Generalized Conjugacy in Hamilton - Jacobi theory for Fully Convex Lagrangians

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1 Introduction

An object of great interest in optimal control, calculus of variations, and the corresponding Hamilton-Jacobi theory is the value function. For an initial cost function $f : I\!R^n \mapsto \overline{I\!R}$ and a running cost $L : I\!R^n \times I\!R^n \mapsto \overline{I\!R}$, also referred to as the Lagrangian, the value function $V : [0, +\infty) \times I\!R^n \mapsto \overline{I\!R}$ is defined to be

$$V(\tau, \xi) = \inf \left\{ f(x(0)) + \int_0^\tau L(x(t), \dot{x}(t))dt | \ x(\tau) = \xi \right\}, \tag{1}$$

where the infimum is taken over all absolutely continuous arcs $x : [0, \tau] \mapsto I\!R^n$,, subject to the terminal constraint $x(\tau) = \xi$. Translation of results from this setting to the one often seen in control theory, where an initial condition and a terminal cost function are considered, involves a simple change of variables, and was addressed by Rockafellar and Wets [10]

The main issue addressed in this paper is whether the knowledge of the value function $V(\overline{\tau}, \cdot)$ at some time $\overline{\tau} > 0$ determines the initial cost function f. To be more precise, suppose that for a given Lagrangian, V_1 and V_2 are two value functions corresponding to initial costs f_1 and f_2. Can we say that

$$V_1(\overline{\tau}, \cdot) = V_2(\overline{\tau}, \cdot) \text{ for some } \overline{\tau} > 0 \quad \Longrightarrow \quad f_1 = f_2. \tag{2}$$

Such a conclusion resembles a "cancelation rule" available for the operation of inf-convolution. Recall that for any two functions $f : I\!R^n \mapsto \overline{I\!R}$ and $g : I\!R^n \mapsto \overline{I\!R}$, their inf-convolution, also known as the epi-addition, is defined as

$$(f \# g)(x) = \inf_{y+z=x} \Big\{ f(y) + g(z) \Big\} = \inf_{y \in \boldsymbol{R}^n} \Big\{ f(y) + f_2(x-y) \Big\}. \tag{3}$$

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When both f and g are proper functions — they do not take on the value $-\infty$ and they are finite somewhere — the inf-convolution is well-defined. If $f_1(\cdot)$, $f_2(\cdot)$ and $g(\cdot)$ are also convex and lsc, with g coercive, the following can be said:

$$f_1 \# g = f_2 \# g \quad \Longrightarrow \quad f_1 = f_2.$$

We recall that a function g is said to be coercive if $\lim_{|x|\to\infty} \frac{g(x)}{|x|} = \infty$. A particular example of such g is provided by a quadratic function $\frac{1}{2\lambda}|\cdot|^2$ for some $\lambda > 0$. The inf-convolution of $f \# g$ is then the Moreau envelope of f with parameter λ:

$$e_\lambda f(x) = \inf_{y \in \boldsymbol{R}^n} \left\{ f(y) + \frac{1}{2\lambda}|y - x|^2 \right\}.$$

The cancelation rule amounts to say that if, for some $\lambda > 0$, Moreau envelopes of $f_1(\cdot)$ and $f_2(\cdot)$ are equal, then actually $f_1 = f_2$.

A direct connection between the discussed cancelation rule and our results is seen by considering Lagrangians independent of the state variable. Indeed, consider a Lagrangian $L(x, v) = g(v)$, where g is a proper, lsc, convex, and coercive function. In this special case, the following formula, which can be traced back to Hopf [4] and Lax [5], holds:

$$V(\tau, \xi) = \inf_{\xi' \in \boldsymbol{R}^n} \left\{ f(\xi') + \tau g\left(\tau^{-1}(\xi - \xi')\right) \right\}. \tag{4}$$

Thus, for any fixed $\tau > 0$, the value function is the inf-convolution $f \# g_\tau$, where $g_\tau(v) = \tau g(\tau^{-1} v)$. Symmetrically, the inf-convolution of $f(\cdot)$ and $g(\cdot)$ can be viewed as the value function at $\tau = 1$.

Coercivity of $g(\cdot)$ translates to that of $g_\tau(\cdot)$ for all $\tau > 0$. Thus, if $V_1(\cdot,\cdot)$ and $V_2(\cdot,\cdot)$ are two value functions corresponding to initial costs $f_1(\cdot)$ and $f_2(\cdot)$ and the Lagrangian $g(\cdot)$, the cancelation rule for inf-convolution implies that (2) holds.

We approach (2) in the general fully convex setting where the Lagrangian depends on both x and v, and is jointly convex in these variables. Assumption of convexity is also in place for the initial cost f. We rely on the Hamilton-Jacobi theory developed for such setting by Rockafellar and Wolenski [10], [11], in particular on a lower envelope representation of the value function involving a "dualizing kernel". This representation allows us to view the initial cost function f and the value function $V(\overline{\tau}, \cdot)$ as functions conjugate to each other, in a framework of "generalized conjugacy", as described by Rockafellar and Wets [9]. We present the necessary background in Section 2.

An affirmative answer to our main question is given in Section 3, subject to persistence of the trajectories of a generalized Hamiltonian system associated with the control problem. Section 4 is devoted to the analysis of the mentioned Hamiltonian system, and direct characterization in terms of the Lagrangian of cases where the trajectories persist. Partial results in these subjects were obtained by Goebel [3].

Let us mentions that the lack of regularity assumptions on the Lagrangian allows our format to express a wide range of optimal control problems, including those with control and mixed constraints. Consider a control problem with linear dynamics $\dot{x}(t) = Ax(t) + Bu(t)$, with the control $u(t)$ constrained to some nonempty, closed, and convex set $U \in \mathbb{R}^k$, and where the cost expression is given by

$$f(x(0)) + \int_0^\tau l(x(t), u(t))dt \tag{5}$$

for some proper lsc and convex functions f and $l : \mathbb{R}^n \times \mathbb{R}^k \mapsto \overline{\mathbb{R}}$. Again, we can consider the value function, as the optimal value in the above control problem, parameterized by (τ, ξ) in the terminal condition $x(\tau) = \xi$. Such value function can be expressed as (1) by defining

$$L(x, v) = \inf_{u \in U} \{l(x, u) \mid v = Ax + Bu\}. \tag{6}$$

Partial answer to the question of which control problems yield a Lagrangian satisfying our assumptions will be given in Section 4, we also refer the reader to [11], [7] and [8].

2 Dualizing kernel and generalized conjugacy

The following assumptions on f and L are in place thorough this Section as well as Section 3.

Assumption 2.1 (basic assumptions)**.**

(A0) The function $f(\cdot)$ is convex, proper and lsc on $\mathbb{R}^n$.

(A1) The function $L(\cdot, \cdot)$ is convex, proper and lsc on $R^n \times R^n$.

(A2) The set $F(x) = \{v \mid L(x, v) < \infty\}$ is nonempty for all x, and there is a constant ρ such that $\mathrm{dist}(0, F(x)) \leq \rho(1 + |x|)$ *for all x.*

(A3) There are constants α and β and a coercive, proper, nondecreasing function $\theta(\cdot)$ on $[0, \infty)$ such that $L(x, v) \geq \theta(\max\{0, |v| - \alpha|x|\}) - \beta|x|$ for all x and v.

These were exactly the assumptions used by Rockafellar and Wolenski [10], [11] in developing the Hamilton-Jacobi theory for convex problems of Bolza. Assumption 2.1 guarantees, among other things, that the value function is a well-defined proper lsc and convex function. Results of [11] which will be used in this paper are summarized in the next theorem.

Theorem 2.2 (envelope representation of the value function)**.** *The dualizing kernel $K : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ defined as*

$$K(\tau, \xi, \eta) = \inf \left\{ \langle x(0), \eta \rangle + \int_0^\tau L(x(t), \dot{x}(t))dt \mid x(\tau) = \xi \right\},$$

is an everywhere finite function, convex in ξ and concave in η. The value function (1) *can be represented as*

$$V(\tau, \xi) = \sup_\eta \{K(\tau, \xi, \eta) - f^*(\eta)\}, \tag{7}$$

where f^* is the function conjugate to the initial cost f.

Note that for a fixed η, $K(\cdot,\cdot,\eta)$ is the value function corresponding to an affine initial cost function $f(x) = \langle \eta, x\rangle$. If a formula reciprocal to (7) was in place, that is if

$$f^*(\eta) = \sup_{\xi} \{K(\tau,\xi,\eta) - V(\tau,\xi)\}. \tag{8}$$

then a recovery of the initial cost f from $V(\tau,\cdot)$ would indeed be possible.

Formulas (7) and (8) can be viewed as a generalized conjugacy relation between the value function $V(\tau,\cdot)$ and the initial cost $f^*(\cdot)$ with respect to the function $K(\tau,\cdot,\cdot)$. Let us present the basic framework of generalized conjugacy. Given any function $\phi : I\!R^n \times I\!R^n \mapsto \overline{I\!R}$ we define the ϕ-conjugate of $f(\cdot)$ as

$$f^{\phi}(y) = \sup_{x} \{\phi(x,y) - f(x)\}, \tag{9}$$

and the ϕ-biconjugate of $f(\cdot)$ as

$$f^{\phi\phi}(x) = \sup_{y} \left\{\phi(x,y) - f^{\phi}(y)\right\}. \tag{10}$$

The standard notion of conjugacy between convex functions is obtained by considering $\phi(x,y) = \langle x,y\rangle$. Directly from the definitions we can obtain that $f \geq f^{\phi\phi}$. Indeed, (10) implies that for all x and y, $f^{\phi}(y) \geq \phi(x,y) - f(x)$, which is equivalent to $f(x) \geq \phi(x,y) - f^{\phi}(y)$. Taking the supremum with respect to y yields $f(x) \geq f^{\phi\phi}(x)$. We now give a condition on subgradients of ϕ sufficient for the last inequality to turn into an equality.

Lemma 2.3 *Let $\phi(x,y)$ be a finite function, concave in x. Assume that for every x and every z there exists y such that $z \in \tilde{\partial}_x \phi(x,y)$. Then, for every proper, lsc and convex function $f(\cdot)$ we have $f^{\phi\phi}(\cdot) = f(\cdot)$.*

Proof. First, note that it is sufficient to show that for every affine function $k(\cdot)$, we have $k^{\phi\phi}(\cdot) = k(\cdot)$. Indeed, suppose that this is true. Pick any proper, lsc and convex function $f(\cdot)$. Let $k(\cdot)$ be an affine function majorized by $f(\cdot)$. We have $f(\cdot) \geq k(\cdot)$, and through duality relationships (9), (10) we get $f^{\phi}(\cdot) \leq k^{\phi}(\cdot)$ and $f^{\phi\phi}(\cdot) \geq k^{\phi\phi}(\cdot)$. By our supposition, the last inequality becomes $f^{\phi\phi}(\cdot) \geq k(\cdot)$. This implies that $f^{\phi\phi} \geq f(\cdot)$, since $f(\cdot)$, being a proper, lsc and convex function, is the supremum of all affine functions it majorizes. But $f(\cdot) \geq f^{\phi\phi}(\cdot)$ is always true, and therefore, $f^{\phi\phi}(\cdot) = f(\cdot)$.

We now show that $k^{\phi\phi}(\cdot) = k(\cdot)$ for any affine function $k(\cdot)$. Let $k(x) = \langle z, x\rangle + b$. By (a), for every x there exists y such that $z \in \tilde{\partial}_x \phi(x,y)$, which is equivalent to $0 \in \tilde{\partial}_x (\phi(x,y) - k(x))$. This is a necessary and sufficient condition for x to be the maximizer in the expression (9) for $k^{\phi}(y)$. Thus, we have that for any x there exists y such that $k(x) + k^{\phi}(y) = \phi(x,y)$. Then

$$k^{\phi\phi}(x) = \sup_{y} \left\{\phi(x,y) - k^{\phi}(y)\right\} \geq k(x).$$

Thus $k^{\phi\phi} = k(x)$. □

It can now be expected that for (8) to hold, a condition similar to the one in Lemma 2.3 will be needed for the subgradients of $K(\tau,\cdot,\cdot)$. These turn out to be closely related to the trajectories of the Hamiltonian system associated with the given Lagrangian.

3 Cost recovery via Hamiltonian trajectories

With every Lagrangian — and every problem of calculus of variations — we can associate a Hamiltonian function, defined as

$$H(x,y) = \sup_{v \in \mathbb{R}^n} \{\langle y, v\rangle - L(x,v)\}. \tag{11}$$

That is, for every fixed x, the convex function $H(x,\cdot)$ is the function conjugate to $L(x,\cdot)$. Under our assumptions, $H(x,y)$ is always finite, concave in x, and, as mentioned, convex in y. For a Lagrangian coming from a control problem (5), the corresponding Hamiltonian function is

$$H(x,y) = \sup_{u \in U} \{\langle y, Ax + Bu\rangle - l(x,u)\} = \langle y, Ax\rangle + h(x, B^* y),$$

where $h(x,\cdot)$ is the convex conjugate of $l(x,\cdot) + \delta_U(\cdot)$. Here, δ_U is the indicator function of the set U. The Hamiltonian plays an important role in the Hamilton-Jacobi theory, where it characterizes the value function as the unique solution to the Hamilton-Jacobi equation, see Galbraith [2].

By a Hamiltonian trajectory on interval $[a,b]$ we will understand a pair of absolutely continuous arcs $x, y : [a,b] \mapsto \mathbb{R}^n$ such that

$$-\dot{y}(t) \in \tilde{\partial}_x H\left(x(t), y(t)\right), \qquad \dot{x}(t) \in \partial_y H\left(x(t), y(t)\right), \tag{12}$$

for almost all $t \in [a,b]$. Above, $\partial_y H(x,y)$ is the subdifferential of the convex function $H(x,\cdot)$ while $\tilde{\partial}_y H(x,y)$ is the subdifferential (in the concave sense) of the concave function $H(\cdot,y)$. Hamiltonian trajectories are involved in optimality conditions and can be used to describe the evolution of the subdifferential of the value function $\partial_\xi V(\tau,\cdot)$ from the subdifferential of the initial costs $f(\cdot)$ – we mention the corresponding result of Rockafellar and Wolenski [10] as Theorem 3.4. The trajectories are also closely related to the subgradients of the dualizing kernel $K(\tau,\cdot,\cdot)$, as it was shown in [11]:

Theorem 3.1 (subgradients of the dualizing kernel)**.** *The following are equivalent:*

- *(a)* $\eta' \in \partial_\xi K(\tau,\xi,\eta)$ *and* $\xi' \in \tilde{\partial}_\eta K(\tau,\xi,\eta)$.
- *(b) there is a Hamiltonian trajectory on* $[0,\tau]$ *from* (ξ',η) *to* (ξ,η').

Equipped with such a characterization of subgradients of $K(\tau,\cdot,\cdot)$, we can apply Lemma 2.3 to the setting of generalized conjugacy between the initial cost f and the value function $V(\tau,\cdot)$.

Theorem 3.2 (recovery of the initial cost)**.** *Assume that there are no Hamiltonian trajectories escaping to infinity on* $[0,\tau]$*. Then formula* (8) *holds.*

Proof. In light of the generalized conjugacy relations (9), (10), and Lemma 2.3, we need to show that for any ξ' and η, there exists ξ so that $\xi' \in \tilde{\partial}_\eta K(\tau,\xi,\eta)$. The set-valued mapping $(x,y) \to \partial_y H(x,y) \times -\tilde{\partial}_x H(x,y)$ has nonempty, compact, and convex values and is outer-semicontinuous. Thus Hamiltonian trajectories exist for every initial point, see Aubin and Cellina [1]. This, and our assumption guarantee that for any point (ξ',η), there exists a Hamiltonian trajectory originating at (ξ',η), with the endpoint at some (ξ,η'). By Theorem 3.1, $\xi' \in \tilde{\partial}_\eta K(\tau,\xi,\eta)$. This finishes the proof. □

Theorem 3.3 shows that, under the assumption that no Hamiltonian trajectories escape to infinity in finite time, the knowledge of the value function $V(\tau,\xi)$ at one time $\tau \geq 0$ actually describes the whole value function.

Corollary 3.3 *Assume that there are no Hamiltonian trajectories escaping to infinity in finite time. The following are equivalent:*

(a) $g_1(x) = g_2(x)$ *for all* $x \in I\!R^n$.

(b) $V_1(\tau,\xi) = V_2(\tau,\xi)$ *for all* $(\tau,\xi) \in [0,+\infty) \times I\!R^n$.

(c) There exists $\overline{\tau} \geq 0$ *such that* $V_1(\overline{\tau},\xi) = V_2(\overline{\tau},\xi)$ *for all* $\xi \in I\!R^n$.

We now present an example where the Hamiltonian trajectories do escape to infinity in finite time, and where the conclusion of Theorem 3.3 fails. The argument will take advantage of the following result of Rockafellar and Wolenski [10]:

Theorem 3.4 (Hamiltonian evolution of subgradients)**.** *A point* (x_t,y_t) *is in the graph of* $\partial_\xi V(t,\cdot)$ *if and only if for some* $(x_0,y_0) \in \operatorname{gph} \partial f(\cdot)$*, there is a Hamiltonian trajectory* $(x(\cdot),y(\cdot))$ *on* $[0,t]$ *with* $(x(0),y(0)) = (x_0,y_0)$, $(x(t),y(t)) = (x_t,y_t)$.

Example 3.5 (Hamiltonian trajectories with finite escape time)**.** Consider the Hamiltonian $H(x,y) = \frac{1}{8}(-x^4+y^4)$, which corresponds to the Lagrangian $L(x,v) = \frac{1}{8}x^4 + \frac{3}{4}2^{\frac{1}{3}}v^{\frac{4}{3}}$. Hamiltonian trajectories are the solutions of the system:

$$\dot{y}(t) = \frac{1}{2}x^3(t), \qquad \dot{x}(t) = \frac{1}{2}p^3(t)$$

The trajectory $(x(\cdot),y(\cdot)$ originating at $(1,1)$ is

$$(x(t),y(t)) = \left((1-t)^{-\frac{1}{2}},(1-t)^{-\frac{1}{2}}\right).$$

This trajectory escapes to infinity in $t=1$. Without direct calculation, we can show that other trajectory $(x'(\cdot),y'(\cdot)$ originating at (x',y') with $x \geq 1$, $y \geq 1$ must also escape to infinity in time $t \leq 1$. A result of Rockafellar [6] states that the function

$$m(t) = \langle x'(t)-x(t), y'(t)-y(t)\rangle$$

is nondecreasing. A slight modification of this result shows that in case of a strictly concave, strictly convex Hamiltonian, $m(t)$ is actually increasing whenever $(x(t), y(t)) \neq (x'(t), y'(t))$. Thus, if $(x', y') \neq (1,1)$, we must have $\langle x'(t) - x(t), y'(t) - y(t) \rangle > 0$. Combining this with the continuity of the trajectories in question, we obtain $x'(t) > x(t)$, $y'(t) > y(t)$ for $t > 0$. This implies that $(x'(\cdot), y'(\cdot 0)$ escapes to infinity in time at most 1. By symmetry, any trajectory originating at (x', y') with $x' \leq -1$, $y \leq -1$ must also escape to infinity in time $t \leq 1$.

Now take any two different nonnegative convex functions $f_1(\cdot)$ and $f_2(\cdot)$ such that $f_i(0) = 0$, $1 \in \partial f_i(1)$, $-1 \in \partial f_i(-1)$, for $i = 1, 2$, and so that $f_1(x) = f_2(x)$ holds for $x \in [-1, 1]$. An example of two such functions is $f_1(x) = \frac{1}{2}x^2$, $f_2(x) = |x|$. By the argument about trajectories escaping to infinity, the graph of $\partial V_i(1, \cdot)$ is the image of the graph $\partial f_i(\cdot)$ restricted to $[-1, 1] \times [-1, 1]$ under the Hamiltonian flow. Our assumptions on f_1 and f_2 guarantee that $\operatorname{gph} \partial f_1$ and $\operatorname{gph} \partial f_2$ agree on $[-1, 1] \times [-1, 1]$. Therefore $\partial V_1(1, \cdot) = \partial V_2(1, \cdot)$. To claim that $V_1(1, \cdot) = V_2(1, \cdot)$ it suffices now to show that the two functions agree at some point. The Lagrangian satisfies $L \geq 0$ and $L(0, 0) = 0$. Thus both value functions are nonnegative, and must equal 0 at $x = 0$.

4 Persistence of Hamiltonian Trajectories

Duality theory can be employed to restate the assumptions on the Lagrangian in terms of the Hamiltonian. Rockafellar and Wolenski showed that $(A1)$, $(A2)$ and $(A3)$ from Assumption 2.1 are equivalent to the following:

Assumption 4.1 (Hamiltonian assumptions). *The function $H(x, y)$ is everywhere finite, concave in x, convex in y, and such that*

(H1) There are constants α and β and a finite, convex function ϕ such that

$$H(x, y) \leq \phi(y) + (\alpha|y| + \beta)|x|,$$

(H2) There are constants α' and β' and a finite, convex function ψ' such that

$$H(x, y) \geq -\psi'(x) - (\alpha'|x| + \beta')|y|.$$

Any function which can be expressed as a sum of a finite concave function of x and a finite convex function of y satisfies the above assumptions. In fact, any Hamiltonian of the form

$$H(x, y) = \langle y, Ax \rangle - f(x) + g(y) \tag{13}$$

with f and g as described satisfies them. This covers the case of a finite Hamiltonian depending on either just x or just y, and Hamiltonians coming from control problems with linear dynamics and cost functionals (5), for which $l(x,u) = f(x) + g^*(u)$ for a coercive g^*. To see that (13) satisfies $(H1)$, note that $f(x) \geq f(0) + \langle v, x\rangle \geq f(0) - |v||x|$ for some chosen $v \in \partial f(0)$ implies

$$H(x,y) = \langle y, Ax\rangle - f(x) + g(B^*y) \leq (\|A\||y| + |v|)\,|x| + g(B^*y) - f(0).$$

Thus $H(x,y)$ satisfies $(H1)$, with $\psi(y) = g(B^*y) - f(0)$. Argument for $(H2)$ is symmetrical.

Another family for which Assumption 4.1 always holds is the family of finite, concave-convex functions which are piecewise linear-quadratic. We recall that a function f is called piecewise linear-quadratic if $\operatorname{dom} f = \{x \mid f(x) < +\infty\}$ can be be represented as a union of finitely many polyhedral sets, relative to each of which f is given by a quadratic function. This family includes Hamiltonians which arise in the setting of extended linear-quadratic control, as introduced in Rockafellar [7].

We now show that any finite piecewise linear-quadratic concave-convex function $H(x,y)$ satisfies Assumption 4.1. By definition, there exist polyhedral sets $S_1, S_2, ...S_m$ with $\bigcup_{i=1}^m S_i = I\!R^n \times I\!R^n$ and such that, on each S_i the function $h(\cdot,\cdot)$ is given by

$$h(z) = \frac{1}{2} z \cdot A_i z + a_i \cdot z + \alpha_i,$$

where $z = (x,y)^*$. We can write express each A_i and a_i as $A_i = \begin{bmatrix} -B_i & C_i \\ D_i & E_i \end{bmatrix}$, $\quad a_i = (b_i, c_i)^*$ for some matrices B_i, C_i, D_i and E_i and vectors b_i and c_i. Since $H(\cdot,\cdot)$ is a concave-convex function, we must have B_i and E_i positive semidefinite. Let $M = \sup\left\{\frac{1}{2}y \cdot E_i y \mid i = 1,2,...m,\ |y| = 1\right\}$. Then, for every $i = 1,2,...m$, $h(x,y) - M|y|^2 \leq \frac{1}{2}x \cdot (C_i + D_i^*)y + b_i \cdot x + c_i \cdot y + \alpha_i \leq |x|N|y| + b|x| + c|y| + \alpha$ where N is defined in similar fashion to M with the matrices $C_i + D_i^*$, $b = \sup_i |b_i|$, $c = \sup_i |c_i|$ and $\alpha = \sup_i |\alpha_i|$. Taking $\phi(y) = M|y|^2 + c|y| + \alpha$ we get that

$$h(x,y) \leq \phi(y) + (N|y| + b)|x|,$$

and this is exactly condition (H1). Symmetrical argument shows (H2).

Lemma 4.2 (linear growth of subdifferential). *For a proper lsc convex function $f : I\!R^n \mapsto I\!R^n$, the following are equivalent:*

(a) $\sup_{v \in \partial f(x)} |v| \leq a|x| + b$ *for some constants* $a, b > 0$.

(b) $f(x) \leq c|x|^2 + d$ *for some constants* $c, d > 0$.

Proof. Assume (a). As f is proper, there exists $\bar{x}$ with $f(\bar{x})$ finite. By the mean value theorem, see for example Rockafellar and Wets [9], there exist $0 < \lambda < 1$ and $v \in \partial f(x_\lambda)$ where $x_\lambda = (1-\lambda)x + \lambda\bar{x}$, such that $f(x) - f(\bar{x}) = \langle v, x - \bar{x}\rangle$. Then $f(x) \leq |f(\bar{x})| + |v|(|x| + |\bar{x}|)$. Using the linear growth assumption and the fact that $|x_\lambda| \leq |x| + |\bar{x}|$ we get $f(x) \leq |f(\bar{x})| + [a(|x| + |\bar{x}|) + b]\,(|x| + |\bar{x}|)$. Elementary analysis shows that the expression on the right can be bounded by $c|x|^2 + d$ for some constants $c, d > 0$.

Now assume (b), and suppose that (a) fails. For every n, we can then find x_n, with $|x_n| > n$, so that for some $v_n \in \partial f(x_n)$, $|v_n| > n|x_n|$. Such a conclusion follows from the fact that, under (b), f is finite-valued and thus ∂f is locally bounded. Fix some $w \in \partial f(0)$. For any $\lambda > 0$ we get

$$f(x_n + \lambda v_n) \geq f(x_n) + \langle v_n, x_n + \lambda v_n - x_n \rangle \geq f(0) + \langle w, x_n \rangle + \lambda |v_n|^2 \geq f(0) - \frac{1}{n}|w||v_n| + \lambda |v_n|^2.$$

We can estimate the norm of $x_n + \lambda v_n$:

$$|x_n + \lambda v_n|^2 \leq |x_n|^2 + 2\lambda |x_n||v_n| + \lambda^2 |v_n|^2 < \frac{1}{n^2}|v_n|^2 + 2\lambda \frac{1}{n}|v_n|^2 + \lambda^2 |v_n|^2 = |v_n|^2 \left(\lambda + \frac{1}{n}\right)^2.$$

Combining the above estimates and the bound in (b), we obtain

$$c|v_n|^2 \left(\lambda + \frac{1}{n}\right)^2 + d > f(0) - \frac{1}{n}|w||v_n| + \lambda |v_n|^2,$$

which, in the limit, yields $c\lambda^2 > \lambda$. Picking $\lambda < \frac{1}{c}$ yields a contradiction. □

Theorem 4.3 (persistence of trajectories)**.** *Any of the following conditions guarantee that no Hamiltonian trajectories escape to infinity in finite time:*

(a) $H(x, y) = -f(x)$ *for some finite convex function* f*, or* $H(x, y) = g(y)$ *for some finite convex function* g*.*

(b) $H(x, y) = -f(x) + g(y)$ *for some finite convex functions* f *and* g*, where either* $\operatorname{rge} \partial f$ *or* $\operatorname{rge} \partial g$ *is bounded.*

(c) $H(x, y) = \langle y, Ax \rangle - f(x) + g(y)$ *for some matrix* A*, and some proper lsc convex functions* f *and* g*, both of which are majorized by* $c|x|^2 + d$*, for some constants* $c, d > 0$ *(so in particular* f *and* g *are finite).*

(d) The Hamiltonian is piecewise linear-quadratic.

(e) The subdifferential of the Hamiltonian is of linear growth – for some constants a*,* b *we have*

$$\sup\left\{(v, z) \mid v \in \tilde{\partial}_x H(x, y), z \in \partial_y H(x, y)\right\} \leq a|(x, y)| + b.$$

Proof. Note that (a) is a special case of (b), with either f or g equal trivially to 0. We show (b), assuming that $\operatorname{rge} \partial f$ is bounded. Suppose $\operatorname{rge} \partial f \in K\mathbb{B}$, fix $T > 0$ and a point (x_0, y_0). Hamiltonian system has the form

$$\dot{y}(t) \in \partial f(x(t)), \qquad \dot{x}(t) \in \partial g(y(t)).$$

Thus any Hamiltonian trajectory originating at (x_0, y_0) satisfies $y(t) \in y_0 + KT\mathbb{B}$ for all those $t \in [0, T]$ for which the trajectory exists. The function g is finite, and thus the local boundedness of ∂g implies that $\dot{x}(t) \in \partial g\,(y_0 + KT\mathbb{B})$ is bounded. That shows that any Hamiltonian trajectory originating at (x_0, y_0) can not escape to infinity in any time less or equal T. By freedom of choice of T and (x_0, y_0), the proof is finished. The case of $\operatorname{rge}\partial g$ bounded is symmetrical.

In (c), the subdifferential of the Hamiltonian is

$$\tilde{\partial}_x H(x,y) = A^* y - \partial f(x), \qquad \partial_y H(x,y) = Ax + \partial g(y).$$

By Lemma 4.2, right-hand sides of the above equations have linear growth, so this is a special case of (e).

Similarly for (d): it can be shown that for such a Hamiltonian, the mapping $(x,y) \to \tilde{\partial}_x H(x,y) \times \partial_y H(x,y)$ is piecewise polyhedral (its graph is a union of finite number of polyhedral sets). Combined with local boundedness of the mapping in question, this yields linear growth. A different approach is as follows: Rockafellar and Wolenski [10] showed that

$$\tilde{\partial}_x H(x,y) \times \partial_y H(x,y) = \operatorname{con}\{(w,v) \mid \exists (x_n, y_n) \to (x,y) \text{ with } \nabla H(x_n, y_n) \to (w,v)\}.$$

By the structure of a piecewise linear-quadratic function, $H(x,y)$ is differentiable almost everywhere, and relative to points of differentiability $\nabla H(x,y)$ is linearly bounded. Combining this with the above formula yields the needed linear growth of $\tilde{\partial}_x H(x,y) \times \partial_y H(x,y)$.

Hamiltonian systems in (c), (d) and (e) can be rewritten as $\dot{z}(t) \in F(z(t))$, for some outer semicontinuous, compact-valued mapping F of linear growth. Thus z satisfies $|\dot{z}(t)| \leq m|z(t)| + n$ for some constants $m, n > 0$, and it is well-known that no such z can escape to infinity in finite time. $\square$

The property used in (b) of Theorem 4.3 — $\operatorname{rge}\partial f$ being bounded — can be equivalently expressed in terms of the conjugate function. As ∂f and ∂f^* are mappings inverse to each other, $\operatorname{rge}\partial f = \operatorname{dom}\partial f^*$ where the latter set represents the points where ∂f^* is nonempty. Also, since the relative interior of $\operatorname{dom} f^*$ is a subset of $\operatorname{dom}\partial f$, $\operatorname{dom} f^*$ is bounded if and only if $\operatorname{dom}\partial f^*$ is. Thus, $\operatorname{rge}\partial f$ is bounded if and only if $\operatorname{dom} f^*$ is.

The property in (c) — f being majorized by $c|x|^2 + d$ — also has an equivalent version, in terms of f^*. A direct calculation yields $f(x) \leq c|x|^2 + d$ if and only if $f^*(x) \geq \frac{1}{4c}|x|^2 + d'$ for some constant d'.

Below, we combine the comments made in this section with the results of Theorem 4.3. Recall that $L(x,v)$ stands for the Lagrangian, while $l(x,u)$ is the function involved in the cost expression (5) for control problems. Conditions(a), (b) and (c) are direct translations of corresponding ones in Theorem 4.3. Both (d) and (e) lead to piecewise linear-quadratic Hamiltonians.

Corollary 4.4 *Conclusions of Corollary 3.3 hold under any of the following conditions:*

(a) $L(x,v) = g(v)$ *for some proper lsc convex and coercive function* g.

(b) $L(x,v) = f(x) + g(v)$ *for some proper lsc convex functions* f *and* g*, with* $\operatorname{dom} g$ *bounded.*

(c) $l(x,u) = f(x) + g(u)$ *for some proper lsc convex functions* f *and* g*, with* $f(x) \leq c|x|^2 + d$*,* $g(u) + \delta_U(u) \geq c|u|^2 + d$ *for some constants* $c, d > 0$*.*

(d) $l(x,u)$ *is any function fitting the format of extended linear-quadratic optimal control, as described by Rockafellar [7].*

(e) $L(x,v)$ *satisfies A1, A2, A3 and is piecewise linear-quadratic.*

In particular, any of the above conditions guarantees that the Lagrangian satisfies A1, A2, A3 of Assumption 2.1. Recall that for (c) and (d) the Lagrangian is given by (6)*.*

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