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# LIPSCHITZIAN PROPERTIES OF MULTIFUNCTIONS

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## 1. INTRODUCTION

A FUNDAMENTAL topic in optimization theory and nonsmooth analysis is the study of sets of the form

$$\Gamma(v) = \{x \mid f_i(v,x) \leqq \text{for } i = 1, \dots, s \quad \text{and} \quad f_i(v,x) = 0 \quad \text{for} \quad i = s+1, \dots, m\}, \tag{1.1}$$

where $f_i$ is a real-valued function on $R^d \times R^n$. Such a set consists of all the points satisfying a certain system of constraints in $R^n$, where the constraints depend on a parameter vector $v \in R^d$. More generally one may consider

$$\Gamma(v) = \{x \mid F(v,x) \in C, (v,x) \in D\}, \tag{1.2}$$

for $F: R^d \times R^n \to R^m$, $C \subset R^m$, $D \subset R^d \times R^m$. A major question is the way that $\Gamma(v)$ varies as $v$ varies. Of particular importance are properties of Lipschitz continuity of the multifunction $\Gamma: v \to \Gamma(v)$ that may be present.

The special case of

$$\Gamma(v) = \{x \mid F(v,x) = 0\} \tag{1.3}$$

is addressed by the classical implicit function theorem when $F$ is smooth. Clarke [2] and Hiriart-Urruty [4] have extended this case to mappings $F$ that are locally Lipschitzian; they give criteria for $\Gamma(v)$ to be single-valued and locally Lipschitzian.

Sets of the form

$$\Gamma(v) = \{x \mid 0 \in f(v,x) + T(x)\} \tag{1.4}$$

have been studied by Robinson [7–11], for certain kinds of multifunctions $T: R^n \rightrightarrows R^m$ and smooth mappings $f: R^d \times R^n \to R^m$. (He allows $v$ and $x$ also to range over spaces more general than the spaces $R^d$ and $R^n$ indicated here.) The condition $0 \in f(v, x) + T(x)$ can be written equivalently as

$$(x, -f(v,x)) \in \operatorname{gph} T, \tag{1.5}$$

so that (1.4) can be viewed as an instance of (1.2) (with $F(v) = (x, -f(v, x))$, $C = \operatorname{gph} T$, $D = R^d \times R^n$). It can also be written in other ways as a special case of (1.2), for instance when $\operatorname{gph} T$ can be described by a system of Lipschitzian constraints.

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What notions of Lipschitz continuity are appropriate in this setting? Consider an arbitrary multifunction $\Gamma: R^d \rightrightarrows R^n$ (assigning to each $v \in R^d$ a set $\Gamma(v) \subset R^n$, which may be empty), and suppose that the image sets $\Gamma(v)$ are closed. The classical notion is that $\Gamma$ is *Lipschitzian relative* to $V$, a subset of $R^d$, if $\Gamma(v)$ is nonempty and compact for every $v \in V$, and there is a constant $\lambda \geqq 0$ (the *modulus* of Lipschitz continuity) such that

$$\operatorname{haus}(\Gamma(v_1), \Gamma(v_2)) \leqq \lambda |v_1 - v_2| \quad \text{for all } v_1, v_2 \in V, \tag{1.6}$$

where "haus" denotes the Hausdorff metric on the space of all nonempty compact subsets of $R^n$:

$$\operatorname{haus}(X_1, X_2) = \min\{\varepsilon \geqq 0 \,|\, X_1 \subset X_2 + \varepsilon B, X_2 \subset X_1 + \varepsilon B\},$$

with $B$ the closed unit ball for the Euclidean norm $|\cdot|$. Condition (1.6) can be written equivalently as

$$\Gamma(v_1) \subset \Gamma(v_2) + \lambda |v_1 - v_2| B \quad \text{for all} \quad v_1, v_2 \in V. \tag{1.7}$$

When the sets $\Gamma(v)$ are unbounded, as is often the case in applications, these notions are not suitable and something else is needed. The following concept was introduced by Aubin [1]: $\Gamma$ is *pseudo-Lipschitzian at* $(\bar{v}, \bar{x})$, where $\bar{x} \in \Gamma(\bar{v})$, if there exist neighborhoods $V$ of $\bar{v}$, $X$ of $\bar{x}$, and a constant $\lambda \geqq 0$ such that

$$\Gamma(v_1) \cap X \subset \Gamma(v_2) + \lambda |v_1 - v_2| B \quad \text{for all} \quad v_1, v_2 \in V. \tag{1.8}$$

A related concept which we introduce here is that $\Gamma$ is *sub-Lipschitzian at* $\bar{v}$ if $\Gamma(\bar{v}) \neq \phi$ and for every compact set $X$ in $R^n$, no matter how large, one has (1.8) for some neighborhood $V$ of $\bar{v}$ and constant $\lambda \geqq 0$.

This paper is focused on the study of multifunctions with these Lipschitzian properties. In Section 2 we clarify the relationship between the properties and express them in terms of the distance function associated with a multifunction. In Sections 3 and 4 we derive conditions that allow these properties to be verified for multifunctions of various constructions. A generalization of Aubin's implicit multifunction theorem [1, Section 3] is obtained in particular.

For multifunctions (1.4) of the kinds investigated by Robinson [8, 9], the results we obtain are complementary to his and somewhat different in spirit. Robinson makes assumptions on the multifunction obtained in (1.4) in place of $\Gamma$ by linearizing $f$ in $x$ at a certain $(\bar{v}, \bar{x})$ but keeping the same $T$. From these he derives bounds of the form

$$\Gamma(v) \subset \Gamma(\bar{v}) + \lambda |v - \bar{v}| B$$

(and more general estimates when $f(v, x)$ is not Lipschitzian in $v$). Such bounds describe an "upper Lipschitzian" behavior of the multifunction $\Gamma$ at the point $\bar{v}$ itself, rather than a Lipschitzian property which compares $\Gamma(v_1)$ and $\Gamma(v_2)$ for arbitrary $v_1$ and $v_2$ in some neighborhood of $\bar{v}$, as pursued here.

## 2. CHARACTERIZATIONS AND INTERRELATIONS

For a multifunction $\Gamma: R^d \rightrightarrows R^n$, we shall use the notation

$$\Gamma(V) = \bigcup_{v \in V} \Gamma(v) \quad \text{for any} \quad V \subset R^d.$$

We shall say that $\Gamma$ is *locally bounded at* $\bar{v}$ if there is a neighborhood $V$ of $\bar{v}$ such that the set $\Gamma(V)$ is bounded.

Furthermore, we shall say that $\Gamma$ is *closed* at $\bar{v}$ if for every $\bar{x} \notin \Gamma(\bar{v})$ there exist neighborhoods $V$ of $\bar{v}$ and $X$ of $\bar{x}$ such that $\Gamma(v) \cap X = \phi$ for all $v \in V$. (Then in particular $\Gamma$ must be *closed-valued* at $\bar{v}$, i.e. the set $\Gamma(\bar{v})$ must be closed.) This is true for every $\bar{v}$ if $\Gamma$ is *of closed graph*, i.e. the set $\operatorname{gph} \Gamma = \{(v, x) \mid x \in \Gamma(v)\}$ is a closed set in $R^d = R^n$.

THEOREM 2.1. Let $\Gamma : R^d \rightrightarrows R^n$ be closed-valued, and let $\bar{v} \in R^d$. Then the following are equivalent:

(a) $\Gamma$ is locally bounded at $\bar{v}$ and sub-Lipschitzian at $\bar{v}$;
(b) on some neighborhood of $\bar{v}$, $\Gamma$ is nonempty-compact-valued and Lipschitzian.

*Proof.* (a) $\Rightarrow$ (b). Local boundedness implies that for some bounded set $X$ and neighborhood $V$ of $\bar{v}$ one has

$$\Gamma(v) \cap X = \Gamma(v) \quad \text{for all} \quad v \in V.$$

Taking $V$ to be a neighborhood with both this property and (1.8), as is possible by definition when $\Gamma$ is sub-Lipschitzian at $\bar{v}$, one obtains the Lipschitz condition (1.7). Also $\Gamma(\bar{v}) \neq \phi$ by definition of "sub-Lipschitzian", and therefore from (1.7) as applied to $v_1 = \bar{v}$, $v_2 = v$, we must have $\Gamma(v) \neq \phi$ for all $v \in V$.

(b) $\Rightarrow$ (c). By assumption there is a neighborhood $V = \bar{v} + \delta B$ of $\bar{v}$ such that $\Gamma(v)$ is nonempty and compact for all $v \in V$, and

$$\Gamma(v_1) \subset \Gamma(v_2) + \lambda |v_1 - v_2| B \quad \text{for all} \quad v_1, v_2 \in V. \tag{2.1}$$

Then trivially (1.8) holds for arbitrary $X$, so $\Gamma$ is sub-Lipschitzian at $\bar{v}$. One has in particular

$$\Gamma(v) \subset \Gamma(\bar{v}) + \lambda |v - \bar{v}| B \subset \Gamma(\bar{v}) + \lambda \delta B$$

for all $v \in V$ by (2.1), so that $\Gamma(V)$ is included in the set $\Gamma(\bar{v}) + \lambda\delta B$, which is bounded. Thus $\Gamma$ is locally bounded at $\bar{v}$. ∎

THEOREM 2.2. Let $\Gamma : R^d \rightrightarrows R^n$ be closed-valued, and let $\bar{v} \in R^d$. Then the following are equivalent:

(a) $\Gamma$ is sub-Lipschitzian at $\bar{v}$;
(b) $\Gamma$ is closed at $\bar{v}$ with $\Gamma(\bar{v}) \neq \phi$ and $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ for every $\bar{x} \in \Gamma(\bar{v})$.

*Proof.* (a) $\Rightarrow$ (b). If $\Gamma$ is sub-Lipschitzian at $\bar{v}$, then from the definition we have that $\Gamma(\bar{v}) \neq \phi$ and for every compact set $X$ there is a neighborhood $V$ of $\bar{v}$ on which (1.8) holds. In particular $X$ can be taken to be a neighborhood of any $\bar{x} \in \Gamma(\bar{v})$, and therefore $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ for any $\bar{x} \in \Gamma(\bar{v})$. To see that $\Gamma$ is closed at $\bar{v}$, consider any $\bar{x} \notin \Gamma(\bar{v})$ and take $X$ to be a compact neighborhood of $\bar{x}$ such that for some $\varepsilon > 0$, $[\Gamma(\bar{v}) + \varepsilon B] \cap X = \phi$. Then for a corresponding neighborhood $V$ as in (1.8) and of the form $V = \bar{v} + \delta B$, we have

$$\Gamma(v) \cap X \subset \Gamma(\bar{v}) + \lambda |v - \bar{v}| B \subset \Gamma(\bar{v}) + \lambda \delta B \quad \text{for all} \quad v \in V.$$

If $\delta$ is chosen small enough that $\lambda\delta \leqq \varepsilon$, we get

$$\Gamma(v) \cap X \subset [\Gamma(\bar{v}) + \varepsilon B] \cap X = \phi \quad \text{for all} \quad v \in V,$$

which is the property of $V$ that is desired.

(b) $\Rightarrow$ (a). In demonstrating that $\Gamma$ is sub-Lipschitzian at $\bar{v}$, it suffices to consider a compact set $X$ large enough that $\Gamma(\bar{v}) \cap X \neq \phi$ and to produce a corresponding neighborhood $V$ of $\bar{v}$ such that (1.8) holds. By assumption there exist for each $\bar{x} \in \Gamma(\bar{v})$ open neighborhoods $V_{\bar{x}}$ of $\bar{v}$, $X_{\bar{x}}$ of $\bar{x}$, and a constant $\lambda_{\bar{x}} \geqq 0$ such that

$$\Gamma(v_1) \cap X_{\bar{x}} \subset \Gamma(v_2) + \lambda_{\bar{x}}|v_1 - v_2|B \quad \text{for all} \quad v_1, v_2 \in V_{\bar{x}}.$$

Because $\Gamma(\bar{v})$ is closed, the set $\Gamma(\bar{v}) \cap X$ is compact, and from the collection of open sets $X_{\bar{x}}$ as $\bar{x}$ ranges over $\Gamma(\bar{v}) \cap X$ we can extract a finite covering of $\Gamma(\bar{v}) \cap X$:

$$\Gamma(v) \cap X \subset \cup_{i=1}^{r} X_{x_i} \quad \text{for certain} \quad x_i \in \Gamma(v) \cap X.$$

Let

$$X' = \cup_{i=1}^{r} X_{x_i}, \qquad V' = \cap_{i=1}^{r} V_{x}^{i}, \qquad \lambda = \max_{i=1,\ldots,r} \lambda_{x_i}.$$

Then $X'$ and $V'$ are opens sets such that $\Gamma(\bar{v}) \cap X \subset X'$, $\bar{v} \in V'$, and

$$\Gamma(v_1) \cap X' \subset \Gamma(v_2) + \lambda|v_1 - v_2|B \quad \text{for all} \quad v_1, v_2 \in V'. \tag{2.2}$$

Consider now the relative complement $X \backslash X'$, which is a compact set with $\Gamma(\bar{v}) \cap [X \backslash X'] = \phi$. Because $\Gamma$ is closed at $\bar{v}$ there exist for any $\bar{x} \in X \backslash X'$ open neighborhoods $V^{\bar{x}}$ of $\bar{v}$ and $X^{\bar{x}}$ of $\bar{x}-$ such that

$$\Gamma(v) \cap X^{\bar{x}} = \phi \quad \text{when} \quad v \in V^{\bar{x}}.$$

From the collection of sets $X^{\bar{x}}$ as $\bar{x}$ ranges over $X \backslash X'$ we can extract a finite covering of $X \backslash X'$.

$$X \backslash X' \subset \cup_{j=1}^{s} X^{x^j} \quad \text{for certain } x^j \in X \backslash X'.$$

Let

$$X'' = \cup_{j=1}^{s} X^{x^j}, \qquad V'' = \cap_{j=1}^{s} V^{x^j}.$$

Then $X''$ and $V''$ are open sets such that $X \backslash X' \subset X''$, $\bar{v} \in V''$, and

$$\Gamma(v) \cap X'' = \phi \quad \text{when} \quad v \in V'',$$

so that

$$\Gamma(v) \cap X \subset \Gamma(v) \cap X' \quad \text{when} \quad v \in V''. \tag{2.3}$$

Let $V = V' \cap V''$. Then $V$ is a neighborhood of $\bar{v}$ for which (1.8) holds by virtue of (2.2) and (2.3). Thus $\Gamma$ is sub-Lipschitzian at $\bar{v}$ relative to $S$. ■

A closed-valued multifunction $\Gamma: R^d \rightrightarrows R^n$ can be identified in a set with its *distance function*

$$d_\Gamma(v, x) := \operatorname{dist}(\Gamma(v), x) := \min_{x \in \Gamma(v)} |x' - x|. \tag{2.4}$$

(By convention this quantity is $\infty$ if $\Gamma(v) = \phi$.) Obviously $\Gamma$ is uniquely determined by $d_\Gamma$, so every property of $\Gamma$ must correspond to a property of $d_\Gamma$ and vice versa. Our next result reveals the property of $d_\Gamma$ that corresponds to pseudo-Lipschitz continuity of $\Gamma$ and indicates clearly why that concept has a natural significance.

THEOREM 2.3. Let $\Gamma: R^d \rightrightarrows R^n$ be closed-valued, and let $\bar{v} \in R^d$, $\bar{x} \in \Gamma(\bar{v})$. Then the following are equivalent:

(a) $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$;
(b) $d_\Gamma$ is Lipschitzian on some neighborhood of $(\bar{v}, \bar{x})$.

*Proof.* Observe first that (b) is equivalent to the existence of neighborhoods $V$ of $\bar{v}$, $X$ of $\bar{x}$, and a constant $\lambda \geqq 0$ such that $d_\Gamma$ is finite on $S \times X$ and

$$d_\Gamma(v_2, x) \leqq d_\Gamma(v_1, x) + \lambda|v_2 - v_1| \quad \text{for all} \quad v_1, v_2 \in V, x \in X. \tag{2.5}$$

This is true because $d_\Gamma(v, x)$ is by nature Lipschitzian in $x$ with modulus 1 for all $v$ such that $\Gamma(v) \neq \phi$.

(b) $\Rightarrow$ (a). If (2.5) holds, then for every $x \in \Gamma(v_1) \cap X$ and $v_1$, $v_2 \in V$ one has

$$\operatorname{dist}(\Gamma(v_2), x) \leqq \operatorname{dist}(\Gamma(v_1), x) + \lambda|v_1 - v_2| = \lambda|v_1 - v_2|,$$

because $\operatorname{dist}(\Gamma(v_1), x) = 0$. Thus $x \in \Gamma(v_2) + \lambda|v_1 - v_2|B$ for every $x \in \Gamma(v_1) \cap X$, as required in (1.8) for pseudo-Lipschitz continuity.

(a) $\Rightarrow$ (b). Suppose (1.8) holds for neighborhoods $V$ of $\bar{v}$ and $X$ of $\bar{x}$. Then

$$\operatorname{dist}(\Gamma(v_2) + \lambda|v_1 - v_2|B, x) \leqq \operatorname{dist}(\Gamma(v_1) \cap X, x) \quad \text{for all} \quad v_1, v_2 \in V, x \in R^n. \tag{2.6}$$

But one also has

$$\operatorname{dist}(\Gamma(v_2), x) - \rho \leqq \operatorname{dist}(\Gamma(v_2) + \rho B, x)$$

for any $\rho \geqq 0$, so (2.6) implies

$$\operatorname{dist}(\Gamma(v_2), x) - \lambda|v_1 - v_2| \leqq \operatorname{dist}(\Gamma(v_1) \cap X, x) \quad \text{for all} \quad v_1, v_2 \in V, x \in R^n. \tag{2.7}$$

It need only be shown now that for certain neighborhoods $V_0$ of $\bar{v}$ and $X_0$ of $\bar{x}$ with $V_0 \subset V$ and $X_0 \subset X$ one has

$$\operatorname{dist}(\Gamma(v) \cap X, x) = \operatorname{dist}(\Gamma(v), x) \quad \text{for all} \quad v \in V_0, x \in X_0. \tag{2.8}$$

Indeed, this in combination with (2.7) will yield

$$\operatorname{dist}(\Gamma(v_2), x) \leqq \operatorname{dist}(\Gamma(v_1), x) + \lambda|v_1 - v_2| \quad \text{for all} \quad v_1, v_2 \in V_0, x \in X_0,$$

which is the desired Lipschitz property (2.5) relative to $V_0$ and $X_0$.

Choose $\varepsilon > 0$ small enough that $\bar{x} + \varepsilon B \subset X$, and let $X_0 = \bar{x} + (1/3)\varepsilon B$. Then $X_0 \subset X$, and for arbitrary $X \in X_0$ one has in fact $x + (2/3)\varepsilon B \subset X$, so that

$$\operatorname{dist}(\Gamma(v) \cap X, x) = \operatorname{dist}(\Gamma(v), x) \text{ when } v \text{ satisfies} \quad \operatorname{dist}(\Gamma(v), x) \leqq (2/3)\varepsilon.$$

Furthermore

$$\operatorname{dist}(\Gamma(v), x) \leqq (2/3)\varepsilon \quad \text{when} \quad \operatorname{dist}(\Gamma(v), x) \leqq (1/3)\varepsilon, x \in X_0,$$

because

$$\operatorname{dist}(\Gamma(v), x) \leqq \operatorname{dist}(\Gamma(v), \bar{x}) + |x - \bar{x}|.$$

Therefore (2.8) does hold for the specified $X_0$ and any neighborhood $V_0$ of $\bar{v}$ with the property that

$$\operatorname{dist}(\Gamma(v), \bar{x}) \leqq (1/3)\varepsilon \quad \text{for all} \quad v \in V_0,$$

if such $V_0$ exists. The existence of such $V_0$ is verified by appealing again to (1.8): one has in particular that

$$\bar{x} \in \Gamma(\bar{v}) \cap X \subset \Gamma(v) + \lambda|v - \bar{v}|B \quad \text{for all} \quad v \in V$$

and consequently

$$\operatorname{dist}(\Gamma(v), \bar{x}) \leqq \lambda|v - \bar{v}| \quad \text{for all} \quad v \in V.$$

Thus one can take $V_0 = \bar{v} + \delta B$ for any $\delta > 0$ small enough that $\bar{v} + \delta B \subset V$ and $\lambda\delta \leqq (1/3)\varepsilon$. ■

COROLLARY 2.4. Let $\Gamma: R^d \rightrightarrows R^n$ be of closed graph, and let $\bar{v} \in R^d$. Then the following are equivalent:

(a) $\Gamma$ is sub-Lipschitzian at $\bar{v}$;

(b) $d_\Gamma$ is locally Lipschitzian relative to some (nonempty) open set containing $\{(\bar{v}, \bar{x}) | \bar{x} \in \Gamma(\bar{v})\}$.

*Proof.* This combines theorem 2.3 with theorem 2.2. ■

*Remark* 2.5. The first part of the proof of theorem 2.5 provides an estimate for the modulus $\lambda$ in the pseudo-Lipschitzian continuity property (1.8) of $\Gamma$ at $(\bar{v}, \bar{x})$: any $\lambda$ such that (2.5) holds will do, where $V \times X$ is a neighborhood of $(\bar{v}, \bar{x})$. The greatest lower bound $\bar{\lambda}$ for all such $\lambda$ is easily identified by the methods of nonsmooth analysis (cf. Clarke [3]) as

$$\bar{\lambda} = \max_{|h| \leqq 1} d_\Gamma^\circ(\bar{v}, \bar{x}; h, 0) = \max\{|z| \mid \exists u \text{ with } (z, u) \in \partial d_\Gamma(\bar{v}, \bar{x})\},$$

where $d_\Gamma^\circ$ denotes the Clarke derivative and $\partial d_\Gamma$ the Clarke subdifferential of $d_\Gamma$.

*Remark* 2.6. Everything in this section (with the exception of the formula in the preceding remark) can readily be generalized to the case where the parameter space is any metric space, rather than just $R^d$ (for instance, a subset of $R^d$).

## 3. SOME CLASSES OF PSEUDO-LIPSCHITZIAN MULTIFUNCTIONS

A sufficient condition for a multifunction of the general form (1.2) to be pseudo-Lipschitzian will be derived now from results in nonsmooth analysis. Let $F: R^d \times R^n \to R^m$ be a locally Lipschitzian function (i.e. single-valued) and consider closed sets $C \subset R^m$ and $D \subset R^d \times R^n$. We shall denote by $N_C(\bar{u})$ the *Clarke normal cone* to $C$ at a point $\bar{u} \in C$ (see Clarke [3] or Rockafellar [12]) and similarly by $N_D(\bar{v}, \bar{x})$ the normal cone to $D$ at a point $(\bar{v}, \bar{x}) \in D$. The set $\partial F(\bar{v}, \bar{x})$ will denote the *Clarke generalized Jacobian* of $F$ at $(\bar{v}, \bar{x})$ (see Clarke [3]); this is a certain nonempty compact convex set of matrices of size $m \times (d+n)$ which reduces to a single matrix (the Jacobian $\nabla F(\bar{v}, \bar{x})$) if and only if $F$ is strictly differentiable at $(\bar{v}, \bar{x})$.

The main result of nonsmooth analysis on which we shall rely is the following.

THEOREM 3.1. (Rockafellar [13].) Let

$$p(v) = \inf_x\{f(v,x) \mid F(v,x) \in C, (v,x) \in D\},$$

$$P(v) = \operatorname{argmin}_x\{f(v,x) \mid F(v,x) \in C, (v(x) \in D\},$$

where $C$ and $D$ are closed, and $f$ and $F$ are locally Lipschitzian. Let $\bar{v}$ be a point at which $p(\bar{v})$ is finite and suppose that for some $\alpha > p(\bar{v})$ the multifunction

$$P_\alpha(v) = \{x \in R^n \mid f(v,x) \leqq \alpha, F(v,x) \in C, (v,x) \in D\} \tag{3.1}$$

is locally bounded around $\bar{v}$. Suppose further that the following constraint qualification holds for every $\bar{x} \in P(\bar{v})$: the only vectors $y \in R^m$ and $z \in R^d$ such that

$$y \in N_C(F(\bar{v},\bar{x})) \quad \text{and} \quad (z,0) \in y\,\partial F(\bar{v},\bar{x}) + N_D(\bar{v},\bar{x}) \tag{3.2}$$

are $y = 0$, $z = 0$.

Then, relative to some neighborhood $V$ of $\bar{v}$, $p$ is finite and locally Lipschitzian, whereas $P$ is nonempty-valued, locally bounded and of closed graph.

*Proof.* This is the special case of [13, theorem 8.3] in which the objective function $f$ is assumed to be locally Lipschitzian and one invokes the criterion in [13, proposition 2.1] for $p$ to be locally Lipschitzian. The graph of $P$ relative to a closed neighborhood $V$ of $v$ is the set of pairs $(v, x) \in V \times R^n$ satisfying $f(v, x) \leqq p(v)$, $F(v, x) \in C$, $(v, x) \in D$, and this is obviously closed when $p$ is continuous on $V$. ■

THEOREM 3.2. Let

$$\Gamma(v) = \{x \mid F(v,x) \in C, (v,x) \in D\}, \tag{3.3}$$

where $C \subset R^m$ and $D \subset R^d \times R^n$ are closed, and $F: R^d \times R^n \to R^m$ is locally Lipschitzian. Let $\bar{v} \in R^d$ and $\bar{x} \in \Gamma(\bar{v})$ be such that the following constraint qualification holds: the only vectors $y \in R^m$ and $z \in R^d$ such that

$$y \in N_C(F(\bar{v},\bar{x})) \quad \text{and} \quad (z,0) \in y\,\partial F(\bar{v},\bar{x}) + N_D(\bar{v},\bar{x}) \tag{3.4}$$

are $y = 0$, $z = 0$.

Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ (and also of closed graph).

*Proof.* It is obvious that $\Gamma$ is of closed graph: $\operatorname{gph}\Gamma = F^{-1}(C) \cap D$. In light of theorem 2.3, we can establish that $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ by demonstrating that the distance function $d_\Gamma$ is Lipschitzian on a neighborhood of $(\bar{v}, \bar{x})$. We shall do this by way of theorem 3.1 and the representation

$$d_\Gamma(v,x) = \min\{|x-w| \mid F(v,x) \in C, (v,x) \in D\}.$$

This can be written as $d_\Gamma(v, x) = p(w, v)$ where

$$p(w,v) = \min\{f_0(w,v,x) \mid F_0(w,v,x) \in C, (w,v,x) \in D_0\}, \tag{3.5}$$

with

$$f_0(w, v, x) = |x - w|, \quad F_0(w, v, x) = F(v, x), \quad D_0 = R^n \times D.$$

We wish to show that $p$ is Lipschitzian on a neighborhood of $(\bar{x}, \bar{v})$, and this can be obtained by verifying that the assumptions in theorem 3.1 are satisfied at $(w, v) = (\bar{x}, \bar{v})$ in the case of (3.5) and the multifunction

$$P(w, v) = \operatorname{argmin}\{f_0(w, v, x) \,|\, F_0(w, v, x) \in C, (w, v, x) \in D\}. \tag{3.6}$$

Note that $p(\bar{x}, \bar{v}) = 0$ and $P(\bar{x}, \bar{v}) = \{\bar{x}\}$, because $\bar{x} \in \Gamma(\bar{v})$. For any $\alpha > 0$ the corresponding multifunction (3.5) is given by

$$p_\alpha(w, v) = \{x \in R^n \,|\, |x - w| \leqq \alpha, x \in \Gamma(v)\} = \Gamma(v) \cap [w + \alpha B].$$

Trivially $P_\alpha$ is locally bounded in a neighborhood of $(w, v) = (\bar{x}, \bar{v})$; in fact $P_\alpha(w, v) \subset [\bar{x} + (\alpha + \varepsilon)B]$ when $|w - \bar{x}| \leqq \varepsilon$.

We need to show now that the following constraint qualification holds at the unique element of $P(\bar{x}, \bar{v})$, namely $\bar{x}$: the only vectors $y \in R^m$ and $(u, z) \in R^n \times R^d$ such that

$$y \in N_C(F_0(\bar{x}, \bar{v}, \bar{x})) \quad \text{and} \quad (u, z, 0) \in \partial F_0(\bar{x}, \bar{v}, \bar{x}) + N_{D_0}(\bar{x}, \bar{v}, \bar{x}) \tag{3.7}$$

are $y = 0$, $(u, z) = (0, 0)$. Here $N_C(F_0(\bar{x}, \bar{v}, \bar{x})) = N_C(F(\bar{v}, \bar{x}))$ and

$$\partial F_0(\bar{x}, \bar{v}, \bar{x}) = (0, \partial F(\bar{v}, \bar{x})), \quad N_{D_0}(\bar{x}, \bar{v}, \bar{x}) = (0, N_D(\bar{v}, \bar{x}))$$

(the last by [13, corollary 2.5.1]), so (3.7) reduces to $u = 0$ and (3.4). The latter implies $y = 0$, $z = 0$, by hypothesis. Therefore the desired constraint qualification is satisfied, and the proof of the theorem is complete. ∎

COROLLARY 3.3. Suppose in theorem 3.2 that the constraint qualification is satisfied at every $\bar{x} \in \Gamma(\bar{v})$, the set $\Gamma(\bar{v})$ being nonempty. Then $\Gamma$ is sub-Lipschitzian at $\bar{v}$.

*Proof.* This follows via theorem 2.2, because $\Gamma$ is of closed graph, hence in particular closed at $\bar{v}$. ∎

COROLLARY 3.4. (Aubin [1, Section 3].) Let $\Gamma: R^d \rightrightarrows R^n$ be a multifunction with closed graph, and let $D = \operatorname{gph} \Gamma$. Let $\bar{v} \in R^d$. Suppose $\bar{x} \in \Gamma(\bar{v})$ is such that the only vector $z$ satisfying $(z, 0) \in N_D(\bar{v}, \bar{x})$ is $z = 0$ (or equivalently, that under the projection $(v, x) \to v$ the Clarke tangent cone $T_D(\bar{v}, \bar{x})$, which is the polar of $N_D(\bar{v}, \bar{x})$ has all of $R^d$ as its image). Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$.

*Proof.* This is the case of theorem 3.2 where the elements $F$ and $C$ do not appear. ∎

COROLLARY 3.5. (Inverse multifunctions.) Let

$$\Gamma(v) = \{x \,|\, F(v, x) \in C\} \tag{3.8}$$

where $F: R^d \times R^n \to R^m$ is locally Lipschitzian and $C \subset R^m$ is closed. Let $(\bar{v}, \bar{x})$ be such that $F(\bar{v}, \bar{x}) \in C$ and the following constraint qualification holds: every matrix $J = (J_v, J_x) \in$

$\partial F(\bar{v}, \bar{x})$ (where $J_v \in R^{m\times d}$ and $J_x \in R^{m\times n}$) has the property that the only $y \in N_C(F(\bar{v}, \bar{x}))$ with $yJ_x = 0$ is $y = 0$.

Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ (and of closed graph).

*Proof.* Take $D = R^d \times R^n$ in theorem 3.2. ■

*Remark* 3.6. A case of corollary 3.5 that deserves particular attention is the one where

$$C = \{w = (w_1, \ldots, w_m) \in R^m | w_i \leqq 0 \quad \text{for} \quad i = 1, \ldots, s, \quad w_i = 0 \quad \text{for} \quad i = s+1, \ldots, m\}.$$

Then for $F(v, x) = (f_1(v, x), \ldots, f_m(v, x))$ we have

$$\Gamma(v) = \{x | f_i(v,x) \leqq 0 \quad \text{for} \quad i = 1, \ldots, \quad \text{and} \quad f_i(v,x) = 0 \quad \text{for} \quad i = s+1, \ldots, m\}. \tag{3.9}$$

Moreover

$$N_C(F(\bar{v}, \bar{x})) = \{y = (y_1, \ldots, y_m) \,|y_i \geqq 0 \quad \text{and} \quad y_i f_i(\bar{v}, \bar{x}) = 0 \quad \text{for} \quad i = 1, \ldots, s\}. \tag{3.10}$$

When the functions $f_i$ are smooth, the constraint qualification in this case is the Mangasarian–Fromovitz condition [5]: the only $y$ as in (3.10) with

$$\textstyle\sum_{i=1}^m y_i \nabla_x f_i(v, \bar{x}) = 0$$

is $y = 0$. This is dual to (and equivalent to) the regularity condition used by Robinson [8] in this situation, as he himself has shown [8, theorem 3]. The result obtained by Robinson under this condition is complementary to ours, however:

$$\operatorname{dist}(\Gamma(v), x) \leqq \mu \operatorname{dist}(F(v,x), C) \quad \text{for} \quad v \in V, x \in X, \tag{3.11}$$

where $V \times X$ is a neighborhood of $(\bar{v}, \bar{x})$ and $\mu \geqq 0$ is a certain constant. In contrast to pseudo-Lipschitz continuity this only compares a $(v, x)$ with the given $(\bar{v}, \bar{x})$, but it does give a bound that in some situations may be more practical.

Corresponding results of Robinson for the more general case contained in theorem 3.2 where $D = R^d \times E$ with $E$ convex (and $F$ still smooth) are also given in [8], and for $E = R^n$ and $F(v, x) = g(x) - v$ in the earlier paper [6]. (Although Robinson's assumptions of smoothness and convexity are more restrictive than ours, he does, on the other hand allow $v$ and $x$ to range over spaces more general than $R^d$ and $R^n$.)

COROLLARY 3.7. Let

$$\Gamma(v) = \{x | F(v,x) = 0\},$$

where $F: R^d \times R^n \to R^m$ is locally Lipschitzian. Let $(\bar{v}, \bar{x})$ be such that $F(\bar{v}, \bar{x}) = 0$ and the following constraint qualification holds: every matrix $J = (J_v, J_x) \in \partial F(\bar{v}, \bar{x})$ (where $J_v \in R^{m\times d}$, $J_x \in R^{m\times n}$) has rank $J_x = m$.

Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ (and also of closed graph).

*Proof.* Take $C = \{0\}$ in corollary 3.4. ■

*Remark* 3.8. When $m = n$ in corollary 3.7, the constraint qualification takes the form that every $J = (J_v, J_x)$ as described has $J_x$ nonsingular. This is the condition in the Lipschitzian implicit function theorem of Hiriart-Urruty [4] (and Clarke [2, 3]). To obtain that result, all that is needed besides the assertion of corollary 3.7 is the proof of single valuedness of $\Gamma$ in such a case. Contrary to what one might expect, however, the cited implicit function does not seem to lend itself in turn to the derivation of the result in corollary 3.7, due to the multiplicity of the Jacobian matrices $J$ at $(\bar{v}, \bar{x})$ when $F$ is not smooth.

*Remark* 3.9. Corollaries 3.5 and 3.7 are obtainable as particular cases of corollary 3.4 when $F$ is continuously differentiable, but not when $F$ is merely locally Lipschitzian. This is because the Clarke tangent cone to the graph of a Lipschitzian function $F$ carries less information about $F$ than the subdifferential $\partial F$ does.

*Remark* 3.10. Conditions for $\Gamma$ to be sub-Lipschitzian in the context of corollaries 3.4, 3.5, and 3.7 can obviously be derived simply by combining these results with theorem 2.2. (Likewise for $\Gamma$ to be Lipschitzian: see theorem 2.1.)

COROLLARY 3.12. Let

$$\Gamma(v) = \{x \mid 0 \in f(v, x) + T(x)\},$$

where $f: R^d \times R^n \to R^m$ is a locally Lipschitzian function and $T: R^n \rightrightarrows R^m$ is of closed graph. Let $\bar{v} \in R^d$ and $\bar{x} \in \Gamma(\bar{v})$ be such that the following constraint qualification holds: for no matrix $J = (J_v, J_x) \in \partial f(\bar{v}, \bar{x})$ does there exist $s \in R^m$ with

$$(sJ_x, s) \in N_{\mathrm{gph}\, T}(\bar{x}, -f(\bar{v}, \bar{x})) \tag{3.12}$$

except for $s = 0$.

Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ (and of closed graph).

*Proof.* Let $F(v, x) = (x, -f(v, x))$, $C = \mathrm{gph}\, T$, $D = R^d \times R^n$. Then $\Gamma$ fits the pattern in theorem 3.2 with

$$(r, s)\partial F(\bar{v}, \bar{x}) = (0, r) - s\partial f(\bar{v}, \bar{x})$$

for all $(r, s) \in R^n \times R^m$. The constraint qualification in theorem 3.2 becomes the condition that the only combinations of $(r, s) \in N_G((\bar{x}, -f(\bar{v}, \bar{x}))$, $z \in R^d$, and $(J_v, J_x) \in \partial f(\bar{v}, \bar{x})$ with $(z, 0) = (-sJ_v, r - sJ_x)$ have $(r, s) = (0, 0)$, $z = 0$. This reduces to the condition stated here. ∎

*Remark* 3.13. In the case where $f$ is continuously differentiable, so that $J_v = \nabla_v f(\bar{v}, \bar{x})$ and $J_x = \nabla_x f(\bar{v}, \bar{x})$, corollary 3.12 can be compared with results of Robinson [6, 8–10]. When $T$ has the special form

$$T(x) = \begin{cases} K & \text{if } x \in E, \\ \phi & \text{if } x \notin E, \end{cases}$$

where $K \subset R^m$ and $E \subset R^n$ are closed sets, the constraint qualification becomes the following: for no matrix $J = (J_v, J_x) \in \partial f(\bar{v}, \bar{x})$ does ~~not~~ there exist $s \in N_K(-f(\bar{v}, \bar{x}))$, $s \neq 0$, such that $sJ_x \in N_E(\bar{x})$. When $K$ and $E$ are convex and $f$ smooth, this condition is dual to the one used by Robinson [8] in obtaining results such as have already been indicated in remark 3.6. (The case of $E = R^n$, $f(v, x) = g(x) - v$, is in Robinson [6].)

On the other hand, when $T = \partial\varphi$ for some closed proper convex function $\varphi$ on $R^n$, or more generally when $T$ is a maximal monotone relation from $R^n$ to $R^n$, we have in corollary 3.12 the framework for a perturbed variational inequality of the kind treated by Robinson on [9, 10]. In this case the normal cone in condition (3.12) is always a (linear) *subspace* $L$ of $R^n \times R^n$ having dimension at least $n$ (see Rockafellar [14]), whereas the linear transformation $s \to (sJ_x, s) = (s\nabla_x(f(\bar{v}, \bar{x}), s)$ has rank exactly $n$ (because $m = n$); its range is a subspace $M$ of $R^n \times R_n$ having dimension $n$. Simple considerations of linear algebra reveal that under these circumstances the constraint qualification in corollary 3.12 fails to be satisfied unless, in particular, the dimension of the subspace $L$ is exactly $n$. Then the graph of $T$ is "strictly smooth" at $(\bar{v}, \bar{x})$ (see Rockafellar [14]). This is a generalized differentiability property of the multifunction $T$ which for $T = \partial\varphi$ corresponds to a generalized second derivative property of $\varphi$ [14, Section 4]. The constraint qualification that results is then more restrictive than the one of Robinson in [9, 10], except in allowing $f$ to be locally Lipschitzian instead of smooth.

A different approach to the case of $T$ maximal monotone would be to use the fact that gph $T$ is a Lipschitzian manifold (as explained in [14]). Associated parameterizations or constraint representations of gph $T$ then make possible other expressions for the multifunction $\Gamma$ in corollary 3.12 to which our results can be applied in a different fashion. This technique will not be pursued here.

*Remark* 3.14. The results of Robinson [6–10] and to some extent Aubin [1], are not only qualitative like those above but furnish *estimates* for the Lipschitzian modulus $\lambda$ that governs each situation. In fact such estimates can also be derived by our techniques. In quoting theorem 3.1 as a special case of [13, theorem 8.3] we omitted the estimate provided by that result for the subgradients of $p$ (in the sense of Clarke [3]):

$$\partial p(\bar{v}) \subset \mathrm{co}\{z \mid \exists \bar{x} \in P(\bar{v}), y \in N_C(F(\bar{v}, \bar{x})), \quad \text{with} \quad (z, 0) \in \partial f(\bar{v}, \bar{x}) + y\partial F(\bar{v}, \bar{x}) + N_D(\bar{v}, \bar{x})\}.$$

Applying this in the context of the proof of theorem 3.2 (where $\partial f_0(\bar{x}, \bar{v}, \bar{x}) = \{(u, 0, -u) \mid |u| \leq 1\}$), one gets

$$\partial d_\Gamma(\bar{v}, \bar{x}) \subset \mathrm{co}\{(z, u) \in N_C(F(\bar{v}, \bar{x}))\partial F(\bar{v}, \bar{x}) + N_D(\bar{v}, \bar{x}) \mid |u| \leq 1\}. \tag{3.13}$$

Then for

$$\bar{\lambda} = \max\{|z| \mid \exists u \quad \text{with} \quad (z, u) \in N_C(F(\bar{v}, \bar{x}), \partial F(\bar{v}, \bar{x}) + N_D(\bar{v}, \bar{x}), |u| \leq 1\} \tag{3.14}$$

one has by remark 2.5 that any $\lambda > \bar{\lambda}$ works, relative to some neighborhood $V \times X$ of $(\bar{v}, \bar{x})$, as the modulus in the pseudo-Lipschitzian continuity property (1.8).

*Remark* 3.15. The results in this section are essentially based on applying [13, theorem 8.3] to get a condition for $d_\Gamma$ to be Lipschitzian at $(\bar{v}, \bar{x})$ and using the fact in theorem 2.3 that this property corresponds to pseudo-Lipschitzian behavior of $\Gamma$ at $(\bar{v}, \bar{x})$. The following generalization would be possible: a certain property of $\Gamma$ corresponds to $d_\Gamma$ being *directionally* Lipschitzian at $(\bar{v}, \bar{x})$ with respect to a vector $(h, k)$ as defined in [12]; criteria for the latter are furnished by [13, theorem 8.3] in terms of the same Lagrange multiplier set-up.

## 4. OPERATIONS THAT PRESERVE LIPSCHITZ CONTINUITY

The conditions in the preceding section serve to identify some important classes multifunctions having properties of Lipschitz continuity, but many other multifunctions can be constructed from these by composition, addition, union, intersection, projection, and so forth. To what extent is Lipschitz continuity preserved when such operations are performed?

We begin with the fundamental operations of composition.

THEOREM 4.1. Let $\Gamma: R^d \rightrightarrows R^n$ be given by

$$\Gamma(v) = \Gamma_1(\Gamma_0(v)) = \bigcup_{u \in \Gamma_0(v)} \Gamma_1(u),$$

where $\Gamma_0: R^d \rightrightarrows R^p$ and $\Gamma_1: R^p \to R^n$ are multifunctions of closed graph such that the multifunction $\Delta: R^d \times R^n \rightrightarrows R^p$ defined by

$$\Delta(v, x) = \{u \in \Gamma_0(v) | x \in \Gamma_1(u)\} = \Gamma_0(v) \cap \Gamma_1^{-1}(x)$$

is locally bounded everywhere. Then $\Gamma$ is of closed graph.

If $\bar{x} \in \Gamma(\bar{v})$ (so that $\Delta(\bar{v}, \bar{x}) \neq \phi$) and for every $\bar{u} \in \Delta(\bar{v}, \bar{x})$ one has $\Gamma_0$ pseudo-Lipschitzian at $(\bar{v}, \bar{u})$ and $\Gamma_1$ pseudo-Lipschitzian at $(\bar{u}, \bar{x})$, then $\Gamma$ is pseudo-Lipschitzian at $\bar{v}$.

*Proof.* First observe that

$$\operatorname{gph} \Gamma = \{(v, x) | \Delta(v, x) \neq \phi\}$$

and that $\Delta$ is of closed graph:

$$\operatorname{gph} \Delta = \{(v, x, u) | (v, u) \in \operatorname{gph} \Gamma_0, (u, x) \in \operatorname{gph} \Gamma_1\}.$$

We can prove the closedness of gph $\Gamma$ by showing that for arbitrary $(\bar{v}, \bar{x}) \in \operatorname{gph} \Gamma$ and compact neighborhoods $V$ of $\bar{v}$ and $X$ of $\bar{x}$ the set

$$\{(v, x) \in V \times X | \Delta(v, x) \neq \phi\} \tag{4.1}$$

is closed. By our assumption that $\Delta$ is locally bounded everywhere, there exists for each $(v, x) \in V \times X$ a neighborhood $W_{v,x}$ such that $\Delta(W_{v,x})$ is bounded. Such neighborhoods cover $V \times X$, which is compact, so finitely many of them cover $V \times X$. Thus there is a compact set $U$ such that

$$\Delta(v, x) \subset U \quad \text{for all} \quad (v, x) \in V \times X.$$

The set (4.1) is then the same as

$$\{(v, x) \in V \times X | \Delta(v, x) \cap U \neq \phi\},$$

which is the image of the compact set $[V \times X \times U] \cap \operatorname{gph} \Delta$ under the projection $(v, x, u) \to (v, x)$. The image of a compact set under a continuous mapping is compact, in particular closed. Therefore (4.1) is a closed set, and the graph of $\Gamma$ is closed as claimed.

Consider now a pair $(\bar{v}, \bar{x})$ such that $\Delta(\bar{v}, \bar{x}) \neq \phi$ and, for every $\bar{u} \in \Delta(\bar{v}, \bar{x})$, $\Gamma_0$ is pseudo-Lipschitzian at $(\bar{v}, \bar{u})$ and $\Gamma_1$ is pseudo-Lipschitzian at $(\bar{u}, \bar{x})$. For every $\bar{u} \in \Delta(\bar{v}, \bar{x})$ there exist open neighborhoods $V_{\bar{u}}$ of $\bar{v}$ and $X_{\bar{u}}$ of $\bar{x}$, and constants $\lambda_{\bar{u}}^0 \geqq \lambda_{\bar{u}}^1 \geqq 0$, such that

$$\Gamma_0(v_1) \cap U_{\bar{u}} \subset \Gamma_0(v_2) + \lambda_{\bar{u}}^0 |v_1 - v_2| B \quad \text{when} \quad v_1, v_2 \in V_{\bar{u}}, \tag{4.2}$$

$$\Gamma_1(u_1) \cap X_{\bar{u}} \subset \Gamma_1(u_2) + \lambda_{\bar{u}}^1 |u_1 - u_2| B \quad \text{when} \quad u_1, u_2 \in U_{\bar{u}}. \tag{4.3}$$

The set $\Delta(\bar{v},\bar{x})$ is compact (because $\Delta$ is of closed graph and locally bounded), and it is covered by the collection of open neighborhoods $U_{\bar{u}}$ as $\bar{u}$ ranges over $\Delta(\bar{v},\bar{x})$, so it is covered by a finite subcollection, corresponding say to points $\bar{u}_i$, $i=1,\ldots,r$. Let

$$U=\cup_{i=1}^{r}U_{\bar{u}_i},\quad V=\cap_{i=1}^{r}V_{\bar{u}_i},\quad X=\cap_{i=1}^{r}X_{\bar{u}_i},\quad \lambda_0=\max_{i=1,\ldots,r}\lambda^0_{\bar{u}_i},\quad \lambda_1=\max_{i=1,\ldots,r}\lambda^1_{\bar{u}_i}.$$

Then from (4.2) we have

$$\Gamma_0(v_1)\cap U\subset\Gamma_0(v_2)+\lambda_0|v_1-v_2|B\quad\text{when}\quad v_1,v_2\in V. \tag{4.4}$$

whereas from (4.3)

$$\Gamma_1(u_1)\cap X\subset\Gamma_1(u_2)+\lambda_1|u_1-u_2|B\quad\text{when}\quad u_1,u_2\in U_{\bar{u}_i}\quad\text{for some } i. \tag{4.5}$$

Returning now to the fact that $\Delta$ is a locally bounded multifunction of closed graph, we note that these properties imply upper semicontinuity of $\Delta$, in particular at $(\bar{v},\bar{x})$: for every $\varepsilon_0>0$ there exist neighborhoods $X_0$ of $X$ and $V_0$ of $V$ such that

$$\Delta(v,x)\subset\Delta(\bar{v},\bar{x})+\varepsilon_0B\quad\text{when}\quad(v,x)\in V_0\times X_0.$$

Here $\Delta(\bar{v},\bar{x})+\varepsilon_0B$ is compact, because $\Delta(\bar{v},\bar{x})$ is compact, and consequently $\Delta(\bar{v},\bar{x})+\varepsilon_0B\subset U$ for $\varepsilon_0$ sufficiently small, because $\Delta(\bar{v},\bar{x})\subset U$ and $U$ is open. Taking $U_0=(\bar{v},\bar{x})+\varepsilon_0B$ for $\varepsilon_0$ sufficiently small, then, we obtain the following: there exist

$$U_0\subset U,\quad V_0\subset V,\quad X_0\subset X,$$

such that $U_0$ is compact, $V_0$ is a neighborhood of $\bar{v}$, $X_0$ is a neighborhood of $\bar{x}$, and

$$\Delta(v,x)\subset U_0\quad\text{when}\quad(v,x)\in V_0\times X_0.$$

The latter means by definition of $\Delta$ that

$$\Gamma_1(\Gamma_0(v)\cap U_0)\cap X_0=\Gamma_1(\Gamma_0(v))\cap X_0\quad\text{when}\quad v\in V_0. \tag{4.6}$$

Remembering that $U$ is the union of the open sets $U_{\bar{u}_i}$, $i=1,\ldots,r$, appearing in (4.5), we note the existence of some $\varepsilon>0$ such that

$$\text{for each } u\in U_0+\varepsilon B,\quad\text{one has } u+\varepsilon B\subset U_{\bar{u}_i}\quad\text{for some } i. \tag{4.7}$$

(Indeed, $U_0+\bar{\varepsilon}B$ for a fixed $\bar{\varepsilon}>0$ sufficiently small is a compact subset of $U$ and is covered by the collection of open sets

$$W_{\varepsilon,i}=\{u\,|\,u+\varepsilon B\subset U_{\bar{u}_i}\}\quad\text{for}\quad\varepsilon>0,\ 1\leqq i\leqq r.$$

Take a finite subcover and the minimum of the corresponding $\varepsilon$ values and $\bar{\varepsilon}$.) Combining (4.7) with (4.5) and the fact that $X_0\subset X$, we obtain

$$\Gamma_1(u')\cap X\subset\Gamma_1(u)+\lambda_1|u'-u|B\quad\text{when}\quad u\in U_0+\varepsilon B,\ |u'-u|\leqq\varepsilon,$$

or in other words

$$\Gamma_1(u+\delta B)\cap X_0\subset\Gamma_1(u)+\lambda_1\delta B\quad\text{when}\quad u\in U_0+\varepsilon B,\ \delta\leqq\varepsilon. \tag{4.8}$$

We can replace $V_0$ by a smaller neighborhood of $v$ if necessary so as to ensure that

$$\lambda_0|v_1-v_2|\leqq\varepsilon\quad\text{when}\quad v_1,v_2\in V_0. \tag{4.9}$$

Updating (4.4) to

$$\Gamma_0(v_1) \cap U_0 \subset \Gamma_0(v_2) + \lambda_0|v_1 - v_2|B \quad \text{when } v_1, v_2 \in V_0, \tag{4.10}$$

as is possible because $U_0 \subset U$, $V_0 \subset V$, and observing that

$$[\Gamma_0(v_2) + \delta B] \cap U_0 \subset [\Gamma_0(v_2) \cap (U_0 + \delta B)] + \delta B, \tag{4.11}$$

we argue now as follows. For arbitrary $v_1$, $v_2 \in V_0$ and the value $\delta = \lambda_0|v_1 - v_2|$, we have successively by (4.6), (4.10) and (4.11) that

$$\begin{aligned}\Gamma_1(\Gamma_0(v_1)) \cap X_0 &= \Gamma_1(\Gamma_0(v_1) \cap U_0) \cap X_0 \\ &\subset \Gamma_1(\Gamma_0(v_2) + \delta B] \cap U_0) \cap X_0 \\ &\subset \Gamma_1([\Gamma_0(v_2) \cap (U_0 + \delta B)] \cap U_0) \cap X.\end{aligned} \tag{4.12}$$

Here $\delta \leqq \varepsilon$ by (4.9), and therefore by (4.8) we have

$$\begin{aligned}&\Gamma_1([\Gamma_0(v_2) \cap (U_0 + \delta B)] + \delta B) \cap X_0 \\ &\quad= \Big[\bigcup_{u \in \Gamma_0(v_2) \cap (U_0 + \delta B)} \Gamma_1(u + \delta B)\Big] \cap X_0 \\ &\quad\subset \bigcup_{u \in \Gamma_0(v_2) \cap (U_0 + \delta B)} [\Gamma_1(u) + \lambda_1 \delta B] \\ &\quad= \Gamma_1(\Gamma_0(v_2) \cap [U_0 + \delta B]) + \lambda_1 \delta B\end{aligned} \tag{4.13}$$

where

$$\Gamma_1(\Gamma_0(v_2) \cap [U_0 + \delta B]) = \Gamma_1(\Gamma_0(v_2)) \tag{4.14}$$

by (4.6), because

$$\Gamma_0(v_2) \cap U_0 \subset \Gamma_0(v_2) \cap [U_0 + \delta B] \subset \Gamma_0(v_2).$$

Putting the chain of (4.12), (4.13) and (4.14) together, we obtain

$$\Gamma(v_1) \cap X_0 \subset \Gamma_1(\Gamma_0(v_2)) + \lambda_1 \delta B$$

whenever $v_1$, $v_2 \in V_0$, with $\delta = \lambda_0|v_1 - v_2|$. In other words,

$$\Gamma(v_1) \cap X_0 \subset \Gamma(v_2) + \lambda_1\lambda_0|v_1 - v_2|B \quad \text{for all} \quad v_1, v_2 \in V_0.$$

This being true for certain neighborhoods $V_0$ of $\bar{v}$ and $X_0$ of $\bar{x}$, we conclude that $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$. ■

*Remark* 4.2. The proof of theorem 4.1 also yields an estimate for the modulus $\lambda$ in the pseudo-Lipschitzian property of $\Gamma$ at $(\bar{v}, \bar{x})$: $\lambda = \lambda_1\lambda_0$ will work for some sufficiently small neighborhood $V \times X$ of $(\bar{v}, \bar{x})$ if $\lambda_0$ works locally for $\Gamma_0$ at each $(\bar{v}, \bar{u})$ with $\bar{u} \in \Delta(\bar{v}, \bar{x})$, and $\lambda_1$ works locally for $\Gamma_1$ at each $(\bar{u}, \bar{x})$ with $\bar{u} \in \Delta(\bar{v}, \bar{x})$.

This sort of estimate could be specialized to the various corollaries that follow.

COROLLARY 4.3. Let $\Gamma(v) = \Gamma_1(G(v))$, where $G: R^d \rightrightarrows R^p$ is a locally Lipschitzian function and $\Gamma_1: R^p \rightrightarrows R^n$ is a multifunction of closed graph. Then $\Gamma$ is of closed graph, and $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ whenever $\Gamma_1$ is pseudo-Lipschitzian at $(G(\bar{v}), \bar{x})$.

*Proof.* Let $\Gamma_0(v) = \{G(v)\}$ in theorem 4.1. One has $\Delta(v, x) = G(v)$ if $G(v) \in \Gamma_1^{-1}(x)$, $\Delta(v, x) = \phi$ otherwise, so $\Delta$ is locally bounded. ■

COROLLARY 4.4. Let $\Gamma(v) = G(\Gamma_0(v))$, where $\Gamma_0: R^d \rightrightarrows R^p$ is a multifunction of closed graph and $G: R^p \to R^n$ is a locally Lipschitzian function. Suppose that the multifunction $\Delta: R^d \times R^n \rightrightarrows R^p$ defined by

$$\Delta(v, x) = \{u \in \Gamma_0(v) \mid G(u) = x\}$$

is locally bounded. Then $\Gamma$ is of closed graph, and $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ whenever $\Gamma_0$ is pseudo-Lipschitzian at every $(\bar{v}, \bar{x})$ with $\bar{u} \in \Delta(\bar{v}, \bar{x})$.

*Proof.* Let $\Gamma_1(v) = \{G(v)\}$ in theorem 4.1. ■

COROLLARY 4.5. Let $\Gamma(v) = \alpha\Gamma_0(v) + a$, where $\Gamma_0: R^d \rightrightarrows R^n$ is a multifunction of closed graph, $\alpha \in R$, $\alpha \neq 0$, and $a \in R^n$. Then $\Gamma$ is of closed graph, and $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \alpha\bar{x} + a)$ whenever $\Gamma_0$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$. (A special case of this is $\Gamma = -\Gamma_0$.)

*Proof.* Let $G(u) = \alpha u + a$ in corollary 4.4. ■

COROLLARY 4.6. Let

$$\Gamma(v) = \{x \in R^n \mid \exists y \quad \text{with} \quad (x, y) \in \Gamma_0(v)\},$$

where $\Gamma_0: R^d \rightrightarrows R^n \times R^m$ is of closed graph and such that the multifunction $\Delta: R^d \times R^n \rightrightarrows R^m$ defined by

$$\Delta(v, x) = \{y \mid (x, y) \in \Gamma_0(v)\}$$

is locally bounded everywhere. Then $\Gamma$ is of closed graph.

If $\bar{x} \in \Gamma(\bar{v})$ and $\Gamma_0$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x}, \bar{y})$ for every $\bar{y} \in \Delta(\bar{v}, \bar{x})$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$.

*Proof.* One has $\Gamma = G(\Gamma_0(v))$ for $G(x, y) = x$. Apply corollary 4.4. ■

THEOREM 4.7. Let

$$\Gamma(v) = (\Gamma_1(v), \Gamma_2(v)) \subset R^{n_1} \times R^{n_2},$$

where $\Gamma_1: R^d \rightrightarrows R^{n_1}$ and $\Gamma_2: R^d \rightrightarrows R^{n_2}$ are multifunctions of closed graph. Then $\Gamma$ is of closed graph.

If $\Gamma_1$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x}_1)$ and $\Gamma_2$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x}_2)$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x}_1, \bar{x}_2)$.

*Proof.* The closedness of gph $\Gamma$ is elementary to verify. The pseudo-Lipschitzian property too follows right from the definition. ■

THEOREM 4.8. Let

$$\Gamma(v) = \Gamma_1(v) - \Gamma_2(v) = \{x_1 - x_2 \mid x_1 \in \Gamma_1(v), x_2 \in \Gamma_2(v)\},$$

where $\Gamma_1 : R^d \rightrightarrows R^n$ and $\Gamma_2 : R^d \rightrightarrows R^n$ are of closed graph and such that the multifunction $\Delta : R^n \rightrightarrows R^n$ defined by

$$\Delta(v) = \Gamma_1(v) \cap \Gamma_2(v)$$

is locally bounded everywhere. Then $\Gamma$ is of closed graph.

If $\bar{x} \in \Gamma(\bar{v})$, and for every $\bar{x}_1 \in \Gamma_1(\bar{v})$ and $\bar{x}_2 \in \Gamma_2(\bar{v})$ with $\bar{x}_1 - \bar{x}_2 = \bar{x}$ one has $\Gamma_1$ pseudo-Lipschitzian at $(\bar{v}, \bar{x}_1)$ and $\Gamma_2$ pseudo-Lipschitzian at $(\bar{v}, \bar{x}_2)$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$.

*Proof.* Let $\Gamma_0(v) = (\Gamma_1(v), \Gamma_2(v))$ and $G(x_1, x_2) = x_1 - x_2$. Then $\Gamma(v) = G(\Gamma_0(v))$. The result then follows from corollary 4.4 and theorem 4.8. ■

COROLLARY 4.9. Let $\Gamma(v) = \Gamma_1(v) + C$ where $\Gamma_1 : R^d \rightrightarrows R^n$ is of closed graph and $C \subset R^n$ is a compact set (for instance, $C = \varepsilon B$). Then $\Gamma$ is of closed graph. If $\Gamma_1$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x}_1)$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ for every $\bar{x} \in [\bar{x}_1 + C]$.

*Proof.* Let $\Gamma_2(v) \equiv C$ in theorem 4.8. ■

COROLLARY 4.10. Let $\Gamma(v) = \Gamma_1(v) + G(v)$ where $\Gamma_1 : \mathrm{R}^d \rightrightarrows R^n$ is of closed graph and $G : R^d \to R^n$ is a locally Lipschitzian function. Then $\Gamma$ is of closed graph. If $\Gamma_1$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x} + G(\bar{v}))$.

*Proof.* Let $\Gamma_2(v) = \{G(v)\}$ in theorem 4.8. ■

THEOREM 4.11. Let

$$\Gamma(v) = \cup_{i=1}^{r} \Gamma_i(v),$$

where $\Gamma_i : R^d \rightrightarrows R^n$ is of closed graph for $i = 1, \ldots, 4$. Then $\Gamma$ is of closed graph. If $\bar{x} \in \Gamma(\bar{v})$ and $\Gamma_i$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ for every $i$ such that $\bar{x} \in \Gamma_i(\bar{v})$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$.

*Proof.* This can easily enough be proved from the definitions, but the pseudo-Lipschitzian property also follows at once from the distance function formula

$$d_\Gamma(v, x) = \min_{i=1}^{r} d_{\Gamma_i}(v, x)$$

and the characterization in theorem 2.3. ■

THEOREM 4.12. Let

$$\Gamma(v) = \cap_{i=1}^{r} \Gamma_i(v),$$

where $\Gamma_i : R^d \rightrightarrows R^n$ is of closed graph for $i = 1, \ldots, r$. Then $\Gamma$ is of closed graph. One has $\Gamma$ pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ if the normal cones

$$N_i = N_{\mathrm{gph}\, \Gamma_i}(\bar{v}, \bar{x}), \ i = 1, \ldots, r,$$

have the following property: the only choice of vectors $(z_i, w_i) \in N_i$ such that $\Sigma_{i=1}^{r} w_i = 0$ is $(z_i, w_i) = (0, 0)$ for $i = 1, \ldots, r$.

*Proof.* Let $D_i = \operatorname{gph}\Gamma_i$, $D = \operatorname{gph}\Gamma$, and observe that $D = \cap_{i=1}^r D_i$. By hypothesis each $D_i$ is closed, so $D$ is closed. Thus $\Gamma$ is of closed graph. One has

$$N_D(\bar{v}, \bar{x}) \subset \sum_{i=1}^{r} N_i$$

by [13, corollary 8.1.1] if the only choice of vectors $(z_i, w_i) \in N_i$ such that $\Sigma_{i=1}^r (z_i, w_i) = (0, 0)$ is $(z_i w_i) = (0, 0)$ for $i = 1, \ldots, r$, Our assumption on the cones $N_i$ ensures this and also that the cone $\Sigma_{i=1}^r N_i$ contains no vector of the form $(z, 0)$ with $z \neq 0$. Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ by corollary 3.4. ■

COROLLARY 4.13. Let

$$\Gamma(v) = \Gamma_1(v) \cap C,$$

where $\Gamma_1 : \mathrm{R}^d \rightrightarrows R^n$ is of closed graph and $C \subset R^n$ is a closed set. Let $\bar{x} \in \Gamma(\bar{v})$ and suppose that the following condition on normal cones holds: the only vector

$$(z, w) \in N_{\operatorname{gph}\Gamma_1}(\bar{v}, \bar{x}) \quad \text{with} \quad -w \in N_C(\bar{x})$$

is $(z, w) = (0, 0)$. Then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$ (and of closed graph).

*Proof.* In theorem 4.12 take $r = 2$, $\Gamma_2(v) \equiv C$. ■

THEOREM 4.14. Let

$$\Gamma(v) = \Gamma_1(v, U) = \bigcup_{u \in U} \Gamma_1(u, v)$$

where $\Gamma_1 : R^d \times R^p \rightrightarrows R^n$ is of closed graph and $U \subset R^p$ is closed. Suppose that the multifunction $\Delta : R^d \times R^n \rightrightarrows R^p$ defined by

$$\Delta(v, x) = \{u \in U \,|\, x \in \Gamma_1(v, u)\}$$

is locally bounded everywhere. Then $\Gamma$ is of closed graph.

If $\bar{v} \in R^d$ and $\bar{x} \in \Gamma(\bar{v})$ are such that $\Gamma_1$ is pseudo-Lipschitzian at $(\bar{v}, \bar{u}, \bar{x})$ for all $\bar{u} \in \Delta(\bar{v}, \bar{x})$, then $\Gamma$ is pseudo-Lipschitzian at $(\bar{v}, \bar{x})$.

*Proof.* Represent $\Gamma$ as $\Gamma_1 \circ \Gamma_0$, where $\Gamma_0 : R^d \rightrightarrows R^d \times \mathrm{R}^p$ is defined by $\Gamma_0(v) = (v, U)$, and apply theorem 4.1. The multifunction

$$(v, x) \rightarrow \Gamma_0(v) \cap \Gamma_1^{-1}(x) = \{(v, u) \,|\, u \in U, x \in \Gamma_1(v, u)\},$$

whose local boundedness is required in theorem 4.1, is locally bounded by our assumption on the multifunction $\Delta$ defined here. ■

COROLLARY 4.15. Let

$$\Gamma(v) = G(v, U) = \{G(v, u) \,|\, u \in U\},$$

where $G : R^d \times R^p \rightarrow R^n$ is a locally Lipschitzian function and $U \subset R^p$ is a nonempty closed set. Suppose that the multifunction $\Delta : R^d \times R^n \rightrightarrows R^p$ defined by

$$\Delta(v, x) = \{u \in U \,|\, G(v, u) = x\}$$

is locally bounded everywhere (as is true in particular if $U$ is bounded, in which $\Gamma$ case itself is locally bounded). Then $\Gamma$ is of closed graph, and everywhere sub-Lipschitzian (actually Lipschitzian if $U$ is bounded).

*Proof.* Take $\Gamma_1(v, u) = \{G(v, u)\}$ in theorem 4.15 and use theorem 2.2. ■

THEOREM 4.16. Let

$$\Gamma(v) = \operatorname{co} \Gamma_0(v) \quad \text{(convex hull)},$$

where $\Gamma_0 : R^d \rightrightarrows R^n$ is locally bounded and of closed graph. Then $\Gamma$ is locally bounded and of closed graph. If $\Gamma_0$ is Lipschitzian at a point $\bar{v}$, then so is $\Gamma$.

*Proof.* Let $U$ be the compact set in $R^{n+1}$ consisting of the vectors $u = (u_0, u_1, \ldots, u_n)$ such that $u_i \geqq 0$, $\Sigma_{i=0}^n u_i = 1$. By Carathéodory's theorem on convex hulls one has

$$\operatorname{co} \Gamma_0(v) = \left\{ \sum_{i=0}^n u_i v_i | x_i \in \Gamma_0(v), u \in U \right\},$$

so that $\Gamma(v) = \Gamma_1(v, U)$ for

$$\Gamma_1(v, u) := \sum_{i=0}^n u_i \Gamma_0(v).$$

Furthermore $\Gamma_1 = G \circ \Gamma_2$, where

$$G(x_0, x_1, \ldots, x_n, u_0, u_1, \ldots, u_n) := \sum_{i=0}^n u_i x_i,$$

$$\Gamma_2(v, u) := (\Gamma_0(v), \ldots, \Gamma_0(v), u).$$

Clearly $\Gamma_2$ inherits closedness and local boundedness from $\Gamma_0$, and $\Gamma_2$ is Lipschitzian at $\bar{v}$ if $\Gamma_0$ is (cf. theorem 4.16 and the characterizations in Section 2). The same then follows for $\Gamma_1$, because $G$ is a locally Lipschitzian function (cf. corollary 4.2). Corollary 4.15 now gives the desired conclusion. ■

COROLLARY 4.17. Let

$$\Gamma(v) = \operatorname{co}\{g_1(v), \ldots, g_r(v)\},$$

where the functions $g_i : R^d \to R^n$ are locally Lipschitzian. Then $\Gamma$ is a locally bounded multifunction of closed graph, and $\Gamma$ is everywhere Lipschitzian.

*Proof.* Let $\Gamma_0(v) = \{g_1(v)\}$ in theorem 4.16. The multifunction $\Gamma_0$ is locally bounded, and by theorem 4.11 it is of closed graph and everywhere Lipschitzian. (The latter also follows easily from the definitions.) ■

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