and numerical, where an infimal (value) function like p can enter into still another optimization problem as the objective or one of the constraint functions.

The parameterized problem (1.7) serves also to bring up other matter that must be dealt with squarely by a modern theory of analysis. Even if the minimum in the problem is attained for some x for each choice of $u \in U$, which is usually not too difficult to guarantee, there may well be more than one such x . The optimal solutions to (1.7) thus form a set $A(u)$ depending on u . There is no escaping this general lack of uniqueness without making restrictions that in many applications would be out of character with the underlying physical, biological, or economic model, and futhermore impossible to check.

This phenomenon certainly provides sharp contrast with what is regarded as normal in classical physical systems. The common view there is that a model is not well formulated unless it leads to both an existence theorem and a uniqueness theorem. In other words, the state set S in such a system is supposed to be supplied with some mechanism which sigles out *one* and *only one* special state x . This notion has to be abandoned in many other contexts. In its absence there is the challenge that in places where one is accustomed to dealing with *functions* (single-valued), one often has to deal with so-called *multifunctions* (multivalued or set valued functions, as will be discussed in Section 2).

Thus in models with parameters, instead of a unique special state which depends on u one has a set $A(u)$. There is every incentive for developing a theory of how $A(u)$ can vary with u : generalized properties of measurability, continuity, smoothness, and so forth. Of course, the same holds also for other kinds of sets that depend on parameters such as $S(u)$ in (1.8), for example. In particular it is necessary to investigate ways in which a sequence of sets in R^n can converge to a set. The answers are helpful not only in treating multifunctions by for setting up numerical methods. A problem of minimizing some function f over a set S , for instance, can presumably be approximated by minimizing the same function over a "nearby" set S , but the sense of such an approximation has to be made exact. Such questions are best handled in the framework provided by the study of *variational systems* that we introduce in Section 3. It is the dependence on the parameters u of the optimization problem (1.7) as a whole, that is now of interest. The properties of the infimal function p or multifunction A of optimal solutions are to be studied in terms of the properties of the class of optimization problems that engender them. To do this we must create a mathematical object that corresponds to an optimization problem, that incorporates both the objective function and the constraints. This brings us to giving the lead role in our analysis to *extended-real-valued functions* , i.e. functions which can take on not only real numbers as values but also ∞ and $-\infty$.

Extended-real-valued functions are nothing new in mathematics, but they have

me to enjoy quite a new range of usefulness based on an attitude that many mathe-
ticians would have balked at in the past : ∞ and $-\infty$ do not have to be treated in symmetric fashion, and indeed in other aspects of theory as well, one should not rry so much about maintaining symmetry with respect to multiplication by -1 . It easy to appreciate how this changed attitude has come about. When a function f pears in an inequality constraint $f(x) \leq 0$ in some application, we usually have interest at all in the opposite direction $f(x) \geq 0$. Thus in studying such a nstraint, there is no need to limit our attention to properties that are formulated mmetrically with respect to f and $-f$. Similarly in the context of variational inciples : when a function f is to be minimized in certain setting, our interest ually stops there, and we do not wish to determine also the points that maximize , i.e. the points that minimize $-f$. Other illustrations could be given, but in ort, there are many situations in which a function that is to be investigated can viewed as having a particular "orientation".

Let us consider the optimization problem

(1.10) find $x \in R^n$ that minimizes $f(x)$,

where f is an extended-real-valued function. If there is a point x where $f(x) = -\infty$ (a circumstance where this may happen is when f is an infimal function of a parameterized optimization problem, cf. (1.9)), then we know at once that x provides the minimum. Points x where $f(x) = \infty$, on the other hand, have almost the opposite significance; they are not even worth contemplating as candidates for providing the minimum, except in the degenerate case when f is identically ∞ . In essence the constraint $f(x) < \infty$ is implicit in such a minimization problem. This being so, we arrive at the possibility of using ∞ constructively to designate the points that are not of interest in a given situation. Each minimization problem of type (1.5) is equivalent to minimizing a certain other function f , called the *essential objective function* over all of R^n , namely

$$(1.11)\quad f(x) = \begin{cases} f_0(x) & \text{if } x \in S \\ \infty & \text{if } x \notin S. \end{cases}$$

or in more detail when S is given the representation (1.1) :

$$(1.12)\quad f(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0 \ , \ i = 1,\ldots,s, \\ & \quad f_i(x) = 0 \ , \ i = s+1,\ldots,m \\ & \quad x \in X \subset R^n \ , \\ +\infty & \text{otherwise.} \end{cases}$$

Thus for theoretical purposes, the study of optimization problems, their general properties as well as their classification, can be undertaken in the framework provided by that of extended-real-valued functions defined on R^n .

However, the traditional approach to function analysis is no longer quite appropriate for this class of functions. The concept of continuity must be replaced that of semicontinuity, and so on. This break with classical analysis
is underscored by the new geometrical viewpoint which must accompany the analysis. The traditional way of applying geometric ideas to functions has been through the geometry of graphs. Such an approach continues to hold much potential in current work on nontraditional topics, but for the treatment of extended-real-valued functions $f : R^n \to \overline{R}$ there is a newer concept of "epigraphs" that has proved to be more fruitful than graphs. It opens a "bridge" between sets and functions that plays such a pivotal role in the mathematical principles, definitions and tools that structure this analysis that we could refer to it as the *epigraphical viewpoint*.

For $f : R^n \to \overline{R}$ we define the *epigraph* of f to be the set

$$(1.13)\quad \text{epi } f : = \{(x,\alpha) \in R^n \times R \mid \alpha \geq f(x)\} ,$$

see Figure 3. The epigraph consists of all points in R^{n+1} that lie "on or above"

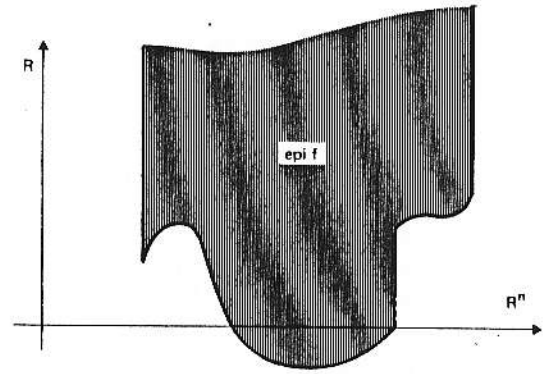

Figure 3. The epigraph of f.

the graph of f , but note : the graph of f is not well defined as a subset of R^{n+1} because $f(x)$ may be ∞ or $-\infty$. The graph of f is really a subset of $R^n \times \overline{R}$. The epigraph, on the other hand, does lie entirely in R^{n+1} by definition, and yet it serves to represent the real-extended-valued function f completely :

(1.14) $f(x) = \inf\{\alpha | (x,\alpha) \in \text{epi } f\}$ for all x.

Turning to epigraphs does condition our view of f. It directs our interest to properties of f that are naturally associated with such subsets of R^{n+1}; also for example to those of sets like

(1.15) $\text{lev}_\alpha f := \{x \in R^n \mid f(x) \leq \alpha\}$

called the α - *level set* of f, instead of the properties of the set $\{x \mid f(x) = \alpha\}$ on which a traditional "two-sided" approach would focus. The level set (1.15) has, of course, a simple geometric meaning in terms of epi f. It corresponds to the horizontal cross-section of epi f at height α. This interplay between the properties of extended-real-valued functions, epigraphs and level sets is nicely illustrated by the following result. Recall that a function $f : R^n \to \overline{R}$ is *lower semicontinuous* at x if

(1.16) $\liminf_{x' \to x} f(x') = f(x)$.

It is *lower semicontinuous* (*l.s.c.*) if this holds for all $x \in R^n$.

1.17. THEOREM. *The following properties of a function* $f : R^n \to \overline{R}$ *are equivalent*

(a) f *is lower semicontinuous*

(b) *the set* epi f *is closed*

(c) *the set* $\text{lev}_\alpha f$ *is closed for all* $\alpha \in R$.

PROOF (a) ⇒ (b). Suppose $(x^\nu,\alpha^\nu) \in \text{epi } f$ and $(x^\nu,\alpha^\nu) \to (x,\alpha)$; then $x^\nu \to x$ and $\alpha^\nu \to \alpha$ with $\alpha^\nu \geq f(x^\nu)$. We must show that $\alpha \geq f(x)$, so that $(x,\alpha) \in \text{epi } f$. The sequence $\{f(x^\nu), \nu=1,\ldots\}$ has least one cluster point in $\overline{R}$. Replacing the sequence $\{(x^\nu,\alpha^\nu), \nu=1,\ldots\}$ by a subsequence if necessary, we can actually suppose that $f(x^\nu) \to \alpha'$ for some $\alpha' \in \overline{R}$. Then $\alpha \geq \alpha'$, but on the other hand

$$\alpha' \geq \liminf_{\nu \to \infty} f(x^\nu).$$

Since we are arguing from (a), we have (1.16); hence $\alpha' \geq f(x)$, and so $\alpha \geq f(x)$ as needed.

(b) ⇒ (c). $\text{lev}_\alpha f$ is then the intersection of two closed sets, namely, epi f and the hyperplane $\{(x,\eta) \in R^{n+1} | \eta = \alpha\}$.

(c) ⇒ (a). Fix any $x \in R^n$ and let

$$\beta := \liminf_{x' \to x} f(x').$$

Then $\beta \leq f(x)$. We must demonstrate that (1.16) holds, and to do this it will suffice to prove that $f(x) \leq \alpha$ when $\beta < \alpha < \infty$. Thus for an arbitrary α satisfying $\beta < \alpha < \infty$ we need only show that $x \in \text{lev}_\alpha f$. We begin by exhibiting a sequence $x^\nu \to x$ such that actually $f(x^\nu) \to \beta$. There is nothing to prove if $\beta = \infty$, so let us suppose $\beta < \infty$. Consider the sequences $\beta_\nu \downarrow \beta$ and $\delta_\nu \downarrow 0$. We know that for all ν

$$\beta_\nu > \inf_{|x'-x| < \delta_\nu} f(x').$$

For every ν, therefore we can select an x^ν such that $|x^\nu - x| < \delta_\nu$ and $\beta_\nu > f(x^\nu)$. For this sequence we do have $x^\nu \to x$ and $f(x^\nu) \to \beta$, because $\beta = \lim \beta_\nu$. Thus for this sequence and for ν sufficiently large $f(x^\nu) \leq \alpha$, since $\beta < \alpha$. Hence $x^\nu \in \text{lev}_\nu f$ is closed under hypothesis (c), so that $x^\nu \to x$ yields the desired relation $x \in \text{lev}_\alpha f$. □

Now seems to be a good time to reiterate that although we have been relying heavily on minimization problems as a source of motivation, there are many applications of the same ideas to problems that are only vaguely related to minimization, if at all. The study of inequality constraints, e.g. $f(x) \leq 0$ or in the form they appear in Viability Theory, has already been mentioned. Convex functions furnish another prime example : such a function f given on a convex set $S \subset R^n$ and then extended to R^n with the value ∞ remains a convex function. Also, every result in this one-sided approach involving epigraphs, lower semicontinuity, and convexity, for example, has its counterpart in terms of *hypographs* $\{(x,\alpha) \in R^{n+1} | \alpha \leq f(x)\}$, upper semicontinuity, and concavity.

If the epigraphical viewpoint is the appropriate approach to the conceptualization of optimization problems as mathematical entities, in terms of extended-real-valued functions, then the study of their dependence on parameters should also follow the same guidelines. And as we shall see in some detail in Section 3, such an approach is indeed the correct one, as confirmed by the wealth of tools and results that reward it. Much more could be said, but at this point let us just introduce the main idea in terms of the parametrized minimization problem (1.7). For each $u \in R^d$, the essential objective function is defined by

(1.18) $$f(x,u) = \begin{cases} F_0(x,u) & \text{if } u \in U \text{ and } x \in S(u), \\ \infty & \text{otherwise.} \end{cases}$$

We have thus an extended-real-valued bivariate function defined on $R^n \times R^d$, but we have only very limited interest in the properties of f jointly in x and u . The properties that we are looking for : continuity, convexity or measurability, to name a few, of the infimal function p and of the multifunction A of optimal solutions, are all conditioned by the properties the *epigraphical multifunction*

$$(1.19)\quad u \to \text{epi } f(\cdot,u) : R^d \rightrightarrows R^{n+1} ,$$

as is demonstrated later on. Thus it is in those terms that we shall proceed, totally in accordance with the observations we have made about "single" optimization problems.

As indicated earlier, Section 2 and 3 are devoted to the study of the parametric dependence of sets (multifunctions) and functions (variational systems). We touch on the key questions of continuity and measurability that are part of this Extended Real Analysis, altough only in a very cursory manner. We also raise some integrability questions. But this picture is not complete; indeed we have often suggested that there is also a need for an appropriate subdifferential calculus that will allow us to manipulate and give meaning in the nonsmooth case to the notion of "derivative". This theory of subdifferentiation, whose earlier development, cf. for example the recent survey Rockafellar, 1983, or the books by Clarke, 1983, or Aubin and Ekeland, 1984, may appear to have only limited intersection with the questions broached here. In fact it is intimately related to the limit notions that surface in the subsequent sections. This, however, is beyond the scope of this introduction and cannot find place here.

2. MULTIFUNCTIONS : CONTINUITY AND MEASURABILITY.

According to strict definition, a *function* $\Gamma : U \to X$, where U and X are arbitrary sets, is nothing more than a subset of $U \times X$ having the property that it contains for each $u \in U$ a *unique* pair (u,x) . Be that as it may, no one really thinks in such terms when going about his everyday business, or we would be accustomed to seeing $(u,x) \in \Gamma$ written at least as often as $x = \Gamma(u)$. In truth there is a very strong feeling for functions as having a more dynamic quality. This is emphasized by the common notation

$$(2.1)\quad u \mapsto \Gamma(u) ,$$

which expresses operationally the assignment of a certain $\Gamma(u)$ to u . The subset of $U \times X$ that is supposed to be the function itself is usually referred to instead as its *graph*. There is no logical distinction between a function and its graphs, yet the use of term "graph" is nevertheless helpful in signaling when a more geometric rather than operational point of view is intended. These observations may help to put the definitions we are to make in a better perspective.

Following the pattern adopted for functions, we define a *multifunction* $\Gamma : U \rightrightarrows X$ technically as just a subset of $U \times X$. Any subset will do, and no extra conditions like those in the case or a "function" are imposed. Even so, we typically refer to the subset in question as the *graph* of Γ and denote it by gph Γ rather than just Γ . For each $u \in U$, we let $\Gamma(u)$ stand for the set of all x such that (u,x) belongs to the graph of Γ , cf. Figure 4.

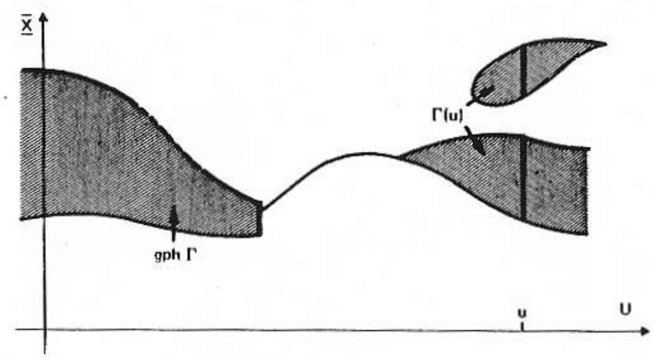

Figure 4. The graph of the multifunction Γ.

Thus

(2.2) $\text{gph}\ \Gamma = \{(u,x) \mid x \in \Gamma(u)\}$.

All of this may seem natural enough, but there are some features of the terminology that might be confusing if not made sufficiently explicit. The notation $\Gamma : U \rightrightarrows X$ in place of $\Gamma : U \to X$ is used to indicate that single-valuedness is *not required*, although it might be present anyway. The possibility that Γ is a function is not excluded. Unfortunately if Γ does happen to be a function, there is a conflict in the way we have defined $\Gamma(u)$ to be a set : do we have $\Gamma(u) = \{x\}$ or $\Gamma(u) = x$? The difference does not make much difference in practice, because the suitable interpretation is usually clear from the context. Coming up with an intricate symbolism to resolve the ambiguity therefore does not seem worth the effort.

A more interesting question is whether a multifunction Γ might not be a true function in a different sense. From the operational point of view we see the symbolism $u \mapsto \Gamma(u)$ as appropriate : to each u assign a certain point $\Gamma(u)$. Since $\Gamma(u)$ is a subset of X , it appears then that we are dealing with a function $\Gamma : U \to 2^X$, where

2^X : = the collection of all subsets of X .

Without denying the usefulness of this approach in many situations, we must observe if such were the total picture, the "graph" of Γ would have to be regarded as a subset of $U \times 2^X$ rather than $U \times X$. Then, it would no longer be accurate to identify functions $\Gamma : U \to X$ as special cases of multifunctions $\Gamma : U \rightrightarrows X$. This would undermine our framework for generalizing from functions to multifunctions. Whenever the need does arise, we shall associate with the multifunction $\Gamma : U \rightrightarrows X$ a function $\gamma : U \to 2^X$ such that the element $\gamma(u) \in 2^X$ identifies the set $\Gamma(u)$.

Let us proceed now with some basic definitions. We have not insisted that a multifunction Γ should have $\Gamma(u)$ nonempty for every u . When $\Gamma(u) = \emptyset$, is said to be *empty-valued* at u . The *effective domain* of Γ is the set

(2.3) $\text{dom}\ \Gamma := \{u | \Gamma(u) \neq \emptyset\}$,

and the range of Γ is

(2.4) $\text{rge}\ \Gamma := \{x | \ u \text{ with } x \in \Gamma(u)\}$.

The *inverse* of a multifunction $\Gamma : U \rightrightarrows X$ is the multifunction $\Gamma^{-1} : X \rightrightarrows U$ obtained by reversing all the ordered pairs in the graph of Γ :

(2.5) $\text{gph}\ \Gamma^{-1} = \{(x,u) | (u,x) \in \text{gph}\ \Gamma\}$.

Evidently

$$u \in \Gamma^{-1}(x) \quad \text{if and only if} \quad x \in \Gamma(u) ,$$

and

(2.6) $\text{dom}\ \Gamma^{-1} = \text{rge}\ \Gamma$, $\text{rge}\ \Gamma^{-1} = \text{dom}\ \Gamma$.

Let us note again that these notions are perfectely consistent with our idea of Γ as corresponding to a subset of $U \times X$, but they do not fit the competing picture sometimes put forward of a set-valued function from U to 2^X , whose inverse would be something quite different.

The *image* of a set $C \subset U$ under Γ is

(2.7) $\Gamma(C) = \cup_{u \in C} \Gamma(u) = \{x | \ \Gamma^{-1}(x) \cap C \neq \emptyset\}$.

In like fashion, the *inverse image* of a set $D \subset X$ under Γ is

(2.8) $\Gamma^{-1}(D) = \cup_{x \in D} \Gamma^{-1}(x) = \{u | \ \Gamma(u) \cap D \neq \emptyset\}$.

In particular

(2.9) $\Gamma(U) = \text{rge}\ \Gamma$ and $\Gamma^{-1}(X) = \text{dom}\ \Gamma$.

We will mostly be concerned in this article with the case when $U \subset R^n$ and $X \subset R^n$ for some d and n . An extra economy of notation is then possible : any multifunction $\Gamma : U \rightrightarrows X$ is in particular a multifunction $\Gamma : R^d \rightrightarrows R^n$. Incidentally, we are not just saying here that $\Gamma : U \rightrightarrows X$ can be *extended* to $\Gamma : R^d \rightrightarrows R^n$, multifunctions have been identified in our set-up with subsets, and any subset of $U \times X$ *is* a subset of $R^d \times R^n$. It follows that, whenever we are in this setting, we will be safe in limiting theoretical discussions of multifunctions to the case $\Gamma : R^d \rightrightarrows R^n$.

Section 1 contains four examples of multifunctions from R^d to R^n that illustrate somewhat different themes but are excellent in providing motivation for the properties that we are going to look at next. The first is the multifunction $u \mapsto S(u)$ defined by (1.8). It associates with each parameter u the set of points satisfying a certain system of constraints; its domain is the set of all u for which the system is consistent. There is little hope for this multifunction $S : R^d \rightrightarrows R^n$ being single-valued except in very special cases. Nonetheless we are not satisfied with looking at the sets $S(u)$ in isolation. We want to know something about the relationship between $S(u')$ and $S(u)$ when u' and u are near to each other.

The second example is the multifunction $t \mapsto \Gamma(t)$ of admissible dynamics which appears in the generic model of viability theory. This multifunction $\Gamma : R \rightrightarrows R^n$, is generally not single-valued unless the system being modeled is such that it is irremediably condemned to die whenever the trajectory hits the boundary of the set of acceptable states S, a rather uninteresting case. If the system is to survive even when it hits the frontier of viability, the set of feasible dynamics must be rich enough to allow for adaptation in critical situations, cf. Aubin, 1984. Of course, we are interested in the measurability of Γ with respect to t, so that at least the differential inclusion $\dot{x}(t) \in \Gamma(t)$ is meaningful, but also the "derivatives" of Γ with respect to t are of interest : they enable us to characterise the changing levels of adaptability of the system under investigation.

The third example is the multifunction $u \mapsto A(u)$, where $A(u)$ is the set of all optimal solutions of the minimization problem (1.7). In this case we are definitely interested in single-valuedness but have to contend with the fact that it cannot usually be counted on for every choice of u. The way that $A(u)$ varies with u is again a prime topic, but the nature of the situation is such that sudden changes are very possible.

The last, and fourth, example is the epigraphical multifonction $u \mapsto \text{epi } f(\cdot,u) = E_f(u)$ as defined by (1.19). The multifunction $E_f : R^d \rightrightarrows R^n$ is never single-valued; in fact if $u \in \text{dom } E_f$, then $E_f(u)$ is always an unbounded set. Like for the multifunction $u \mapsto S(u)$ we are interested in the relationship between $E_f(u')$ and $E_f(u)$ when u and u' are near to each other. Or course, we expect to see a strong relationship between the multifunctions $u \mapsto S(u)$ and $u \mapsto E_f(u)$, the latter having possibly a slightly more sedate behavior, due to the dampening effect of the objective function whose dependence on parameters is usually less erratic than that generated by the intersections of sets that depend on parameters, since

$$S(u) = \cap_{i=1}^{m} S_i(u) \cap X ,$$

where $S_i(u) = \text{lev}_0 F_i(\cdot,u)$ for $i = 1,\ldots,s$, and $S_i(u) = \{x | F_i(x,u) = 0\}$ for $i = s+1,\ldots,m$.

In meeting the challenge raised by such examples we will need to pin down the various ideas of how a sequence of sets in R^n might be said to converge to set in R^n. Because the sets we encounter are frequently unbounded, in order to study their convergence behavior adequately in the large we shall need to adjoin to the space R^n a kind of "boundary at infinity" consisting of ideal points which represent possible directions of divergence. Here this will be done in the simplest possible way, namely by relying on a 1-point compactification of R^n. This is not the only way, and not always the best way of dealing with directions of divergence, but it is all we can cover in this Introduction, and it does provide us with the results that we need in our study of variational systems in Section 3.

Although there are many instances when results about arbitrary multifunctions are of interest, all the examples that we have dealt with suggest that the bread-and-butter case is when the multifunction Γ is *closed-valued*, i.e. for all u, the set $\Gamma(u)$ is closed subset of R^n. We shall restrict ourselves to that case, not just because all examples mentioned are of that type, but also because it allow us to escape a number of technicalities that would overburden this presentation. This means that henceforth we may restrict ourselves to limiting properties of sequences of *closed* sets.

Let $\{S^\nu, \nu = 1\ldots\}$ be such a sequence of closed subsets of R^n. Then its *limit inferior* is the set

$$(2.10)\quad \liminf_{\nu\to\infty} S^\nu : = \{x = \lim_{\nu\to\infty} x^\nu | x^\nu \in S^\nu \text{ for } \nu = 1,\ldots\}$$

$$= \cap_{\{\nu_k\}\subset \mathbb{N}} \text{cl } (\cup_k S^{\nu_k})$$

where by $\{\nu_k\} \subset \mathbb{N} = \{1,2,\ldots\}$ we mean that the intersection is with respect to all subsequences contained in $\mathbb{N}$. By cl, of course, we refer to the *closure* operation. Similarly, the *limit superior* of the sequence is the set

$$(2.11)\quad \limsup_{\nu\to\infty} S^\nu : = \{x = \lim_{k\to\infty} x^k | x^k \in S^{\nu_k} \text{ for some } \{\nu_k\} \subset \mathbb{N}\}$$

$$= \cap_{\nu=1}^{\infty} \text{cl } (\cup_{\mu=\nu}^{\infty} S^\mu)$$

Since the limit inferior consists of the limit points of all possible sequences $\{x^\nu, \nu = 1,\ldots\}$ with $x^\nu \in S^\nu$ for all ν, and the limit superior consist of the cluster points of the same sequences, we necessarily have

$$(2.12)\quad \liminf_{\nu\to\infty} S^\nu \subset \limsup_{\nu\to\infty} S^\nu .$$

The *limit* $\lim_{\nu\to\infty} S$ is said to exist, if actually equality holds in (2.12), in which case we set

$$(2.13)\quad \lim_{\nu\to\infty} S^\nu := \liminf_{\nu\to\infty} S^\nu = \limsup_{\nu\to\infty} S^\nu .$$

Of course, we do not need to restrict the definition of limits to sequences of sets, and for our purpose a somewhat more general definition serves us better later on. Let $\mathcal{H}$ be a *filter* on an index space N, i.e. a collection of nonempty subsets H of N closed under inclusion and finite intersections. We deal only with two types of filters, namely neighborhood systems $N(u)$ of points u in R^d and the Fréchet filter on $\mathbb{N}$ (which can be viewed as the neighborhood system $N(\infty)$ of the point ∞ at infinity in $\mathbb{N}$), but it is convenient to deal with both cases at once. The *grill* $\ddot{\mathcal{H}}$ of a filter $\mathcal{H}$ is defined by

$$(2.14)\quad \ddot{\mathcal{H}} := \{H \subset N \mid H \cap H' \neq \emptyset \text{ for all } H' \in \mathcal{H}\} .$$

There is a natural duality between a filter and its grill since, we have that

$$(2.15)\quad \mathcal{H} = \{H \subset N \mid H \cap H' \neq \emptyset \text{ for all } H' \in \ddot{\mathcal{H}}\} ,$$

as can easily be verified. For the filter $N(x)$ and the Fréchet filter $N(\infty)$ which are of primary interest to us, we have that $H \in N(\infty)$ if and only if H contains all execpt possibly a finite number of the elements of $\mathbb{N} = \{1,2,\dots\}$, whereas $H \in \ddot{N}(\infty)$ if and only if $H \subset \mathbb{N}$ is not finite, i.e. $\ddot{N}(\infty)$ corresponds to all subsequences of $\mathbb{N}$; also for $u \in R^d$, $N(u)$ = collection of all sets having u in their interior, whereas $\ddot{N}(u)$ = collection of all sets having u in their closure.

Extending the notion of limits of sets to that of filtered families $\{S^\nu, \nu \in (N,\mathcal{H})\}$, we get for the *limit inferior*

$$(2.16)\quad \liminf_{\nu\in N} S^\nu = \cap_{H\in\ddot{\mathcal{H}}} \operatorname{cl}(\cup_{\nu\in H} S^\nu)$$

and for the *limit superior*

$$(2.17)\quad \limsup_{\nu\in N} S^\nu = \cap_{H\in\mathcal{H}} \operatorname{cl}(\cup_{\nu\in H} S^\nu) .$$

Since $\mathcal{H} \subset \ddot{\mathcal{H}}$, it is again true that

$$(2.18)\quad \liminf_{\nu\in N} S^\nu \subset \limsup_{\nu\in N} S^\nu .$$

The *limit* is said to exist if equality holds, and as before it is then denoted by $\lim_{\nu\in N} S^\nu$ and defined to be the common limit.

We are now in a position to introduce continuity concepts for multifunctions. A multifunction $\Gamma : R^d \rightrightarrows R^n$ is *lower semicontinuous* (l.sc.) *at* u if for the filtered family

$$(\Gamma(u'),\ u' \in (R^d, N(u)) ,$$

we have

$$(2.19)\quad \liminf_{u'\to u} \Gamma(u') \supset \Gamma(u) ;$$

the notation $u' \to u$ suggests the filtering process by the neighborhood system of u (in the same way that $\nu\to\infty$ suggests the filtering process by the neighborhood system of the point at infinity). Similary, Γ is *upper semicontinuous* (*u.sc.*) at u if

$$(2.20)\quad \limsup_{u'\to u} \Gamma(u') \subset \Gamma(u) .$$

Finally, Γ is *continuous* at u if it is both lower and upper semicontinuous, or equivalently if

$$(2.21)\quad \lim_{u'\to u} \Gamma(u') = \Gamma(u) .$$

The multifunction Γ is said to be *lower* or *upper semicontinuous*, or *continuous*, if the corresponding property holds at all u in R^d.

Because of the topological properties of R^d, in particular the fact that neighborhood systems have a countable base, the definitions of lower and upper semicontinuity for multifunctions can also be rephrased in terms of sequences. This yields the following, which brings the definitions of lower and upper semicontinuity in line with the sequential definitions of lower and upper limits of sequences of sets that we gave first.

2.22 PROPOSITION. *A closed-valued multifunction* $\Gamma = R^d \rightrightarrows R^n$ *is lower semicontinuous at* u *if and only if for every sequence* $\{x^\nu, \nu = 1,\dots\}$ *converging to* u *and every*

$x \in \Gamma(u)$, *there exists a sequence* $\{x^\nu, \nu = 1,\ldots\}$ *converging to* x *such that* $x^\nu \in \Gamma(u^\nu)$ *for all* ν *except possibly for a finite number of* ν .

It is upper semicontinuous at u *if and only if for all the sequences* $\{u^\nu, \nu = 1,\ldots\}$ *converging to* u , *the cluster points of all possible sequences* $\{x^\nu, \nu = 1,\ldots\}$ *with* $x^\nu \in \Gamma(u^\nu)$ *belong to* $\Gamma(u)$.

2.23. THEOREM. *A closed-valued multifunction* $\Gamma : R^d \rightrightarrows R^n$ *is upper semicontinuous if and only if* $\text{gph}\,\Gamma \subset R^d \times R^n$ *is closed, or also, if and only if its inverse* $\Gamma^{-1} : R^n \rightrightarrows R^d$ *is upper semicontinuous.*

PROOF. Note simply that $\text{gph}\,\Gamma$ is closed if and only if every cluster point (u,x) of a sequence $\{(u^\nu,x^\nu),\nu = 1,\ldots\}$ in $\text{gph}\,\Gamma$ also belongs to $\text{gph}\,\Gamma$, or equivalently, if and only if any cluster point x of a sequence $\{x^\nu, \nu = 1,\ldots\}$, with $x^\nu \in \Gamma(u^\nu)$ and $u^\nu \to u$, belongs to $\Gamma(u)$. This is the characterization of upper semicontinuity provided by Proposition 2.22 .

The second assertion involving the inverse Γ^{-1} follows simply from the first one via (2.5); the graph of Γ^{-1} is just the subset of $R^n \times R^d$ obtained by reversing the order of all the pairs (u,x) in the graph of Γ . □

As an illustration of this idea, the constraint multifunction $u \mapsto S(u)$ defined by (1.8) is u.sc. if the sets U and X are closed and the functions $(x,u) \mapsto F_i(x,u)$ are continuous relative to $X \times U$. Indeed, the graph of the inverse of this multifunction is the intersection of several subsets of $U \times X$ of the form $\{(x,u) | F_i(x,u) \leq 0\}$; these sets are closed under our assuptions, and hence so is the graph in question.

Short of continuity of the multifunction $u \mapsto A(u)$ of optimal solutions to (1.7) which is difficult to guarantee, it is in the upper semicontinuity of this multifunction that we are interested in. Indeed, it is precisely this property which allows us to assert that whenever x^ν is an optimal solution of (1.7) for $\nu = 1,\ldots,$ with ν in place of u , and the u^ν converge to a certain $\bar{u}$, then any cluster point $\bar{x}$ of the sequence $\{x^\nu, \nu = 1,\ldots\}$ is an optimal solution of (1.7) with $u = \hat{u}$. In this sense the x^ν are approximate solutions of the limit problem.

The following characterization of upper and lower limits of filtered collection of sets leads to the construction of a topology on the *(hyper) space* $\mathcal{F}$ of closed subsets of R^n , the subspace of 2^X which is of interest here, and which allows us to relate the continuity of a multifunction $\Gamma : R^d \rightrightarrows R^n$ to that of the associated function $\gamma : R^d \to \mathcal{F}$. It is convenient in what follows to rely on the following notation.

$\mathcal{F}$: = the hyperspace of closed subsets of R^n ,

$\mathcal{G}$: = the hyperspace of open subsets of R^n ,

$\mathcal{K}$: = the hyperspace of compact subsets of R^n .

2.24. THEOREM. *Let* $\{S^\nu, \nu \in (N,\mathcal{H})\}$ *be a filtered family of closed subsets of* R^n , *and* S *a closed subset of* R^n . *Then*

$$S \subset \liminf S^\nu$$

if and only if for all $G \in \mathcal{G}$

$$(2.25) \quad S \cap G \neq \emptyset \Rightarrow \textit{for some } H \in \mathcal{H} ,\ S^\nu \cap G \neq \emptyset \textit{ for all } \nu \in H .$$

Also

$$S \supset \lim_\nu \sup S^\nu$$

if and only if for all $K \in \mathcal{K}$

$$(2.26) \quad S \cap K = \emptyset \Rightarrow \textit{for some } H \in \mathcal{H} ,\ S^\nu \cap K = \emptyset \textit{ for all } \nu \in H .$$

PROOF. $S \subset \lim_\nu\inf S^\nu$ if and only if for all $G \in \mathcal{G}$ such that $S \cap G \neq \emptyset$ we have that for all $H \in \ddot{\mathcal{H}}$

$$G \cap (\cup_{\nu\in H} S^\nu) \neq \emptyset ,$$

as follows from (2.16) and the fact that an open set G meets $\text{cl}\,D$ if and only if $G \cap D \neq \emptyset$. Now to say that the relation holds for all $H \in \ddot{\mathcal{H}}$ means that for some $H \in \mathcal{H}$, G must meet every set S^ν with $\nu \in H$, as follows from the duality between $\mathcal{H}$ and $\ddot{\mathcal{H}}$, in particular (2.15). And this now yields (2.25).

To prove the second half of the theorem, we observe that $S \supset \lim\sup_\nu S^\nu$ if and only if for every compact $K \in \mathcal{K}$ such that $S \cap K \neq \emptyset$, we have for some $H \in \mathcal{H}$

$$K \cap \text{cl}(\cup_{\nu\in H} S^\nu)$$

as follows from (2.17) and the fact that $\mathcal{H}$ is closed under inclusion. This clearly implies (2.26). On the other hand, if (2.26) holds but there is some points y in $K \cap \text{cl}(\cup_{\nu\in H} S^\nu)$, then this y belong to $K \cap \mathcal{H}$, as follos from the structure of $\mathcal{H}$, and hence $K \cap S \neq \emptyset$, contradicting our assumption. □

We are now in a position to build on the hyperspace $\mathcal{F}$ a topology $\mathfrak{T}$ consistent with the convergence of closed sets introduced here. A *subbase of* $\mathfrak{T}$ consists of the families of sets

$$\{\mathcal{F}_G\ ,\ G \in \mathcal{G}\} \quad \text{and} \quad \{\mathcal{F}^K\ ,\ K \in \mathcal{K}\}\ ,$$

where for any subset $Q \subset R^n$

$$\mathcal{F}_Q : = \{F \in \mathcal{F}\ |F \cap Q \neq \emptyset\}$$

and

$$\mathcal{F}^Q : = \{F \in \mathcal{F}\ |F \cap Q = \emptyset\}\ .$$

A base of open sets in $\mathfrak{T}$ is thus all sets of the type

(2.27) $\mathcal{F}^K_{G_1,\dots,G_p} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \dots \cap \mathcal{F}_{G_p}$ with p finite, where $K \in \mathcal{K}$ and for $i = 1,\dots, p$, $G \in \mathcal{G}$, (this is the collection of finite intersections of elements of the subbase); note that for any finite collection of compact sets $K_1,\dots,K_q$, one has $\mathcal{F}^{K_1} \cap \dots \cap \mathcal{F}^{K_q} = \mathcal{F}^K$, where K is the compact set $\cup_{i=1}^q K_i$. Of course we have :

2.28. COROLLARY. *Let* $\{S^\nu,\ \nu \in (N, \mathcal{H})\}$ *a filtered family in* $\mathcal{F}$. *Then* $S = \mathfrak{T} - \lim_\nu S^\nu$ *if and only if for the corresponding family of closed subsets of* R^n , $S = \lim S^\nu$.

PROOF. In view of the structural properties of the basis of $\mathfrak{T}$, one has $S = \mathfrak{T} - \lim_\nu S^\nu$ if and only if conditions (2.25) and (2.26) are satisfied for the corresponding collection of sets $\{S;\ S_\nu\ \nu \in N\}$. □

Translating all of this into the terminology of multifunctions, it becomes :

2.29. COROLLARY. *Let* $\Gamma : R^d \rightrightarrows R^n$ *be a closed-valued multifunction.*

Then Γ *is l.sc. at* u *if and only if to every open set* G *that meets* $\Gamma(u)$ *there corresponds a neighborhood* $V \in \mathcal{N}(u)$ *such that* $\Gamma(u') \cap G$ *is nonempty for all* $u' \in V$.

Γ *is u.sc. at* u *if and only if to every compact set* K *that "misses"* $\Gamma(u)$, i.e. *with* $\Gamma(u) \cap K = \emptyset$; *there corresponds a neighborhood* $V \in \mathcal{N}(u)$ *such that* $\Gamma(u') \cap K = \emptyset$ *for all* $u' \in V$.

Of course, it is continuous at u *if and only if the two preceding conditions are satisfied, or equivalently if the map* $\gamma : R^d \to \mathcal{F}$ *associated with* Γ , *is continuous at* u *with respect to the topology* $\mathfrak{T}$ *on* $\mathcal{F}$.

There are number of properties of the topological space $(\mathcal{F}, \mathfrak{T})$ that turn out to be useful in the sequel. First, it is *separated* (Hausdorff), as can easily be verified.

Second, it admits a *countable base*. Consider the base generated by the open and closed balls with rational center (in Q^n) and rational radius :

$$\text{(2.30)} \quad \mathcal{F}^{B_1 \cup B_2 \dots \cup B_q}_{B^\circ_1,\dots,B^\circ_p} \quad \text{with } p \text{ and } q \text{ finite,}$$

where B_i denotes a closed ball, and B°_i an open ball (not necessarily related to B°_i). Clearly it is a countable base for a topology on $\mathcal{F}$; all what needs to be verified is that it actually generates $\mathfrak{T}$. Since the sets generated by the finite union of closed balls form a subclass of $\mathfrak{T}$, and the open balls are in $\mathcal{G}$, the topology generated by the base of open sets of type (2.30) is possibly coarser than $\mathfrak{T}$. That it is actually as fine as $\mathfrak{T}$ can be argued as follows. Consider F in $\mathcal{F}$ and $\mathcal{F}^K_{G_1,\dots,G_1}$ in the fundamental $\mathfrak{T}$ - neighborhood system of F ; we need to exhibit a neighborhood of F of type (2.30) contained in $\mathcal{F}^K_{G_1,\dots,G_p}$. We obtain this from the topological properties of R^n , namely (i) that if K is compact set separated from F , then there is a finite cover of K by closed rational balls whose union also fails to meet F , say $B_1,\dots,B_q$, and (ii) if G_i is an open set that meets F , then it contains an open rational ball, say B°_i , that also meets F .

Third, $(\mathcal{F}, \mathfrak{T})$ is a *compact* space. We derive this by relying on Alexander's characterization of compactness, see Kelley, 1955. Observe that the family of sets

$$\{\mathcal{F}_K\ ,\ K \in \mathcal{K}\} \quad \text{and} \quad \{\mathcal{F}^G\ ,\ G \in \mathcal{G}\}$$

is a subbase of closed sets for the topology $\mathfrak{T}$; they are the complements of the open sets used in our original construction of $\mathfrak{T}$. We need to show that any arbitrary collection of elements of this subbase with empty intersection contains a finite subcollection with the same property. Let $\{K_i\ ,\ i \in I\}$ be compact and $\{G_j\ ,\ j \in J\}$ open subsets of R^n such that

$$(\cap_{i \in I}\ \mathcal{F}_{K_i}) \cap (\cap_{j \in J}\ \mathcal{F}^{G_j}) = \emptyset\ ,$$

where I and J are arbitrary index sets. With $G : = \cup_{j \in J} G_j$, the above can be rewritten as

$$\cap_{i \in I} (\mathcal{F}_{K_i} \cap \mathcal{F}^{G}) = \emptyset ,$$

which holds if and only if for some K in $\{K_i , i \in I\}$

$$\mathcal{F}_K \cap \mathcal{F}^G = \emptyset ,$$

or also, if and only if $K \subset G = \cup_{j \in J} G_j$. Since K is compact, there is a cover of K by a finite number of elements $G_1, \ldots G_p$ of $\{G_j , j \in J\}$. This yields the desired finite collection, since

$$\mathcal{F}_K \cap (\mathcal{F}^{G_1} \cap \ldots \cap \mathcal{F}^{G_p}) = \emptyset .$$

Summarizing our results, we have

2.31. THEOREM. $(\mathcal{F}, \mathfrak{T})$ *is a separated, compact topological space with countable base, and hence also metrizable.*

Metrizability is a direct consequence of the preceeding properties; consult Kelley, 1955, for example. We shall actually exhibit a metric consistent with $\mathfrak{T}$, but before we do this, let us record one of the main implications of this theorem.

2.32. COROLLARY. *Given any filtered family* $\{S^\nu, \nu \in (N, \mathcal{H})\}$ *of closed subsets of* R^n , *there always exists a subfamily* $\{S^\nu, \nu \in (N', \mathcal{H})\}$ *that converges,* i.e. *such that* $\lim_{\nu \in N'} S^\nu$ *exists, possibly the empty set.*

Before we begin with the construction of a metric on $\mathcal{F}$, it is useful to record the following special case of the second part of theorem 2.24 that characterizes convergence to the empty set.

2.33. LEMMA. *Suppose* $\{S^\nu, \nu \in (N, \mathcal{H})\}$ *is a filtered family of subsets of* R^n . *Then the condition*

$$\limsup_{\nu \in N} S^\nu = \lim_{\nu \in N} S^\nu = \emptyset$$

holds if and only if to every $\varepsilon > 0$, *there corresponds* $H \in \mathcal{H}$ *such that*

$$\varepsilon^{-1} B \cap S^\nu = \emptyset \quad \textit{for all} \quad \nu \in H ,$$

where $\varepsilon^{-1}B$ *is the closed ball of radius* ε^{-1} *and center at* 0 .

The Hausdorff distance between two nonempty sets C and D of a metric space is defined as follows :

$$\text{haus}\,(C,D) := \inf \{\varepsilon \mid \varepsilon^\circ C \supset D , \varepsilon^\circ D \supset C\}$$

where $\varepsilon^\circ Q$ is for $\varepsilon > 0$, the ε-*enlargement* of the set Q , i.e.

$$\varepsilon^\circ Q := \{ y \mid \text{dist}\,(x,Q) < \varepsilon\} .$$

Here dist is the distance function on the underlying metric space. The Hausdorff distance is nonnegative; haus $(C,D) = 0$ if and only if $C = D$. The triangle inequality follows directly from the fact that for any set E , $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$:

$$\varepsilon_1^\circ C \supset E \text{ and } \varepsilon_2^\circ E \supset D \text{ implies } (\varepsilon_1 + \varepsilon_2)^\circ C \supset D ,$$

whereas

$$\varepsilon_1^\circ E \supset D \text{ and } \varepsilon_2^\circ D \supset E \text{ implies } (\varepsilon_1 + \varepsilon_2)^\circ D \supset C .$$

In order to express the convergence of sets in terms of a metric, we resort to a one point compactification of R^n , which we render concrete by means of the stereographic projection of R^n on the sphere $S^n \subset R^{n+1}$, see Figure 5. The *stereographic*

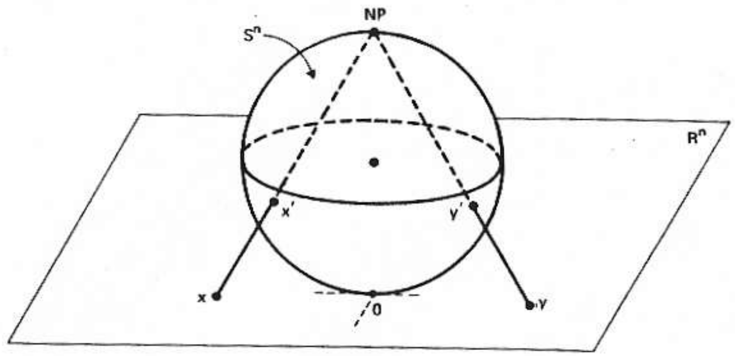

Figure 5. Stereographic projection of x, y on S^n.

distance between two points x and y of R^n is

$$\text{dist}^S(x,y) = \text{haus}(\{x'\} \cup NP, \{y'\} \cup NP)$$

where x' and y' are the stereographic projection of x and y on S^n and NP is the north pole of S^n ; haus is the Hausdorff distance in R^{n+1} . Between x and $\emptyset$, one has

$$\text{dist}^S(x,\emptyset) = \text{haus}(\{x'\} \cup NP, NP) .$$

Of course, we have identified the north pole NP of S^n with the empty subset of R^n . The *stereographic Hausdorff distance* between two arbitrary subsets C , D of R^n is given by.

$$\text{haus}^S(C,D) = \text{haus}(C' \cup NP, D' \cup NP)$$

where C' and D' are the stereographic projections of C and D on S^n . If C and D are closed, then so are C' and D' ; in fact $C' \cup NP$ and $D' \cup NP$ are then nonempty compact subsets of the compact sphere S^n . The stereographic Hausdorff distance haus^S is thus a metric on $\mathcal{F}$ which is bounded above by the diameter of S^n . It remains to show that this metric is consistent with the topology of set convergence.

2.35. THEOREM. *Consider* $\{S^\nu, \nu \in (N,\mathcal{H})\}$ *, a filtered family of closed subsets of* R^n *. Then* $\lim_{\nu \in N} S^\nu = S$ *if and only if*

$$\lim_{\nu \in N} \text{haus}^S(S^\nu, S) = 0 .$$

PROOF. As a direct consequence of the definition of lower and upper limits and Lemma 2.33 we have that $\lim_{\nu\in N} S^\nu = S$ if and only if $\lim_{\nu \in N}(S^{\nu\prime} \cup NP) = S' \cup NP$, where $S^{\nu\prime}$ and S' are the stereographic projections of S^ν and S respectively. To complete the proof it suffices to use the definition of "haus" and apply the following lemma. □

2.36. LEMMA. *Consider* $\{S^\nu, \nu \in (N,\mathcal{H})\}$ *, a filtered family of nonempty closed subsets of a compact set* $D \subset R^n$ *. Then the* S^ν *converge to the nonempty set* $S \subset D$ *if and only if*

$$\text{haus}(S^\nu, S) \to 0 .$$

PROOF. We begin by showing that $S \subset \text{cl}(\cup_{\nu\in H} S^\nu)$ for all $H \in \ddot{\mathcal{H}}$ if and only if for all $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that for all $\nu \in H$, $S \subset \varepsilon^\circ S^\nu$. The "if" part is self evident. The "only if" part is argued by contradiction. Suppose that there exists an $\varepsilon > 0$ such that for all $H \in \mathcal{H}$ there always is some $\nu \in H$ such that $S \not\subset \varepsilon^\circ S^\nu$, or equivalently that there exists $H' \in \ddot{\mathcal{H}}$ --using here (2.14)-- such that for all $\nu \in H'$, $S \not\subset \varepsilon^\circ S^\nu$. This means that $S \not\subset \text{cl}(\cup_{\nu\in H'} S^\nu)$ and consequently

$$S \not\subset \cap_{H\in\ddot{\mathcal{H}}} \text{cl}(\cup_{\nu\in H} S^\nu) = \liminf_{\nu\in N} S^\nu ,$$

contradicting the hypothesis.

Next we show that $S^\nu \supset \limsup_{\nu\in N} S$ if and only if for all $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that, for all $\nu \in H$, one has $S^\nu \subset \varepsilon^\circ S$. For the "if" part, note simply that

$$S = \text{cl}\, S = \cap_{\varepsilon>0} \varepsilon^\circ S \subset \cap_{\varepsilon>0} \text{cl}(\cup_{\nu\in H_\varepsilon} S^\nu) \subset \cap_{H\in\mathcal{H}} (\text{cl}\cup_{\nu\in H} S^\nu) ,$$

where H_ε denotes a member of $\mathcal{H}$ so that $S^\nu \subset \varepsilon^\circ S$ for all $\nu \in H_\varepsilon$. For the only if part we appeal to Theorem 2.24 which implies that if $S \supset \limsup_{\nu\in N} S^\nu$, then to every $\varepsilon > 0$ there corresponds $H \in \mathcal{H}$ such that $(D\setminus \varepsilon^\circ S) \cap S^\nu = \emptyset$, since $D\setminus \varepsilon^\circ S$ is compact and clearly $(D\setminus \varepsilon^\circ S) \cap S = \emptyset$. But this is the same as the assertion that $S^\nu \subset \varepsilon^\circ S$ for all $\nu \in H$.

So far we have shown that

$$S = \lim_{\nu\in N} S^\nu = \liminf_{\nu\in N} S^\nu = \limsup_{\nu\in N} S^\nu$$

if and only if to every $\varepsilon > 0$, there corresponds $H \in \mathcal{H}$ such that

$$\varepsilon^\circ S^\nu \supset S \quad \text{and} \quad \varepsilon^\circ S \supset S^\nu \quad \text{for all } \nu \in H .$$

(We used the fact that $\mathcal{H}$ is closed under finite intersections). Equivalently in view of the definition of the Hausdorff distance, the latter condition means that

$$\text{haus}(S^\nu, S) \leq \varepsilon \quad \text{for all } \nu \in H .$$

Hence S^ν converges to S if and only if $\text{haus}(S^\nu, S) \to 0$. □

For multifunctions we have the following version of Theorem 2.35 and Lemma 2.36.

2.37. COROLLARY. *A closed-valued multifunction* $\Gamma : R^d \rightrightarrows R^n$ *is continuous at* u *if and onlu if*

$$\lim_{u' \to u} \text{haus}^S(\Gamma(u'), \Gamma(u)) = 0 .$$

If Γ *is compact-nonempty-valued on a neighborhood of* u *on which it is uniformly bounded, it is continuous at* u *if and only if*

$$\lim_{u' \to u} \text{haus}(\Gamma(u'), \Gamma(u)) = 0 .$$

Let us make a couple of observations about continuity of multifunctions before we proceed any further. When the multifunction is uniformly bounded, or in a number of other related cases, the definition of continuity introduced here is consistent with what is expected. However, if Γ is not uniformly bounded then continuity may very well not be all what we expect from such a concept. Consider for example the multifunction $\Gamma : R \rightrightarrows R$ with

$$\Gamma(u) = \begin{cases} \{|u^{-1}|\} & \text{if } u \neq 0 , \\ \emptyset & \text{if } u = 0 . \end{cases}$$

It is easy to verify that multifunction is continuous, in particular at $u = 0$, see Figure 6. So is the multifunction Γ' with

$$\Gamma'(u) = \begin{cases} \{u^{-1}\} & \text{if } u \neq 0 , \\ \emptyset & \text{if } u = 0 ; \end{cases}$$

see Figure 6. The difference is that Γ and Γ' do not tend to the compactification point "∞" in the same fashion. (In order to avoid an example in which Γ and Γ' are empty at 0 , simply condiser $\Gamma_1 = \Gamma \cup 0$ and $\Gamma_1' = \Gamma' \cup 0$, where 0 is the constant multifunction which takes on the value $\{0\}$ everywhere). Naturally the stereographic projections of the graphs of these multifunctions on S^1 restores our sense of propriety, the projected graph of Γ' is "continuous" at 0 . But nonetheless, we are still confronted with the fact that for u sufficiently close to 0 , the sets $\Gamma'(u)$ and $\Gamma'(-u)$ are very "far" apart, whereas $\Gamma(u)$ and $\Gamma(-u')$ are very close to each, and no distinction is made as far as continuity at 0 is concer-

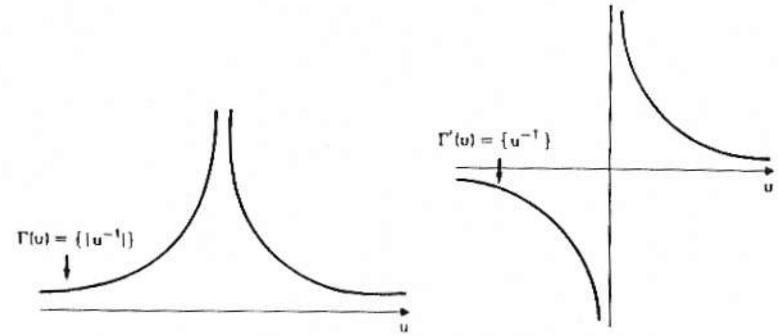

Figure 6. Continuous multifunctions.

ned between Γ and Γ' . In many applications, it is imperative to distinguish between the directions of recession, i.e. the asymptotic behavior at ∞ of the values of the multifunction. This leads us to a more stringent notion of continuity for multifunctions requiring in addition "continuity" of the directions of recession; this is elaborated in Rockafellar and Wets, 1985.

Continuity of the function $\gamma : R^d \to \mathcal{F}$, associated to the multifunction $\Gamma : R^d \rightrightarrows R^n$, has been defined in terms of the topology $\mathfrak{T}$. In view of what precedes, in particular Theorem 2.24, it is easy to see that lower semicontinuity of Γ corresponds to continuity of γ with respect to the topology $\mathfrak{T}_{lsc}$ with base $\{ \mathcal{F}_G , G \in \mathcal{G} \}$, and upper semicontinuity of Γ corresponds to continuity of γ with respect to $\mathfrak{T}_{usc}$ generated by the base $\{ \mathcal{F}^K , K \in \mathcal{K} \}$.

We now leave the subset of continuity per se and turn to measurability. As in the case of functions, measurability for multifunctions is a way of qualifying the dependence of the values on a certain parameter. The basic difference between measurability and continuity is that the underlying structure is that of a measure space instead of a topological space, which in a sense is coarser. Again we shall limit ourselves to multifunctions whose values are closed subsets of R^n , but we allow the domain to be an arbitrary set Ω equipped with a sigmafield $\mathcal{A}$. Of course $(\Omega, \mathcal{A})$ could be R^d , equipped with the Borel field, in particular with $d = 1$ when the parameter in question is time. Or it could be a sample space with $\mathcal{A}$ the sigma-field of events, and so on.

It will not be possible here to review all that is known about measurable multifunctions; for that the reader should consult for example Rockafellar, 1976, Castaing and Valadier, 1977, and the bibliography of Wagner, 1977, supplemented by Ioffe, 1978. We shall restrict ourselves to bringing out the connections between the concept of measurability and that of continuity for multifunctions. It is standard procedure to begin with a definition of measurability for closed-valued multifunctions and then study its implications, in particular what measurability means in terms of the associated function with values in $\mathcal{F}$. Here we go about it exactly in the opposite way.

Since $(\mathcal{F}, \text{haus}^s)$, or equivalently $(\mathcal{F}, \mathfrak{T})$, is a metric space, there is a natural concept of measurability for functions γ defined on a measure space $(\Omega, \mathcal{A})$ and having values in $\mathcal{F}$. Namely, let $\mathcal{B}$ be the Borel field generated by the class of $\mathfrak{T}$- open sets on $\mathcal{F}$. From the construction of $\mathfrak{T}$ and its properties (Theorem 2.31), the following classes of sets :

$$\{\mathcal{F}^K, K \in \mathcal{K}\}, \{\mathcal{F}_G, G \in \mathcal{G}\},$$

$$\{\mathcal{F}^B, B \text{ a closed ball}\}, \{\mathcal{F}_{B^\circ}, B^\circ \text{ an open ball}\},$$

$$\{\mathcal{F}^B, B \text{ a closed rational ball}\},$$

$$\{\mathcal{F}_{B^\circ}, B^\circ \text{ an open rational ball}\},$$

and their complements are all generators of $\mathcal{B}$ (taking complements and forming countable unions). Consequently we have that $\gamma : (\Omega, \mathcal{A}) \to (\mathcal{F}, \mathcal{B})$ is measurable if and only if

(i) for all $G \in \mathcal{G}$, $\gamma^{-1}(\mathcal{F}_G) \in \mathcal{A}$,

(ii) for all $K \in \mathcal{K}$, $\gamma^{-1}(\mathcal{F}_K) \in \mathcal{A}$,

(iii) for all $F \in \mathcal{F}$, $\gamma^{-1}(\mathcal{F}_F) \in \mathcal{A}$,

(iv) for all closed rational balls B, $\gamma^{-1}(\mathcal{F}_B) \in \mathcal{A}$, and so on.

Since each function $\gamma : \Omega \to \mathcal{F}$ is associated with a closed-valued multifunction $\Gamma : \Omega \to R^n$, we say that Γ is *measurable* if γ is measurable. Since for any set $D \subset R^n$

$$\gamma^{-1}(\mathcal{F}_D) = \{w \in \Omega | \gamma(w) \in \mathcal{F}_D\} = \{w | \Gamma(w) \cap D \neq \emptyset\} = \Gamma^{-1}(D),$$

the preceding characterizations of measurability of γ yield :

2.39. PROPOSITION. *The following are equivalent :*

(i) Γ *is measurable;*

(ii) *for every open set* $G \subset R^n$, $\Gamma^{-1}(G)$ *is measurable;*

(iii) *for every compact set* $K \subset R^n$, $\Gamma^{-1}(K)$ *is measurable;*

(iv) *for every closed set* $F \subset R^n$, $\Gamma^{-1}(F)$ *is measurable;*

(v) *for every closed rational ball* B, $\Gamma^{-1}(B)$ *is measurable.*

Of course this list could be continued at infinitum, all that is needed is to identif a class of sets $\{D \in \mathcal{D}\}$ such that the sets $\{\mathcal{F}_D, D \in \mathcal{D}\}$, or their complements, generate $\mathcal{B}$.

However, there is more to measurable multifunctions than the characterizations obtained as a direct consequence of the measurability of the associated functions (with values in the hyperspace $\mathcal{F}$). A countable (possibly finite) collection of measurable functions

$$\{x^\nu : \text{dom}\,\Gamma \to R^n, \nu = 1, \ldots\}$$

is said to be a *Castaing representation* of the multifunction $\Gamma : \Omega \rightrightarrows R^d$ if dom Γ is measurable $(\in \mathcal{A})$ and for all $w \in \text{dom}\,\Gamma$

$$\Gamma(w) = \text{cl} \bigcup_{\nu=1}^{\infty} x^\nu(w),$$

i.e. the set $\{x^\nu(w), \nu = 1, \ldots\}$ is dense in $\Gamma(w)$. It is remarkable that measurability of Γ can be expressed in terms of the existence of Castaing representations. Indeed we have :

2.40. THEOREM. *A closed-valued multifunction* $\Gamma : \Omega \rightrightarrows R^n$ *is measurable if and only if it admits a Castaing representation.*

PROOF. See Rockafellar, 1976. Theorem 1B, for example. □

2.41. COROLLARY. (Theorem of Measurable Selections). *If* $\Gamma : \Omega \rightrightarrows R^n$ *is a closed-valued measurable multifunction, then there is at least one measurable selector*, i.e. *a measurable function* $x : \text{dom}\,\Gamma \to R^n$ *such that for all* $w \in \text{dom}\,\Gamma$, $x(w) \in \Gamma(w)$.

3. VARIATIONAL SYSTEMS : EPICONTINUITY AND NORMAL INTEGRANDS

Problems rarely occur in isolation. In almost every situation modeled in applied mathematics, there are various sets and functions that describe any particular instance of the model, but these depend on certain parameters. When the parameters are varied, a whole family of problems having the same structure is generated. The interesting thing is that parametrization in this sense is not useful merely for the sake of giving treatment to similar situations. The parameters often have a fundamental role in the analysis. This is all the more true in applications involving time, random variables, or numerical approximations, where exact values may not be available for the parameters. It is inescapable then that the study of one instance of a problem entails the study of other possible instances.

We have seen in Section 2 how the notion of a set that depends on parameters can be investigated in terms of multifunctions. In this section we take up in parallel fashion the notion of a function that depends on parameters, the goal being to develop the epigraphical approach explained in Section 1.

Minimization problems are again good reference point for motivation. Such problems often depend on parameters and can be modeled abstractly as in Section 1, for instance as

(3.1) find $x \in R^n$ that minimizes $f(x,u)$,

where $f : R^n \times R^d \to \overline{R}$ is the essential objective function of a certain parameterized optimization problem, say

$$f(x,u) = \begin{cases} F_0(x,u) & \text{if } F_i(x,u) \leq 0 , \quad i = 1,\ldots,s, \\ & \quad F_i(x,u) = 0 , \quad i = s+1,\ldots,m, \\ & \quad x \in X \subset R^n , \quad u \in U \subset R^d ; \\ +\infty & \text{otherwise.} \end{cases}$$

where the F_i are real-valued functions on the product $X \times U$ of closed sets X and U , as in (1.8).

Generalizations are immediate. There is no reason why we cannot start with an arbitrary function $f : R^n \times R^d \to \overline{R}$ and consider (3.1) as parameterized by u ranging over R^d . The beauty is that although constraints do not have to appear explicitly, they are present nonetheless. Likewise, although no restriction on the parameter u has to be mentioned, we implicitly need $u \in U \subset R^d$. The specification of f thus embodies at the same time the specification of a constraint system depending on parameters, and it does so in a very flexible manner whose virtues should be

easy to appreciate by now.

If a minimization problem can be described by a single function, then a problem that depends on a parameter vector $u \in R^d$ must correspond to a function that depends on u . As already suggested in Section 1, a function $f : R^n \times R^d \to \overline{R}$ does not tell quite the right story. What we have in mind is rather a correspondence that assigns to each u a function $f(\cdot,u)$ on R^n , i.e. a function-valued mapping $u \to f(\cdot,u)$, and we want to study how certain objects associated with $f(\cdot,u)$ --its epigraph, its infimum, etc.-- depend on u . Mappings which assign to each $u \in R^d$ a function on R^n have been called *bifunctions* in convex analysis, but here we are going to adopt a different terminology that appears more appropriate to the general setting that we deal with.

By a *variational system* on R^n with parameter space R^d we shall mean a parameterized family of extended-real-valued functions :

(3.2) $\mathcal{F} = \{f_u = R^n \to \overline{R} \mid u \in R^d\}$.

(There should be no confusion between the use of the symbol $\mathcal{F}$ to denote here a variational system and in Section 2 the class of closed subsets of R^n). The definition could obviously be generalized, and it will be when we consider normal integrands, to an arbitrary measure space instead of R^d . The function f_u is called the *valuate* of the system $\mathcal{F}$ corresponding to u . We speak of the function

(3.3) $f(x,u) = f_u(x)$ on $R^n \times R^d$

as the *conjuctive function* associated with $\mathcal{F}$.

In principle, of course, the parameterized family (3.2) is identical to a mapping $u \mapsto f_u$, the first stage of a two-stage correspondence

$$u \mapsto f(\cdot,u) \mapsto f(x,u) .$$

This is reflected in an alternative notation that we shall sometimes use to indicate a variational system $\mathcal{F}$ on R^n parameterized by R^d , namely

(3.4) $\mathcal{F} : R^d \nearrow R^n$, $\mathcal{F}(u) = f_u$.

(The symbol $\nearrow$ is intended as a reminder that $\mathcal{F}$ assigns to each $u \in R^d$ a *function on* R^n).

This terminology and notation is designed to give us flexibility emphasizing distinctly different roles for u and x in treating a quantity $f(x,u)$ that depends both on x and u but not quite in the same manner. In the context of the variational system $\mathcal{F}$, properties of $f(x,u)$ with respect to x will typically be properties of the individual valuates f, while properties with respect to u will typically involve the way function f_u as a *whole* depends on u, not just the way $f_u(x)$ depends on u for fixed x.

In some situations, for example, we may wish f_u to be a continuous function on R^n for each fixed u. Then we talk about a *variational system* $\mathcal{F}$ *with continuous valuates.* On the other hand, we may wish the entire function f_u to depend continuously on u in the sense of its epigraphs in R^{n+1} varying continuously with u. Then we talk instead about an *epicontinuous variational system* $\mathcal{F}$; this concept will only be defined later on, but the reader can easily get the picture. In the classical setting the distinction does not take on such proportions, because, for instance, continuous dependence of f_u on u can often be expressed adequately in the pointwise sense of $f_u(x)$ depending continuously on u for each x. That is *not* true here, due to the way ∞ is being used to represent constraints, as well as for other reasons.

The situations where in dealing with $f(x,u)$ we prefer to think of a variational system $\mathcal{F}$ are largely the ones where properties of the epigraphical multifunction

$$u \mapsto \text{epi } f(u,\cdot)$$

assume importance. It should be clear that this is not limited to situations where minimization in x is at issue. In any framework where the epigraphical approach to analysis is natural, and several have been mentioned in Section 1, the idea of a variational system is likewise natural. The word "variation" is intended to refer primarily to the dependence of a function f_u on a parameter vector u, but the suggestion of a relationship to variational problems is a welcome coincidence.

We shall denote by epi $\mathcal{F}$ the *epigraphical multifunction* $u \mapsto \text{epi } f_u$ associated with a variational system $\mathcal{F} = \{f_u : R^n \to \overline{R} \mid u \in R^d\}$; thus

$$(3.5) \quad \text{epi}\,\mathcal{F} \ : R^d \rightrightarrows R^{n+1}, \quad (\text{epi } \mathcal{F})(u) = \text{epi } f_u .$$

The graph of epi $\mathcal{F}$ is the set

$$(3.6) \quad \text{hyper } \mathcal{F} \ : = \{(u,x,\alpha) \in R^d \times R^{n+1} \mid \alpha \geq f_u(x)\} ,$$

which is called the *hypergraph* of $\mathcal{F}$. It is identical to the epigraph of the conjuctive function f of $\mathcal{F}$:

$$(3.7) \quad \text{hyper } \mathcal{F} = \text{gph (epi } \mathcal{F}\text{)} = \text{epi } f .$$

Other useful multifunctions associated with $\mathcal{F}$ are the *domain multifunction*

$$(3.8) \quad \text{dom } \mathcal{F} \ : R^d \rightrightarrows R^n , \ (\text{dom } \mathcal{F})(u) : = \text{dom } f_u ,$$

and the *level set multifunction*

$$(3.9) \quad \text{lev}_\alpha \ \mathcal{F} \ : R^d \rightrightarrows R^n , \ (\text{lev}_\alpha \ \mathcal{F})(u) : = \text{lev}_\alpha \ f_u .$$

We shall restrict our analysis to *variational system with lower semicontinuous (l.sc.) valuates*, i.e. such that for all $u \in R^d$, the function $x \mapsto f_u(x)$ is l.sc. or equivalently (Theorem 1.17) such that the epigraphical multifunction $u \to (\text{epi } \mathcal{F})(u)$ is closed-valued. This is in keeping with the case considered in Section 2, and for practical purposes it covers all the applications of interest, certainly all those mentioned in Section 1.

In parallel to the development in Section 2, we begin with the concept of epi-limits for sequences of (l.sc.) functions. Let $\{f;f_\nu,\nu = 1,\ldots\}$ be a collection of extended-real-valued function defined on R^n. We say that the f_ν *epi-converge to* f *at* x if

$$(3.10) \quad \liminf_{\nu\to\infty} f_\nu(x) \geq f(x) \quad \text{for all sequences} \quad x^\nu \to x ,$$

and

$$(3.11) \quad \limsup_{\nu\to\infty} f_\nu(x^\nu) \leq f(x) \quad \text{for some sequence} \quad x^\nu \to x .$$

The functions $\{f_\nu ,\nu = 1,\ldots\}$ *epi-converge* to f, equivalently

$$f = \text{epi-}\lim_{\nu\to\infty} f_\nu \quad ,$$

if the conditions (3.10) and (3.11) hold for all $x \in R^n$. Although closely connected to the notion of pointwise convergence it is neither stronger nor weaker. In fact, certain sequences of functions have different pointwise and epi-limits. Consider the sequence

$$f_\nu(x) = \begin{cases} 0 & \text{if } x = \nu^{-1} , \\ 1 & \text{if } x \neq \nu^{-1} , \end{cases}$$

that pointwise converges to the function

$$f'(x) = 1 \quad \text{for all} \quad x$$

and epi-converges to

$$f(x) = \begin{cases} 0 & \text{if } x = 0 , \\ 1 & \text{if } x \neq 0 . \end{cases}$$

The epi-limit takes into account the behavior of the f in the neighborhood of 0, whereas the pointwise limit restricts attention to what happens with the f_ν at the point 0.

Following the same pattern as that for the limits of sequences of sets, we can associate with any sequence of functions $\{f_\nu, \nu = 1, \ldots\}$ a lower and upper epi-limit, and declare that the limit exists if both are equal (at x). This approach will allow us to transpose our results about sequences of closed sets and multifunction to this new context : sequences of l.sc. functions and variational systems. We may as well work directly with filtered families of functions $\{f_\nu, \nu \in (N, \mathcal{H})\}$ recalling naturally that the case of sequences is just $N = \mathbb{N}$ with $\mathcal{H}$ the Fréchet filter, cf. Section 2.

The *upper epi-limit at* x of a filtered family $\{f_\nu, \nu \in (N, \mathcal{H})\}$ is

$$\text{(3.12)} \quad (\text{epi-}\lim_{\nu \in N} \sup f_\nu)(x) := \sup_{V \in \mathcal{N}(x)} \limsup_{\nu \in N} \inf_{x' \in V} f_\nu(x') ,$$

and its *lower epi-limit at* x is

$$\text{(3.13)} \quad (\text{epi-}\lim_{\nu \in N} \inf f_\nu)(x) := \sup_{V \in \mathcal{N}(x)} \liminf_{\nu \in N} \inf_{x' \in V} f_\nu(x') .$$

Clearly

$$\text{(3.14)} \quad (\text{epi-}\lim_{\nu \in N} \inf f_\nu)(x) \leq (\text{epi-}\lim_{\nu \in N} \sup f_\nu)(x) .$$

The *epi-limit*,

$$(\text{epi-}\lim_{\nu \in N} f_\nu)(x) ,$$

is said to exist if equality holds in (3.14). Thus a function f is the *epi-limit* of the filtered family $\{f_\nu, \nu \in (N, \mathcal{H})\}$ if for all x

$$\text{(3.15)} \quad (\text{epi-}\lim_{\nu \in N} \sup f_\nu)(x) \leq (\text{epi-}\lim_{\nu \in N} \inf f_\nu)(x)$$

which we more simply write as

$$f = \text{epi-}\lim_{\nu \in N} f_\nu .$$

It is an easy exercise to verify that in the sequential case (3.12) and (3.13) can be expressed in the following terms :

$$(\text{epi-}\lim_{\nu \in N} \sup f_\nu)(x) = \inf_{x^\nu \to x} \limsup_{\nu \to \infty} f_\nu(x^\nu) ,$$

and

$$(\text{epi-}\lim_{\nu \in N} \inf f_\nu)(x) = \inf_{\substack{x^k \to x \\ \{\nu_k\} \subset \mathbb{N}}} \liminf_{k \to \infty} f_{\nu_k}(x^k) .$$

These allow us to recognize immediately in this case the equivalence between the original definition, (3.10) and (3.11), and that via limit functions.

The terminology "epi-convergence", "epi-limit", etc. find its justification in the following result.

3.16. THEOREM. *Suppose* $\{f_\nu, \nu \in (N, \mathcal{H})\}$ *is a filtered family of l.sc. extended-real-valued functions defined on* R^n. *Then*

$$\text{(3.17)} \quad \text{epi}\,(\text{epi-}\lim_{\nu \in N} \sup f_\nu) = \liminf_{\nu \in N} \text{epi}\, f_\nu ,$$

and

$$\text{(3.18)} \quad \text{epi}\,(\text{epi-}\lim_{\nu \in N} \inf f_\nu) = \limsup_{\nu \in N} \text{epi}\, f_\nu .$$

PROOF. Recall that for a collection of filtered extended-reals $\{\alpha_\nu, \nu \in (N, \mathcal{H})\}$ we have

$$\liminf_{\nu \in N} \alpha_\nu = \sup_{H \in \mathcal{H}} \inf_{\nu \in H} \alpha_\nu ,$$

and

$$\limsup_{\nu \in N} \alpha_\nu = \inf_{H \in \mathcal{H}} \sup_{\nu \in H} \alpha_\nu .$$

Thus

$$\text{epi}(\text{epi-}\limsup_{\nu \in N} f_\nu) = \{(x,\alpha) | \alpha \geq \sup_{V \in N(x), H \in \ddot{\mathcal{H}}} \inf_{\nu \in H, y \in V} f_\nu(y)\}$$

$$= \{(x,\alpha) | \forall (H \in \ddot{\mathcal{H}}, V \in N(x), \varepsilon > 0)\ \exists (\nu \in H, y \in V) \text{ with } f_\nu(y) < \alpha + \varepsilon\}$$

$$= \{(x,\alpha) | \forall (H \in \ddot{\mathcal{H}}, V \in N(x), \varepsilon > 0), [V \times (-\infty, \alpha+\varepsilon)] \cap (\cup_{\nu \in H} \text{epi } f_\nu) \neq \emptyset\}$$

$$= \{(x,\alpha) | \forall H \in \ddot{\mathcal{H}}, (x,\alpha) \in \text{cl}(\cup_{\nu \in H} \text{epi } f_\nu)\}$$

$$= \cap_{H \in \ddot{\mathcal{H}}} \text{cl}(\cup_{\nu \in H} \text{epi } f_\nu) = \liminf_{\nu \in H} \text{epi } f_\nu$$

The last equality comes from the definition of the upper limit of the filtered family of closed sets (2.17). The proof of (3.17) is identical except that $\ddot{\mathcal{H}}$ needs to be replaced by the filter $\mathcal{H}$. □

This theorem implies that the limit functions are necessarily lower semicontinuous and means that continuity questions can be addressed in the framework provided by the theory of multifunctions. A variational system $\mathcal{F} = \{f_u | u \in R^d\} : R^d \nearrow R^n$ with l.sc. valuates is *upper epi-semicontinuous at* u if for the filtered family

$$(f_{u'}, u' \in (R^d , N(u)))$$

we have

$$(3.19) \quad (\text{epi-}\limsup_{u' \to u} f_{u'}) \leq f_{u'}$$

or equivalently

$$\liminf_{u' \to u} \text{epi } f_{u'} \supset \text{epi } f_u \quad ;$$

as in Section 2, the notation $u' \to u$ suggests the filtering process by the neighborhood system $N(u)$. Similary $\mathcal{F} = \{f_u | u \in R^d\} : R^d \nearrow R^n$ is *lower epi-semicontinuous at* u if

$$(3.20) \quad (\text{epi-}\liminf_{u' \to u} f_{u'}) \geq f_u \quad ,$$

or equivalently

$$\limsup_{u' \to u} \text{epi } f_{u'} \subset \text{epi } f_u \quad .$$

Finally, $\mathcal{F}$ is *epicontinuous at* u if is both lower and upper epi-semicontinuous at u , i.e. if

$$(3.21) \quad \text{epi-}\limsup_{u' \to u} f_{u'} \leq f_u \leq \text{epi-}\liminf_{u' \to u} f_{u'} \quad .$$

The variational system $\mathcal{F}$ is *lower* or *upper epi-semicontinuous* or *epicontinuous* if the corresponding property holds for all u in R^d .

Every result of Section 2, in particular every characterization of semicontinuity for multifunctions, can now be translated in terms of variational systems. We do not intend to do so except in one particular instance which is of direct interest in the description of the dependence on u of the infima and the optimal solutions of variational systems.

3.22. THEOREM. *Suppose* f *and* $\{f_\nu , \nu \in (N,\mathcal{H})\}$ *, a filtered family, are l.sc. extended-real-valued functions defined on* R^n *. Then*

$$f \geq \text{epi}\limsup_{\nu \in N} f_\nu$$

if and only if for all open $G \subset R^n$

$$(3.23) \quad \limsup_{\nu \in N} (\inf_G f_\nu) \leq \inf_G f \ .$$

Also,

$$f \leq \text{epi-}\liminf_{\nu \in N} f_\nu$$

if and only if for all compact $K \subset R^n$

(3.24) $\liminf_{\nu\in N} (\inf_K f_\nu) \geq \inf_K f$.

PROOF. We apply Theorem 2.24 to epi f and the filtered family $\{\text{epi } f_\nu, \nu \in (N,\mathcal{H})\}$. We have

$$f \geq \text{epi-}\limsup_{\nu\in N} f_\nu \quad \text{if and only if} \quad \text{epi } f \subset \liminf_{\nu\in N} \text{epi } f_\nu$$

by (3.17), or if and only if for all open $G' \in R^{n+1}$ the condition $\text{epi } f \cap G' \neq \emptyset$ implies that for some $H \in \mathcal{H}$ one has $\text{epi } f_\nu \cap G' \neq \emptyset$ for all $\nu \in H$, as follows from (2.25). Since these are epigraphs, and the open sets $G' \subset R^{n+1}$ can be generated by the open sets $G \times (a',a)$ with G an open subset of R^n, we can reexpress the preceding implication as :

$$[\inf_G f > a] \Rightarrow [\text{for some } H \in \mathcal{H}, \inf_G f_\nu > a \text{ for all } \nu \in H] .$$

But this holds if and only if (3.23) holds.

The proof of (3.24) is identical, except this time we rely on (3.18) and (2.26).□

There are numerous corollaries to this theorem, in particular about the convergence of infima. We shall come to these, but first let us rework this result in the terminology of variational systems and study its implications for the construction of an epi-topology on the space of lower semicontinuous functions.

For a function $f : R^n \to \bar{R}$, we define

$$\inf f : = \inf_{x\in R^n} f(x) .$$

For an arbitrary subset D of R^n, we write

$$\inf_D f : = \inf_{x\in D\subset R^n} f(x) .$$

The infinimum of f on D, $\inf_D f$, may be a real number, or $-\infty$ (if f is not bounded below) or even ∞ (if $D \cap \text{dom } f = \emptyset$). The set of points that minimize f is denoted by

$$\text{argmin } f : = \{x \in R^n | f(x) \leq \inf f < \infty\} ;$$

thus in particular

$$\text{argmin } f = \emptyset \quad \text{if} \quad \text{dom } f = \emptyset .$$

This convention is dictated by the desire to have argmin f be the set of *optimal* solutions in the minimization for which dom f is the set of *feasible* solution. We do not want to consider as "feasible" the points at x with $f(x) = \infty$ and certainly do not want the possibility of a point being "optimal" without even being "feasible". The points that are *nearly optimal* or *ε-optimal* for some $\varepsilon > 0$ belong to the set

$$\varepsilon\text{-argmin } f : = \{x \in R^n | f(x) \leq \inf f + \varepsilon < \infty\} .$$

By the way, it is customary in optimization theory to write "min f" in place of "inf f", and speak of *minimum* in place of *infimum*, as an indication that the infimum is actually attained at some point x. We shall also have recourse to this convention if we want to insist on the existence of a minimum.

3.25. COROLLARY. *Consider a variational system* $F : R^d \rightrightarrows R^n$. *Then* F *is epicontinuous at* u *if and only if*

$$\limsup_{u'\to u} (\inf_G f_{u'}) \leq \inf_G f_u \quad \textit{for all open } G \subset R^n ,$$

and

$$\liminf_{u'\to u} (\inf_K f_{u'}) \geq \inf_K f_u \quad \textit{for all compact } K \subset R^n .$$

Theorem 3.22, in particular its proof, suggests the use of the following sets as an (open) base for the epi-topology "epi" on $SC(R^n)$, the *space of all l.sc. extended-real-valued functions on* R^n :

(3.26) $\{f \in SC(R^n) \mid \inf_G f < a' , G \subset R^n \text{ open}, a' \in R\}$

and

(3.27) $\{f \in SC(R^n) \mid \inf_K f > a , K \subset R^n \text{ compact}, a \in R\}$.

Indeed, in the *space of epigraphs* E these open sets correspond to

$$\{E \in E \mid E \cap (G \times (-\infty, a')) \neq \emptyset \text{ , G open, } a' \in R\}$$

and

$$\{E \in E \mid E \cap (K \times (-\infty, a]) = \emptyset \text{ , K compact, } a \in R\} \text{ .}$$

This topology "epi" is nothing other than the topology $\mathcal{T}$ on the closed subsets of R^{n+1} --see Section 2-- relative to E . It is easy to verify that E is a $\mathcal{T}$-closed subset of the hyperspace of closed subsets of R^{n+1} . Thus as a consequence of Theorem 2.31 we obtain :

3.28. THEOREM. $(SC(R^n))$, epi) *is a metrizable, compact topological space with countable base.*

3.29. COROLLARY. *Given any filtered family* $\{f_\nu \in SC(R^n), \nu \in (N, \mathcal{H})\}$ *there always exists a subfamily* $\{f_\nu, \nu \in (N', \mathcal{H})\}$ *that epi-converges, i.e. such that*

$$\text{epi-}\lim_{\nu \in N'} f_\nu$$

exists.

Theorem 3.22 suggests still another way of generating the epi-topology, namely as the coarsest topology on $SC(R^n)$ such that

(3.30) for all open $G \subset R^n$, $f \mapsto \inf_G f$ is u.sc. ,

and

(3.31) for all compact $K \subset R^n$, $f \mapsto \inf_K f$ is l.sc. .

The resemblance of this characterization of the epi-topology to that of the so-called vague topology has led Vervaat, 1982, to refer to the epi-topology as the inf-vague topology.

We can of course, as in Section 2, exhibit a metric on $SC(R^n)$ compatible with the epi-topology, in fact

$$\text{epi-dist } (f,g) = \text{haus}^S(\text{epi } f, \text{ epi } g)$$

will do. Convergence rates can then be considered. And if we think of f and g as the essential objective functions of two optimization problems, this metric gives us a concrete way of measuring the goodness of fit when g approximates f . However, at this time there is no operational calculus which allows us to work easily with the epi-distance as defined above.

From the foregoing it may appear that the epigraphical approach to variational systems is to be justified on the grounds of esthetics. In fact it is because of its applications, some of which we detail next, that it is gaining its key role in Extended Real Analysis. For more about this, consult the articles in this Volume by Attouch, 1984, and De Giorgi, 1984, and the references given there.

3.32. COROLLARY. *Suppose* f *and* $\{f_\nu, \nu \in (N, \mathcal{H})\}$ *, a filtered family, are l.sc. extended-real-valued functions defined on* R^n *and such that*

$$f \geq \text{epi-}\limsup_{\nu \in N} f_\nu \text{ .}$$

Then

$$\text{(3.33)} \quad \limsup_{\nu \in N} (\inf f_\nu) \leq \inf f \text{ .}$$

Moreover, if actually

$$f = \text{epi-}\lim_{\nu \in N} f_\nu$$

and there exist $H \in \mathcal{H}$ *and a compact set* $K \subset R^n$ *such that for all* $\nu \in H$, $\text{dom } f_\nu \subset K$ *, then*

$$\text{(3.34)} \quad \lim_{\nu \in N} (\inf f_\nu) = \inf f \text{ .}$$

PROOF. The first inequality (3.33) follows from (3.23) with $G = R^n$. From this and (3.24) we get (3.34), since the assumptions imply that $\inf f_\nu = \inf_K f_\nu$. □

To rephrase this in terms of variational systems, let us introduce the *infimal function*

$$u \mapsto (\inf \mathcal{F})(u) := \inf f_u : R^d \to \overline{R}$$

associated with a variational system .

3.35. COROLLARY. *Suppose* $\mathcal{F} = \{f_u : R^n \to \overline{R} | u \in R^d\}$ *is a variational system with l.sc. valuates. If it is upper epi-semicontinuous at* $\bar{u}$ *then the infimal function* $u \mapsto \inf \mathcal{F}(u)$ *is upper semicontinuous at* $\bar{u}$ *. Moreover if the variational system is epicontinuous at* $\bar{u}$ *, and if there exists a neighborhood* V *of* $\bar{u}$ *such that on* V *the domain multifunction* $u \mapsto \text{dom } \mathcal{F}(u)$ *is uniformly bounded, then the infimal function is continuous at* $\bar{u}$.

Corollary 3.32, and its version Corollary 3.35 for variational systems, which already cover a wide variety of applications can be refined in a number of ways. What is bothersome is that the equality (3.34), or equivalently the continuity of the infimal function, is obtained under uniform boundedness of the effective domains of the functions near f or $f_{\bar{u}}$. If we think of these functions as the essential objective functions of optimization problems, this would mean that the corresponding sets of feasible solutions are not only bounded but all are contained in the same bounded set. There are many ways of improving on these results; in fact it is possible to obtain conditions that are both necessary and sufficient for the convergence of the infima. For a detailed analysis, we refer to Salinetti and Wets, 1984. Here we content ourselves with suggesting how such conditions can be obtained. Suppose that the collection $\{f_\nu, \nu \in (N, \mathcal{H})\}$ epi-converges to f. In view of (3.33) all that is needed is to show that

$$\inf f \leq \liminf_{\nu \in N} (\inf f_\nu) .$$

We know that this inequality holds if the infima are taken with respect to a compact set instead of all of R^n. In particular we have that for any compact $K \subset R^n$.

$$\inf f \leq \inf_K f \leq \liminf_{\nu \in N} (\inf_K f_\nu) .$$

and the question would be settled if we could assert : *that for every* $\varepsilon > 0$ *there exists a compact* K *such that*

$$\liminf_{\nu \in N} (\inf_K f_\nu) \leq \liminf_{\nu \in N} (\inf f_\nu + \varepsilon) .$$

This is clearly a sufficient condition for the convergence of the infima. That it is also necessary --excluding the cases when the infima are not finite-- requires a little bit more work. The meaning here is clear : what we need is that up to an arbitrary ε, the minimization could as well take place on a bounded region, which may depend on ε. In the terminology of variational systems we have shown :

3.36. PROPOSITION. *Consider a variational system* $\mathcal{F} : R^d \rightrightarrows R^n$ *with l.sc. valuates which is epicontinuous at* $\bar{u}$ *. Suppose that for all* $\varepsilon > 0$ *, there exist* $V \in N(\bar{u})$ *and* $K \subset R^n$ *such that*

$$\inf_K f_u \leq \inf f_u + \varepsilon$$

for all $u \in V$ *. Then the infimal function* $(\inf \mathcal{F})$ *is continuous at* $\bar{u}$.

Epi-continuity does not just garantee continuity properties for the infimal function, but also for the multifunction of optimal solutions $u \mapsto (\text{argmin } \mathcal{F})(u)$. As usual, we first state our results for filtered families of l.sc. functions.

3.37. THEOREM. *Suppose* $\{f_\nu : R^n \to \overline{R}, \nu \in (N, \mathcal{H})\}$ *is a filtered family of lower semicontinuous functions epi-convergent to* f *. Then*

$$(3.38) \quad \limsup_{\nu \in N} (\text{argmin } f_\nu) \subset \text{argmin } f .$$

Moreover, assuming argmin f *nonempty, one has*

$$(3.39) \quad \text{argmin } f = \bigcap_{\varepsilon > 0} \liminf_{\nu \in N} (\varepsilon\text{-argmin } f_\nu)$$

if and only if

$$\lim_{\nu \in N} (\inf f_\nu) = \min f .$$

PROOF. We shall prove somewhat more than (3.38), which will be used in the sequel, namely : for all $\varepsilon \geq 0$,

$$\limsup_{\nu \in N} (\varepsilon\text{-argmin } f_\nu) \subset \varepsilon\text{-argmin } f .$$

Suppose $N' \subset N$ and

$$\{x^\nu \in \varepsilon\text{-argmin } f_\nu, \nu \in (N, \mathcal{H})\}$$

is a filtered collection of points converging to x. The preceding inclusion will be proved if we show that $x \in \varepsilon$-argmin f. But this follows from (3.15) and (3.33) since they imply

$$f(x) \leq (\text{epi-}\lim_{\nu\in N} f_\nu)(x) \leq \liminf_{\nu\in N} f_\nu(x^\nu)$$

$$\leq \limsup_{\nu\in N} f_\nu(x^\nu) \leq \limsup_{\nu\in N} (\inf f_\nu + \varepsilon) \leq \inf f + \varepsilon .$$

To prove the second assertion, let us first assume that

$$\lim_{\nu\in N} (\inf f_\nu) = \inf f .$$

In view of the above, for all $\varepsilon > 0$

$$\liminf_{\nu\in N} (\varepsilon\text{-argmin } f_\nu) \subset \limsup_{\nu\in N} (\varepsilon\text{-argmin } f_\nu) \subset \varepsilon\text{-argmin } f .$$

Due also to the fact that

$$\text{argmin } f = \cap_{\varepsilon>0}\ \varepsilon\text{-argmin } f ,$$

there remains only to show that

$$\text{argmin } f \subset \cap_{\varepsilon>0} \liminf_{\nu\in N} (\varepsilon\text{-argmin } f_\nu) .$$

For any $x \in \text{argmin } f_\nu$, it follows from the definition of epi-convergence --combining (3.13) with (3.12)-- that there exists $N' \in \mathcal{H}$ and $\{x^\nu, \nu \in (N', \mathcal{H})\}$ such that

$$x = \lim_{\nu\in N'} x^\nu \quad \text{and} \quad \lim_{\nu\in N'} f_\nu(x^\nu) = f(x) .$$

If, for some (filtered) collection $\varepsilon_\nu \downarrow 0$, we have that $x^\nu \in \varepsilon_\nu\text{-argmin } f_\nu$, we are done. Otherwise, there exists $H' \in \mathcal{H}$ such that for some $\varepsilon' > 0$ and all $\nu \in H'$,

$$f_\nu(x^\nu) > \inf f_\nu + \varepsilon' .$$

Taking limits on H' (with $\mathcal{H}$ restricted to H'), we obtain

$$f(x) = \lim_{\nu\in H'} f_\nu(x^\nu) \geq \varepsilon' + \lim_{\nu\in H'} (\inf f_\nu) = \varepsilon' + \min f > f(x) ,$$

a clear contradiction.

Let us now assume that (3.39) holds and $x \in \text{argmin } f$. This implies that there exist $\varepsilon_\nu \downarrow 0$ and $x^\nu \to x$ such that for all $\nu \in N$

$$x_\nu \in \varepsilon_\nu\text{-argmin } f_\nu .$$

From the definition of epi-convergence, in particular (3.12), it follows that

$$\min f = f(x) \leq \liminf_\nu f_\nu(x^\nu) \leq \liminf_\nu (\inf f_\nu + \varepsilon_\nu) = \liminf_{\nu\in N} (\inf f_\nu) .$$

This combined with (3.33) yields the convergence of the infima. □

Although epi-convergence gives us directly the important relations (3.38) and (3.39), to obtain the actual convergence of the argmin f_ν to argmin f we need an additional condition. Two examples illustrate some of the difficulties.

3.40. EXAMPLE Let $f(x) = \max \{0,|x|-1\}$, and for $\nu = 1,\ldots,$

$$f_\nu(x) = \max \{f(x), \nu^{-1} x^2\} .$$

It is easy to verify that the f_ν epi-converge to f with

$$\text{argmin } f_\nu = \{0\} , \text{ for } \nu = 1,\ldots$$

But the latter definitely do not converge to

$$\text{argmin } f = [-1,1] .$$

3.41. EXAMPLE. For $\nu = 1,\ldots,$ let

$$f_\nu(x) = 1 , \text{ except that } f_\nu(0) = \nu^{-1} , f_\nu(\nu) = 0 ,$$

and

$$f(x) = 1 \text{ except that } f(0) = 0 .$$

Then the f_ν epi-converge to f, and the sets argmin f and argmin f_ν are singletons, but

$$\lim_{\nu\to\infty} (\text{argmin } f_\nu) = \lim_{\nu\to\infty} \{\nu\} = \emptyset \neq \{0\} = \text{argmin } f .$$

The following sufficient condition is due to Dolecki, 1983.

3.42. PROPOSITION. *Suppose* f *and* $\{f_\nu : R^n \to R,\ \nu \in (N,\mathcal{H})\}$, *a filtered family, are l.sc. functions such that :*

$$\text{epi-}\limsup_{\nu\in N} f_\nu \leq f \quad , \quad \liminf_{\nu\in N} (\inf f_\nu) \geq \inf f > -\infty \quad ,$$

and to every $V \in \mathcal{N}(x)$ *with* $x \in$ argmin f, *there corresponds* $\delta > 0$, $W \in \mathcal{N}(x)$ *and* $H \in \mathcal{H}$ *such that for all* $\nu \in H$

$$V \cap \text{argmin } f_\nu = \emptyset \Rightarrow \inf_W f_\nu \geq \inf f_\nu + \delta .$$

Then

$$\liminf_{\nu\in N} (\text{argmin } f_\nu) \supset \text{argmin } f$$

PROOF. Let $\alpha_\nu := \inf f_\nu$, and note that from the assumptions, using (3.23) with $G = R^n$, it follows that

$$\lim_{\nu\in N} \alpha_\nu = \alpha := \inf f > -\infty .$$

Suppose

$$x \in \text{argmin } f \quad \text{but} \quad x \notin \liminf_{\nu\in N} (\text{argmin } f_\nu) .$$

Corollary 2.29 tells us that there exist $V \in \mathcal{N}(x)$, $H \in \mathcal{H}$ such that for all $\nu \in H$, $V \cap \text{argmin } f_\nu = \emptyset$. But this then means that there exist an open neighborhood $W \in \mathcal{N}(x)$ and $\delta > 0$ such that $\inf_W f_\nu \geq \alpha_\nu + \delta$. Taking lim sup on both sides, using (3.23) and the fact that $x \in$ argmin f, we obtain :

$$\alpha = \inf_W f \geq \limsup_{\nu\in H} (\inf f_\nu) \geq \delta + \lim_{\nu\in N} \alpha_\nu = \delta + \alpha$$

a clear contradiction. Thus

$$x \in \liminf_{\nu\in N} (\text{argmin } f_\nu) . \square$$

The condition of Proposition 3.42 imposed a restriction on the way the f_ν approach f in the neighborhood of the points that minimize f, whenever they are not in argmin f they cannot "sneak up" on the latter. Following Dolecki, we shall say that the collection $\{f_\nu, \nu \in (N,\mathcal{H})\}$ is of *decisive growth* at x if for every $V \in \mathcal{N}(x)$ there correspond $\delta > 0$, $H \in \mathcal{H}$ and $W \in \mathcal{N}(x)$, such that for all $\nu \in H$

(3.43) $\quad V \cap \text{argmin } f = \emptyset \Rightarrow \inf_W f_\nu \geq \delta + \inf f_\nu .$

Rephrasing our results in terms of variational systems, we get

3.44. COROLLARY. *Suppose* $\mathcal{F} = \{f_u : R^n \to \overline{R} \mid u \in R^d\}$ *is a variational system with l.sc. valuates, epicontinuous at* $\bar{u} \in R^d$. *The multifunction of optimal solutions*

$$u \to (\text{argmin } \mathcal{F})(u) = \text{argmin } f_u$$

is upper semicontinuous at $\bar{u}$. *Moreover, if the* $\{f_u, u \in (R^d, \mathcal{N}(\bar{u}))\}$ *grow decisively at every* $x \in \text{argmin } f_{\bar{u}}$ *and* $\inf f_u \to \inf f_{\bar{u}} > -\infty$, *then this multifunction* (argmin $\mathcal{F}$) *is continuous at* $\bar{u}$.

Of course, we have only been able to exhibit *some* of the consequences of epi-continuity. Much more could be said, in particular in the convex case. There are also corresponding concepts for bivariate functions : epi/hypo-convergence that guarantees the convergence of saddle points, lopsided convergence connected with the convergence of min/sup points. The definition of Γ - convergence, introduced by De Giorgi, extends these concepts to multivariate functions; for further details and references consult the forthcoming book of Attouch, 1985.

Measurability, or more precisely measurable dependence on parameters, of a variational system is again handled in the epigraphical setting. As in the multifunction case in Section 2, we allow the parameters w to lie in an (abstract) space equipped with a sigma-field $\mathcal{A}$. A variational system $\mathcal{F} = \{f_w : R^n \to \overline{R} \mid w \in \Omega\}$ with l.sc. valuates is *epimeasurable* if the epigraphical multifunction $w \mapsto (\text{epi } \mathcal{F})(w) = \text{epi } f_w$ is a (closed-valued) measurable multifunction. The conjuctive function

$$(x,w) \mapsto (f,w) : R^n \times \Omega \to \overline{R}$$

is then called a *normal integrand*. It is really not possible to review to any extent the theory of normal integrands and their integrals; for that the reader could refer to Rockafellar, 1976, Castaing and Valadier, 1977, who deal mostly with the convex case, and Papageorgiou, 1983, who extends many results to the nonconvex infinite-dimensional setting. We shall limit ourselves to a few properties, in particular those of the infimal function and the multifunction of optimal solutions. We begin with a general result which leads up to the construction of integral functionals.

3.45. THEOREM. *Let* $\mathcal{F} = \{f_w : R^n \to \overline{R} | w \in \Omega\}$ *be an epimeasurable variational system with l.sc. valuates. Then the associated normal integrand* $f : R^n \times \Omega \to \overline{R}$ *is* $\mathcal{B}^n \otimes \mathcal{A}$ *- measurable, where* $\mathcal{B}^n$ *is the Borel field on* R^n *. Moreover, the function*

$$w \mapsto f(x(w),w)$$

is measurable for any measurable function $w \mapsto x(w) : \Omega \to R^n$ *.*

PROOF. For any $\alpha \in R$, the level set multifunction (3.9) is a closed-valued measurable multifunction. Indeed

$$(\mathrm{lev}_\alpha \mathcal{F})^{-1}(F) = (\mathrm{epi}\ \mathcal{F})^{-1}(F \times \{\alpha\})$$

for any closed set $F \subset R^n$. Since $(\mathrm{epi}\ \mathcal{F})$ is a measurable multifunction, it follows from Proposition 2.39 that the set on the left is measurable $(\in \mathcal{A})$. This holds for all closed sets F , hence --again by Proposition 2.39-- we have that $(\mathrm{lev}_\nu \mathcal{F})$ is measurable. This implies that $\mathrm{gph}(\mathrm{lev}_\alpha \mathcal{F})$ is a measurable subset of $R^n \times \Omega$. Indeed

$$(3.46)\quad \mathrm{gph}(\mathrm{lev}_\alpha \mathcal{F}) = \cap_{k=1}^\infty \cap_{i=1}^\infty [B_{ik} \times (\mathrm{lev}_\alpha \mathcal{F})^{-1}(B_{ik})]$$

where $\{B_{ik}, i \in \mathbb{N}, k \in \mathbb{N}\}$ is the collection of all rational balls with centers indexed by i and having radius k^{-1} . Because $B_{ik} \in \mathcal{B}^n$ and $(\mathrm{lev}_\alpha \mathcal{F})^{-1}(B_{ik}) \in \mathcal{A}$ (Proposition 2.39.v), one has

$$B_{ik} \times (\mathrm{lev}_\alpha \mathcal{F})^{-1}(B_{ik}) \in \mathcal{B}^n \times \mathcal{A}$$

and hence, in view of (3.46),

$$\mathrm{gph}(\mathrm{lev}_\alpha \mathcal{F}) \in \mathcal{B}^n \otimes \mathcal{A} .$$

Since this holds for all $\alpha \in R$, it proves that f is $\mathcal{B}^n \otimes \mathcal{A}$ - measurable.

Now, to see that $w \mapsto f(x(w),w)$ is measurable whenever $x(\cdot)$ is measurable, all that is needed is to observe that the map $w \mapsto (x(w),w)$ from $(\Omega, \mathcal{A})$ into $(R^n \times \Omega, \mathcal{B}^n \otimes \mathcal{A})$ is measurable. □

3.47. THEOREM. *Let* $\mathcal{F} = \{f_w : R^n \to \overline{R} | w \in \Omega\}$ *be an epimeasurable variational system with l.sc. valuates. Then the infimal function*

$$w \mapsto (\inf \mathcal{F})(w) = \inf f_w$$

is measurable, and the multifunction of optimal solutions

$$w \mapsto (\mathrm{argmin}\ \mathcal{F})(w) : \Omega \rightrightarrows R^n$$

is a closed-valued measurable multifunction.

PROOF. For $\beta \in R$,

$$(\inf \mathcal{F})^{-1}(-\infty, \beta) = \{w | \inf_w < \beta\} = (\mathrm{epi}\ \mathcal{F})^{-1}(R^n \times (-\infty, \beta)).$$

These sets belong to $\mathcal{A}$, since the epigraphical multifunction $\mathrm{epi}\ \mathcal{F}$ is a closed-valued measurable multifunction and $R^n \times (-\infty, \beta)$ is open, cf. Proposition 2.39. Since this holds for all β , $\inf \mathcal{F}$ is measurable.

It is easy to verify that the function g defined by

$$g(x,w) = f(x,w) - \inf f_w$$

is a normal integrand; we use the convention that $\infty - \infty = \infty$. Then $w \mapsto (\mathrm{epi}\ g(\cdot,w))$ is a closed-valued measurable multifunction, and in the proof of Theorem 3.45 we have shown that this implies

$$w \mapsto \mathrm{lev}_0\ g(\cdot,w) = (\mathrm{argmin}\ \mathcal{F})(w)$$

is a closed-valued measurable multifunction.□

3.48. COROLLARY. (Existence of Measurable Solutions). *Let* $\mathcal{F} = \{f_w : R^n \to \bar{R} | w \in \Omega\}$ *be an epimeasurable variational system with l.sc. valuates. Then there exists a measurable function*

$$w \mapsto x^*(w) : \mathrm{dom}(\mathrm{argmin}\ \mathcal{F}) \to R^n$$

such that $x^*(w)$ *minimizes* f_w *whenever* $\mathrm{argmin}\ f_w \neq \emptyset$.

PROOF. Simply use the previous result in conjunction with Corollary 2.41 about Measurable Selections.□

We have gone as far as this introduction allows us to, in showing that the epigraphical approach to variational problems is dictated by the intrinsic nature of such problems as well as the type of properties we are interested in. Of course, this is not the whole story, and it would not be possible to summarize even sketchily its many other features. To terminate, let us just suggest the theory of integration that goes with this approach. Let μ denote a nonnegative, sigma-finite measure on $(\Omega, \mathcal{A})$. For any normal integrand f on $R^n \times \Omega$ and any measurable function $x : \Omega \to R^n$, we have $f(x(\cdot), \cdot)$ measurable (Theorem 3.45), and therefore the integral

$$I_f(x) = \int_\Omega f(x(w), w)\mu(dw)$$

is a well defined value in $\bar{R}$ under the usual convention : if neither the positive nor the negative part of the integrand is summable, we set $I_f(x) = \infty$. We can also think of I_f as the integral functional of a variational system parameterized by w , and write more suggestively $I_{\mathcal{F}}(x)$. The theory of integral functionals provides us with the tools that are needed to study problems of the calculus of variations (there $\mu(dw) = dt$) in its modern version optimal control theory, involving (hard) constraints on the control and the state of the system, problems in stochastic optimization (there μ is a probability measure), problems in economics involving infinite horizons (then $\mu(dw)$ may correspond to a discounting coefficient), and so on. It may appear from the definition of I_f that except for some manipulations involving ∞ and $-\infty$ we have returne to a classical definition. This, however, is misleading. The calculus for integral functionals shows that the key role is played by the epigraphical multifunction. For example the definition of Radon-Nikodym derivatives (conditional expectations) as well as the calculation of subdifferentials all pass through the corresponding notions for the integral of the epigraphical multifunction. This point is very much brought home in the recent work of Giner, 1984, and Papageorgiou, 1983.

REFERENCES

H. ATTOUCH, 1984 : Variational properties of epi-convergence. Applications to limit analysis problems in mechanics and duality theory, in *Multifunction and Integrands: Stochastic Analysis, Approximation and Optimization*, ed. G. Salinetti, Springer Verlag Lecture Notes in Mathematics, Berlin.

H. ATTOUCH, 1985 : *Variational convergences for Functions and Operators*, Research Notes in Mathematics, Pitman Ltd, London (to appear).

J-P. AUBIN, 1984 : An introduction to viability theory, in *Multifunctions and Integrands : Stochastic Analysis, Approximation and Optimization*, ed. G. Salinetti, Springer Verlag Lecture Notes in Mathematics, Berlin.

J-P. AUBIN and I. EKELAND, 1984 : *Applied Nonlinear Analysis*, Wiley-Interscience, New-York.

C. CASTAING and M. VALADIER, 1977 : *Convex Analysis and Measurables Multifunctions*, Springer Verlag Lecture Notes in Mathematics, 580, Berlin.

F. CLARKE, 1983 : *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New-York.

F. DE GIORGI, 1984 : G-Operators, Γ-convergence and their applications, in *Multifunctions and Integrands : Stochastic Analysis, Approximation and Optimization*, ed. G. Salinetti, Springer Verlag Lecture Notes in Mathematics, Berlin.

S. DOLECKI, 1983 : Convergence of global minima and infima, Manuscript, Universita di Trento.

E. GINER, 1983 : Sous differentiabilité des fonctionelles intégrales (II), Manuscript, Univ. Toulouse.

A. IOFFE, 1978 : Survey of measurable selection theorems : Russian Literature supplement : *SIAM J. Control and Optimization*, 16, 728-123.

J. KELLEY, 1955 : *General Topology*, van Norstrand.

N. PAPAGEORGIOU, 1983 : Stochastic nonsmooth analysis and optimization I and II, Manuscript, Harvard Univ.

R.T. ROCKAFELLAR, 1976 : Integral functionals, normal integrands and measurable selections, in *Nonlinear Operators and the Calculus of Variations*, ed. L. Waelbroeck, Springer-Verlag Lecture Notes in Mathematics, 543, Berlin.

R.T. ROCKAFELLAR, 1983 : Generalized subgradients and mathematical programming, in *Mathematical Programming : The State of the Art 1982*, eds. A. Bachem, M. Grötschel and B. Korte, Springer-Verlag, Berlin.

R.T. ROCKAFELLAR and R.J-B. WETS, 1985 : *Extended Real Analysis*, in preparation.

G. SALINETTI and R.J-B. WETS, 1984 : Convergence of infima, especially stochastic infima, in preparation.

W. VERWAT, 1982 : Random upper semicontinuous functions and extremal process, manuscript, University of Nijmegen.

D. WAGNER, 1977 : Survey of measurable selection theorems, *SIAM J. Control and Optimization*, 15, 859-903.

EXTENSION OF THE CLASS OF MARKOV CONTROLS.

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INTRODUCTION.

In control theory, for example when deriving existence theorems or optimality criteria, it is often necessary to extend the class of controls without changing the value of the problem. There are a number of well-known methods for doing this which are based on the convexity of integrals of measurable multifunctions and which are related to randomized and relaxed controls.

This paper is devoted to some new theorems of this kind for control problems involving stochastic difference equations with mixed constraints on phase coordinates and controls.

The results presented here are generalizations and extensions of earlier results obtained by the author [1].

1. STATEMENT OF THE PROBLEM

Let s_t be a Markov process defined on a measurable space (S,E). Assume that s_t has a transition function $P_t(s_t,ds_{t+1})$, $t = 0,1,\ldots$ and initial distribution $P_0(ds_0)$.

Consider the following problem:

$$\sum_{t=0}^{\tau-1} E\Phi^{t+1}(s_t,s_{t+1},y_t,u_t) \to \max$$