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Generalized Deviations in Risk Analysis

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Abstract General deviation measures are introduced and studied systematically for their potential applications to risk management in areas like portfolio optimization and engineering. Such measures include standard deviation as a special case but need not be symmetric with respect to ups and downs. Their properties are explored with a mind to generating a large assortment of examples and assessing which may exhibit superior behavior. Connections are shown with coherent risk measures in the sense of Artzner, Delbaen, Eber and Heath, when those are applied to the difference between a random variable and its expectation, instead of to the random variable itself. However, the correspondence is only one-to-one when both classes are restricted by properties called lower range dominance, on the one hand, and strict expectation boundedness on the other. Dual characterizations in terms of sets called risk envelopes are fully provided.

Key words risk management – deviation measures – coherent risk measures – value-at-risk – conditional value-at-risk – portfolio optimization – convex analysis

1 Introduction

The uncertainty inherent in a random variable is most commonly measured by its standard deviation, although other indicators, such as mean absolute deviation, have sometimes been utilized instead. In many situations, however, there is interest in treating the extent to which a random variable falls short of its expected value differently from the extent to which it exceeds its expected value. This suggests the study we undertake here, of general deviation measures that resemble the classical ones in some major respects, but need not be symmetric between "ups" and "downs."

The analysis of risky portfolios in finance provides one of the major motivations. How far could one go in replacing standard deviation, around which so much of that subject revolves, by something else, and what advantages, if any might be gained? In what way would the optimality of a portfolio change, and what would it mean in connection with the concepts of capital asset pricing (CAPM)? How would it tie in with techniques that utilize value-at-risk (VaR) and conditional value-at-risk (CVaR)? Of course, risk analysis can go far beyond portfolios, and advances in the subject can be beneficial in other areas of management and engineering. Another potential area of application for general deviation measures could be alternative forms of statistical regression tailored to various specifications.

Our aim is to develop a theory of deviation measures axiomatically, presenting key examples and tracing the relationships with concepts like that of coherent risk measures, which have recently attracted much attention. An accompanying theme is the characterization of deviation measures through duality. This is essential groundwork for understanding problems in which deviation is minimized subject to constraints, and optimality conditions have to be derived, although we will not go into such derivations yet here.

For applications in finance, many researchers have already delved into particular deviations other than standard deviation, in one aspect or another. Markowitz [16] suggested the use of a downside form of standard deviation. Possible advantages of mean absolute deviation and its downside version, most notably in relation to linear programming computations of optimal portfolios, have been explored in [9], [12], [13], [17], [27]. What contrasts here with those contributions is our general axiomatic approach, opening up and elucidating a larger territory. In particular, this effort is intended as support for our work in [24] on building a broader form of basic portfolio theory. Deviation-like axioms have also been considered recently by De Giorgi [6], but from another, more special point of view, having only partial overlap with what we consider essential for a "deviation."

Dual characterizations are well known for coherent risk measures in the sense of Artzner, Delbaen, Eber and Heath [3], but they do not translate fully to our context. Although *some* deviation measures in our sense correspond, under a certain transformation, with *some* coherent risk measures, a direct study of duality with respect to deviation is needed to get a complete picture.

The plan of the paper is as follows. After fixing some notation in the remainder of this section, we go on in Section 2 to define deviation measures axiomatically and develop their dual counterparts, which we call "risk envelopes." They too are described axiomatically. Key issues of geometry and semicontinuity are clarified. Questions are answered about how operations performed on deviation measures affect the associated risk envelopes.

Next, in Section 3, we take up the relationship between deviation measures and coherent risk measures. We demonstrate a one-to-one correspondence between deviation measures and "strictly expectation bounded risk measures," which in general form a class that neither includes, nor is included in, the class of coherent risk measures, although there is a major intersection. By bringing out the distinction between these classes, we hope to turn a brighter light on a part of risk analysis that has seemed to be little appreciated. We show that the deviation measures that correspond to risk measures in the intersection of the two classes are characterized by the heretofore unidentified property of being "lower range dominated."

Also in Section 3, we use the relation to risk measures to pin down a number of valuable examples of deviation measures that arise from CVaR, mixed CVaR, and worst-case CVaR. We provide a rigorous proof of the extent to which mixed CVaR has a "spectral" representation in the sense of Acerbi [1]. (In [10] there is an allusion to such a connection, but without details.)

Finally, in Section 4, we produce another wide array of examples of deviation measures by introducing "basic error functionals" and applying them in two different ways. For these deviation measures we fully determine the risk envelopes and the exact cases in which lower range dominance is enjoyed. That has the side benefit of simultaneously yielding new examples of coherent risk measures.

In line with this agenda, we adopt the setting of a probability space $(\Omega, \mathcal{M}, P)$, where the elements ω of Ω represent future states, or individual scenarios (perhaps just finitely many); $\mathcal{M}$ is the field of measurable subsets of Ω, and P is a probability measure on $\mathcal{M}$. We treat as random variables (r.v.'s) the functions $X : \Omega \to I\!R$ that belong to the linear space $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, P)$, i.e., the (measurable) functions for which the mean and variance

$$\mu(X) = EX = \int_\Omega X(\omega) dP(\omega),$$
$$\sigma^2(X) = E[X - EX]^2 = \int_\Omega [X(\omega) - \mu(X)]^2 dP(\omega),$$

exist (i.e., these integrals are well defined). The inner product in $\mathcal{L}^2(\Omega)$ between any X and Y is $E[XY]$, which can be identified with $\mathrm{covar}(X,Y) + \mu(X)\mu(Y)$. The corresponding norm is $||X|| = \big(E[X^2]\big)^{1/2}$. Convergence $X_k \to X$ of a sequence of r.v.'s X_k for $k = 1, 2, \dots,$ to an r.v. X in the sense of $||X_k - X|| \to 0$ is equivalent to having both $\mu(X_k - X) \to 0$ and $\sigma(X_k - X) \to 0$.

In particular, of course, the space $\mathcal{L}^2(\Omega)$ contains all constant r.v.'s, $X \equiv C$. To assist in working with such r.v.'s, the letter C will always stand for a constant in the real numbers $I\!R$, and when C or a number like 0, 1, or -1 appears in the position of some r.v. X, it will signify the r.v. has that constant value almost surely. Similarly, an inequality like $X \geq C$ or $X \leq C$, or $X_1 \leq X_2$ is to be viewed in the sense of holding almost surely.

The *essential* infimum and supremum of X will be denoted simply by $\inf X$ and $\sup X$.

Our choice of $\mathcal{L}^2(\Omega)$ rather than some other linear space of random variables is dictated by a desire to maintain easy access to tools associated with duality. It is natural too because we want to operate in a context centered on comparisons between nonstandard deviations and an underlying standard deviation. However, this does not prevent us from working with the general $\mathcal{L}^p$ norms

$$||X||_p = \begin{cases} \big(E\big[|X|^p\big]\big)^{1/p} & \text{for } p \in [1,\infty), \\ \sup|X| & \text{for } p = \infty, \end{cases}$$

where $||X||_2 = ||X||$. These expressions are well defined for $X \in \mathcal{L}^2(\Omega)$, except that they could be ∞. Of course, they are sure to be finite for all $X \in \mathcal{L}^2(\Omega)$ when the probability space is *essentially finite* in the sense that the probability measure P takes on only finitely many different values on the field $\mathcal{M}$ of measurable sets, which holds in particular when Ω is a finite set (as comes up in scenario models, for instance). But in the complementary case of the probability space being *essentially infinite*, $||X||_p$ is bound to take on ∞ for some $X \in \mathcal{L}^2(\Omega)$ when $p > 2$.

In dealing with possibly ∞-valued functionals $\mathcal{F}$ on $\mathcal{L}^2(\Omega)$ like $\mathcal{F}(X) = ||X||_p$ or $\mathcal{F}(X) = \sup X$, it will be valuable to have the notion of *lower semicontinuity*. This means that all the subsets of $\mathcal{L}^2(\Omega)$ having the form $\{\, X \,|\, \mathcal{F}(X) \le c\}$ for $c \in I\!R$ are closed, or in other words that

$$X_k \to X \ \text{ with } \ \mathcal{F}(X_k) \le c \ \text{ implies } \ \mathcal{F}(X) \le c. \tag{1}$$

Upper semicontinuity corresponds to the opposite inequalities. The combination of lower semicontinuity with upper semicontinuity is continuity, i.e., the property that $X_k \to X$ entails $\mathcal{F}(X_k) \to \mathcal{F}(X)$.

For $\mathcal{F}(X) = ||X||_p$ we have continuity when $p \in [1,2]$ or when the probability space is essentially finite, but otherwise only lower semicontinuity. For $\mathcal{F}(X) = \sup X$ we have lower semicontinuity, whereas for $\mathcal{F}(X) = \inf X$ we have upper semicontinuity. Later we will additionally be involved, for instance, with expressions like $||aX_+ + bX_-||_p$ for $a \ge 0$ and $b \ge 0$, where the notation is used that

$$X = X_+ - X_- \ \text{ with } \ X_+ = \max\{0, X\},\ X_- = \max\{0, -X\}. \tag{2}$$

Also of fundamental interest to us below will be certain other quantities defined in terms of the distribution function F_X of X, namely

$$F_X(z) = P\{X \le z\}.$$

The *value-at-risk* of X for any $\alpha \in (0,1)$ is

$$\mathrm{VaR}_\alpha(X) = -\inf\{\, z \,|\, F_X(z) > \alpha\}. \tag{3}$$

The corresponding *conditional value-at-risk* is given by

$$\begin{aligned} \mathrm{CVaR}_\alpha(X) &= -E[X|X \le -\mathrm{VaR}_\alpha(X)] \\ &\qquad \text{when } F_X \text{ is continuous at } -\mathrm{VaR}_\alpha(X), \end{aligned} \tag{4}$$

but requires a more subtle definition to be able to handle possible discontinuities. One approach, first followed by Pflug [18] is to rely even for discontinuous F_X on the minimization formula we developed for the case of continuous F_X in [21], where the term "conditional value-at-risk" was coined:

$$\mathrm{CVaR}_\alpha(X) = \min_C \left\{ C + \alpha^{-1} E[X + C]_- \right\}, \tag{5}$$

We proved in [22] that, for possibly discontinuous F_X, this approach coordinates with the original one by being equivalent to replacing the conditional expectation in (4) by the "generalized α-tail" expectation

$$\begin{aligned} &\mathrm{CVaR}_\alpha(X) = -\int_{-\infty}^{\infty} z dF_X^\alpha(z), \\ &\qquad \text{where } F_X^\alpha(z) = \begin{cases} \alpha^{-1} F_X(z) & \text{when } F_X(z) < \alpha, \\ 1 & \text{when } F_X(z) \ge \alpha. \end{cases} \end{aligned} \tag{6}$$

A different track was followed by Acerbi [1], in observing that CVaR can equivalently be expressed as a VaR average:

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\beta(X) d\beta. \tag{7}$$

(Acerbi began by using the expression on the right as the definition of a functional he called "expected shortfall," but subsequently realized its connection with CVaR; cf. also [2], [25]. Because of (7), Föllmer and Schied [10] have proposed yet another name: "average value-at-risk.") Limit analysis in these formulas as α tends to 0 or 1 leads to the conventions that

$$\mathrm{CVaR}_0(X) = -\inf X, \qquad \mathrm{CVaR}_1(X) = -EX. \tag{8}$$

2 Deviation Measures

In focusing on generalized deviations, our goal is to investigate functionals $\mathcal{D}$ on $\mathcal{L}^2(\Omega)$ that obey certain axioms taken from the properties of standard deviation.

Definition 1 (general deviation measures). *By a* deviation measure *will be meant any functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ satisfying*

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ *for all X and constants C,*
(D2) $\mathcal{D}(0) = 0$, *and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,*
(D3) $\mathcal{D}(X + X') \le \mathcal{D}(X) + \mathcal{D}(X')$ *for all X and X',*
(D4) $\mathcal{D}(X) \ge 0$ *for all X, with $\mathcal{D}(X) > 0$ for nonconstant X.*

Under these axioms, $\mathcal{D}(X)$ depends only on $X - EX$ (from the case of D1 where $C = -EX$), and it vanishes only if $X - EX = 0$ (as seen from D4 with $X - EX$ in place of X; note that the property in D4 for constant X already follows from D1 and D2). This captures the idea that $\mathcal{D}$ measures the degree of *uncertainty* in X. Indeed, $\mathcal{D}$ acts as a sort of norm on the "pure uncertainty" subspace of $\mathcal{L}^2(\Omega)$ consisting of all X with $EX = 0$, except that the symmetry required by the definition of a norm — the additional condition that $\mathcal{D}(-X) = \mathcal{D}(X)$ for all X — may be absent. Note, however, that if $\mathcal{D}$ is a deviation measure then so too are its *reflection* $\tilde{\mathcal{D}}$ and its *symmetrization* $\tilde{\tilde{\mathcal{D}}}$, given by

$$\tilde{\mathcal{D}}(X) = \mathcal{D}(-X), \qquad \tilde{\tilde{\mathcal{D}}}(X) = \tfrac{1}{2}[\mathcal{D}(X) + \tilde{\mathcal{D}}(X)].$$

Axiom D2 is *positive homogeneity*. The combination of D2 with D3 is the property known as *sublinearity*. It implies that $\mathcal{D}$ is a *convex* functional on $\mathcal{L}^2(\Omega)$.

Definition 1 allows a deviation measure $\mathcal{D}$ to have $\mathcal{D}(X) = \infty$ for some r.v.'s X. When this is excluded, we have a *finite* deviation measure. In D2 and D3, infinite values are to be handled in the obvious way: $\alpha + \infty = \infty$ for any $\alpha \in (-\infty, \infty]$, and $\lambda\infty = \infty$ for any $\lambda > 0$, whereas $0\infty = 0$. (These conventions are standard in convex analysis [19].)

Example 1 (standard deviation and semideviations). *For* $\mathcal{D}(X) = \sigma(X) = ||X - EX||$, *all the properties D1, D2, D3 and D4 hold. This deviation measure is symmetric. The standard upper and lower* semi*deviations* σ_+ *and* σ_-, *where*

$$\sigma_+(X) = ||[X - EX]_+||, \qquad \sigma_-(X) = ||[X - EX]_-||,$$

satisfy all the conditions as well, but are not symmetric. All three are finite and continuous.

Example 2 (range-based deviations). *A deviation measure that is lower semicontinuous is defined by*

$$\mathcal{D}(X) = EX - \inf X = \sup[EX - X].$$

It gives the size of the lower range *of* X. *Its reflection,*

$$\tilde{\mathcal{D}}(X) = \sup X - EX = \sup[X - EX],$$

giving the size of the upper range *of* X, *is likewise a deviation measure that is lower semicontinuous, and so too is their sum, giving the* full range *of* X,

$$\mathcal{D}(X) + \tilde{\mathcal{D}}(X) = \sup X - \inf X.$$

Unless the probability space is essentially finite, these deviation measures can take on ∞ *and need not be continuous.*

It is interesting to note, before proceeding, that axiom D1 in Definition 1 could be replaced by a seemingly much weaker property, complementary to D4 (subsuming its equality part), without changing the class of functionals $\mathcal{D}$ that is described.

Proposition 1 (simplified axioms). *Under D3, the requirements in D1 and D2 are implied by the simpler conditions that*
(D1′) $\mathcal{D}(C) = 0$ *for constants* C,
(D2′) $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all* X *and all* $\lambda > 0$.

Proof Obviously D1′ furnishes the condition $\mathcal{D}(0) = 0$ needed to convert D2′ into D2. Then for any X and constant C we have through D3 that $\mathcal{D}(X) = \mathcal{D}(X+C-C) \le \mathcal{D}(X+C)+\mathcal{D}(-C) = \mathcal{D}(X+C) \le \mathcal{D}(X)+\mathcal{D}(C) = \mathcal{D}(X)$. This chain yields $\mathcal{D}(X) \le \mathcal{D}(X+C) \le \mathcal{D}(X)$, so we have D1. □

Proposition 2 (continuity of deviation measures). *A finite deviation measure* $\mathcal{D}$ *on* $\mathcal{L}^2(\Omega)$ *that is lower semicontinuous must be continuous.*

Proof In view of deviation measures being convex functionals in particular, this merely specializes a result that is known for any convex functional on a Banach space such as $\mathcal{L}^2(\Omega)$; see Rockafellar [20, Corollary 8B]. □

Another property of a deviation measure will sometimes have a major role along with D1–D4 in the theory we are building up, most notably in connection with duality.

Definition 2 (lower range dominance). *A deviation measure* $\mathcal{D}$ *will be called* lower range dominated *when it satisfies*
(D5) $\mathcal{D}(X) \le EX - \inf X$ *for all* X.

In singling out the downside of X for special attention, the lower range dominance property D5 is clearly directed toward concerns about outcomes of X possibly falling short of EX. This will be seen below to support a probabilistic interpretation of $\mathcal{D}$ in terms of downside risk. In the deviation measure examples so far, lower range dominance holds of course for $\mathcal{D}(X) = EX - \inf X$ but not for $\mathcal{D}(X) = \sup X - EX$; it holds for $\mathcal{D}(X) = \sigma_-(X)$ but not for $\mathcal{D}(X) = \sigma(X)$ or $\mathcal{D}(X) = \sigma_+(X)$.

Theorem 1 (dual characterization of deviation measures). *A functional* $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty)$ *is a lower semicontinuous deviation measure if and only if it has a representation of the form*

$$\mathcal{D}(X) = EX - \inf_{Q \in \mathcal{Q}} E[XQ] \tag{9}$$

in terms of a subset $\mathcal{Q}$ *of* $\mathcal{L}^2(\Omega)$ *such that*
(Q1) $\mathcal{Q}$ *is nonempty, closed and convex,*
(Q2) *for every nonconstant* X *there is some* $Q \in \mathcal{Q}$ *with* $E[XQ] < EX$,
(Q3) $EQ = 1$ *for all* $Q \in \mathcal{Q}$.

In this representation, $\mathcal{Q}$ is uniquely determined by $\mathcal{D}$ through

$$\mathcal{Q} = \left\{ Q \,\middle|\, \mathcal{D}(X) \geq EX - E[XQ] \text{ for all } X \right\}, \tag{10}$$

and the finiteness of $\mathcal{D}$ is equivalent to the boundedness of $\mathcal{Q}$. Furthermore, $\mathcal{D}$ is lower range dominated if and only if $\mathcal{Q}$ has the additional property that

(Q4) $Q \geq 0$ *for all* $Q \in \mathcal{Q}$.

Proof It is well known in convex analysis (cf. [5], [19]) that the lower semicontinuous functionals $\mathcal{D}$ on $\mathcal{L}^2(\Omega)$ satisfying D2 and D3 are the support functions of the nonempty closed convex subsets $\mathcal{Y}$ of $\mathcal{L}^2(\Omega)$. In our present notation, this refers to a representation of the type

$$\mathcal{D}(X) = \sup_{Y \in \mathcal{Y}} E[XY] \tag{11}$$

in terms of such a set $\mathcal{Y}$, from which $\mathcal{Y}$ can be recovered by

$$\mathcal{Y} = \left\{ Y \,\middle|\, \mathcal{D}(Y) \geq E[XY] \text{ for all } X \right\}. \tag{12}$$

For $\mathcal{D}$ to satisfy axiom D1, the extra condition on $\mathcal{Y}$ is that $EY = 0$ for all $Y \in \mathcal{Y}$. These properties of $\mathcal{Y}$ translate into conditions Q1 and Q3 on $\mathcal{Q}$ under the arrangement that $\mathcal{Q} = \left\{ Q = 1 - Y \,\middle|\, Y \in \mathcal{Y} \right\}$, or conversely $\mathcal{Y} = \left\{ Y = 1 - Q \,\middle|\, Q \in \mathcal{Q} \right\}$. Then $E[XY] = E[X(1-Q)] = EX - E[XQ]$, whereby the formulas in (11) and (12) translate into (9) and (10). Axiom D4 is equivalent then to Q2.

In (11), the finiteness of $\mathcal{D}$ means that, for every X, the continuous linear functional on $\mathcal{L}^2$ associated with X is bounded from above on the set $\mathcal{Y}$. In functional analysis, that property is known to correspond by the Banach-Steinhaus theorem to $\mathcal{Y}$ be bounded (with respect to the norm). The boundedness of $\mathcal{Y}$ is equivalent to the boundedness of $\mathcal{Q}$ under the specified transformation between them.

As for the claim about D5, we have $\mathcal{D}(X) \leq EX - \inf X$ under formula (9) if and only if $EX - \inf_{Q \in \mathcal{Q}} E[XQ] \leq EX - \inf X$, or in other words $\inf_{Q \in \mathcal{Q}} E[XQ] \geq \inf X$. This holds for every $X \in \mathcal{L}^2(\Omega)$ when $\mathcal{Q}$ satisfies Q4, but it fails otherwise, since if $\mathcal{Q}$ contains a Q_0 with $\inf Q_0 < 0$ we can take $X_0 = -\min\{Q_0, 0\}$ and get $E[X_0 Q_0] < 0$ even though $\inf X_0 \geq 0$. □

Definition 3 (risk envelopes). *The uniquely determined set $\mathcal{Q}$ in Theorem 1 will be called the* risk envelope *corresponding to the lower semicontinuous deviation measure $\mathcal{D}$.*

The risk envelope $\mathcal{Q}$ can be interpreted most clearly when $\mathcal{D}$ is lower range dominated, since that corresponds by Theorem 1 to Q4 being satisfied along with Q1–Q3. Through Q3 and Q4, each $Q \in \mathcal{Q}$ may be regarded as the *density relative to P of some probability measure P' on Ω*:

$$Q = \frac{dP'}{dP}, \qquad P' = QP, \tag{13}$$

with expectation functional

$$E_{P'}(X) = \int_{\Omega} X(\omega) d\mathcal{P}'(\omega) = \int_{\Omega} X(\omega) Q(\omega) d\mathcal{P}(\omega) = E[XQ]. \tag{14}$$

We can think of these probability measures as designating alternatives to the underlying probability measure P which a modeler might wish to take into account. The difference

$$EX - E[XQ] = EX - E_{P'}X$$

assesses how much worse the expectation of X might be under P' than under P. In this sense, $\mathcal{D}$ performs a worst-case analysis over the probability alternatives that have been selected. Observe, in the same vein, that from (9) we get

$$\mathcal{D}(X) + \tilde{\mathcal{D}}(X) = \sup_{Q \in \mathcal{Q}} E[XQ] - \inf_{Q \in \mathcal{Q}} E[XQ],$$

which assesses the difference between the best and worst possible expectations for X relative to the risk envelope $\mathcal{Q}$.

A richer range of examples will emerge in due course, but the most extreme case of a lower range dominated deviation measure can immediately be seen as associated with the largest possible set $\mathcal{Q}$ satisfying Q1–Q4:

$$\mathcal{D}(X) = EX - \inf X \text{ corresponds to } \mathcal{Q} = \left\{ Q \,\middle|\, Q \geq 0,\, EQ = 1 \right\}. \tag{15}$$

This deviation measure thus performs a worst-case analysis of $EX - E_{P'}X$ by looking at *all* the possible alternatives P' to P in this scheme. In contrast,

$$\mathcal{D}(X) = \sigma(X) \text{ corresponds to } \mathcal{Q} = \left\{ Q \,\middle|\, \sigma(Q-1) \leq 1,\, EQ = 1 \right\}, \tag{16}$$

as can be seen from the fact that $\sigma(X) = \sup \left\{ E[XY] \,\middle|\, \sigma(Y) \leq 1,\, EY = 0 \right\}$ by way of the change of variables $Y = Q - 1$ utilized in the proof of Theorem 1. In this case the elements Q of $\mathcal{Q}$ fail to necessarily satisfy $Q \geq 0$, as is consonant with the lack of lower range dominance of $\mathcal{D} = \sigma$. Identification of the risk envelopes associated with $\mathcal{D} = \sigma_+$ or $\mathcal{D} = \sigma_-$ will have to wait until later (in Example 6).

In the framework of (13) and (14) the probability measure P itself corresponds to the density function $Q = 1$. From (10), every risk envelope $\mathcal{Q}$ has to have this constant density as one of its elements. Thus, in the probabilistic interpretation when $\mathcal{Q}$, associated with a lower semicontinuous deviation measure $\mathcal{D}$, satisfies Q4, the specified collection of probability measures P' must in particular include P. In fact, through Q2 it must constitute a kind of neighborhood of P, in the sense explained next.

Proposition 3 (risk envelope geometry). *Conditions Q1–Q3 mean that $\mathcal{Q}$ is a closed, convex subset of the closed hyperplane $\mathcal{H}_0 = \left\{ Q \,\middle|\, EQ = 1 \right\}$ in $\mathcal{L}^2(\Omega)$ that contains the constant 1 in its quasi-interior relative to $\mathcal{H}_0$; in other words, $1 \in \mathcal{Q}$ but every closed hyperplane $\mathcal{H} \neq \mathcal{H}_0$ containing 1 has elements of $\mathcal{Q}$ in both of its associated open half-spaces.*

Proof We already know from Theorem 1, via the observation just made, that risk envelopes $\mathcal{Q}$ comprise a class of closed, convex sets $\mathcal{Q} \subset \mathcal{H}_0$ containing 1. These properties correspond through the representation in (9) to $\mathcal{D}$ being a nonnegative, lower semicontinuous functional that satisfies D1, D2 and D3. The issue is what additional feature of $\mathcal{Q}$ makes the inequalities in D4 be strict.

For this, we recall that the closed hyperplanes $\mathcal{H}$ in $\mathcal{L}^2(\Omega)$ are the subsets expressible in the form $\mathcal{H} = \{ Q \,|\, E[XQ] = c \}$ for some $X \neq 0$ and c. Having $1 \in \mathcal{H}$ amounts to having $c = EX$, whereas having $\mathcal{H} \neq \mathcal{H}_0$ amounts to X being nonconstant. Axiom D4, in requiring for every nonconstant X the existence of some $Q \in \mathcal{Q}$ for which $E[XQ] < EX$, also requires, through the application to $X' = -X$, the existence for each such X of some $Q \in \mathcal{Q}$ for which $E[XQ] > EX$. Thus it is equivalent requiring, for each hyperplane $\mathcal{H} = \{ Q \,|\, E[XQ] = EX \}$ with X nonconstant, the existence of elements of $\mathcal{Q}$ lying on both sides of $\mathcal{H}$. $\square$

It is worth noting that, although Q1 is essential in getting the one-to-one correspondence between functionals $\mathcal{D}$ and sets $\mathcal{Q}$ in Theorem 1, formula (9) would still define a lower semicontinuous deviation measure $\mathcal{D}$ if $\mathcal{Q}$ merely satisfied Q2 and Q3. The risk envelope associated with $\mathcal{D}$ would then be the closure of the convex hull of $\mathcal{Q}$.

Proposition 4 (operations on deviation measures). *Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be deviation measures for which the associated risk envelopes are $\mathcal{Q}_1, \dots, \mathcal{Q}_m$.*

(a) *If $\mathcal{D}(X) = \lambda_1 \mathcal{D}_1(X) + \cdots + \lambda_m \mathcal{D}_m(X)$ with coefficients $\lambda_k > 0$, $\lambda_1 + \cdots + \lambda_m = 1$ then $\mathcal{D}$ is another deviation measure. It is lower range dominated if each $\mathcal{D}_k$ is lower range dominated. The corresponding risk envelope is*

$$\mathcal{Q} = \text{ closure of } \lambda_1 \mathcal{Q}_1 + \cdots + \lambda_m \mathcal{Q}_m, \tag{17}$$

where set is closed (and the closure operation thus superfluous) when all but perhaps one of the $\mathcal{D}_k$'s is finite.

(b) *If $\mathcal{D} = \max\{ \mathcal{D}_1(X), \dots, \mathcal{D}_m(X) \}$, then $\mathcal{D}$ is another deviation measure. It is lower range dominated if each $\mathcal{D}_k$ is lower range dominated. The corresponding risk envelope is*

$$\mathcal{Q} = \text{closure of the convex hull of } \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_m, \tag{18}$$

where convex hull is itself already closed (and the closure operation thus superfluous) when all of the $\mathcal{D}_k$'s are finite.

(c) *If $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$ and a deviation measure $\mathcal{D}_0$ with risk envelope $\mathcal{Q}_0$, then $\mathcal{D}$ is a deviation measure with risk envelope*

$$\mathcal{Q} = \left\{ Q \,\middle|\, Q = (1-\lambda) + \lambda Q_0 \text{ for some } Q_0 \in \mathcal{Q}_0 \right\}. \tag{19}$$

Proof The claims about $\mathcal{D}$ are in each case elementary consequences of the definitions. The risk envelope formulas immediately come out of the uniqueness of the representation in Theorem 1 by observing that the $\mathcal{Q}$ satisfies Q1, Q2 and Q3, and yields the designated $\mathcal{D}$ through (9).

When $\mathcal{D}_k$ is finite, the set $\mathcal{Q}_k$ is bounded by Theorem 1. Since $\mathcal{Q}_k$ is closed and convex, it must therefore be weakly compact, inasmuch as closedness coincides with weak closedness for convex sets (due to the separation principle), and weakly closed, bounded, convex subsets of $\mathcal{L}^2(\Omega)$ are always weakly compact. Any multiple $\lambda_k\mathcal{Q}_k$ is then likewise convex and weakly compact. The sum of finitely many weakly closed sets, all but perhaps one of which is weakly compact, is known to be another weakly closed set, and convexity is preserved under addition as well. On the other hand, the union of any finite collection of weakly compact sets is again weakly compact, and its convex hull is therefore weakly compact. □

3 Relation to Coherent Risk Measures

Although deviation measures are designed for applications to problems involving risk, they are not "risk measures" in the sense proposed by Artzner, Delbaen, Eber and Heath [3] in their landmark paper. The connection between deviation measures and risk measures is close, but a crucial distinction must be appreciated clearly. Instead of measuring the uncertainty in X, in the sense of nonconstancy, a risk measure evaluates the "overall seriousness of possible losses" associated with X, where *a loss is an outcome below* 0, in contrast to a gain, which is an outcome above 0. In applying a risk measure, this orientation is crucial; if the concern is over the extent to which a given r.v. X might have outcomes $X(\omega)$ that drop below a threshold C, one needs to replace X by $X - C$.

The touchstone for this section is the following concept of "coherent" risk measure, adopted essentially from [3].

Definition 4 (coherent risk measures). *By a* coherent risk measure *will be meant any functional* $\mathcal{R} : \mathcal{L}^2(\Omega) \to (-\infty, \infty]$ *satisfying*
(R1) $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ *for all* X *and constants* C*,*
(R2) $\mathcal{R}(0) = 0$*, and* $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ *for all* X *and all* $\lambda > 0$*,*
(R3) $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ *for all* X *and* X'*,*
(R4) $\mathcal{R}(X) \leq \mathcal{R}(X')$ *when* $X \geq X'$*.*

Once more, R2 is *positive homogeneity*, R3 is *subadditivity*, and the combination of R2 and R3 is *sublinearity*, implying *convexity.* Again too, $\mathcal{R}(X) = \infty$ is allowed (under the arithmetic conventions already noted, which now affect R1 as well as R2 and R3). Property R4 is *monotonicity.*

Interestingly, by an argument along the lines of Proposition 1, axioms R1 and R2 could be replaced equivalently by the simpler conditions

(R1') $\mathcal{R}(C) = -C$ for constants C,
(R2') $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ for all X and $\lambda > 0$.

Actually, only *finite* risk measures were considered originally in [3], and axiom R1 was slightly different, involving investment in a reference r.v.;

moreover the state space Ω was assumed to be a finite set. Extensions to the present context and formulation were made, however, in follow-up work by Delbaen [7], [8]. That format was adopted also in the presentation of the subject furnished by Föllmer and Schied in [10].

The rationale behind coherent risk measures was very well argued in [3], but how are they related to the deviation measures we have introduced? Although the R2 and R3 for a risk measure agree with the D2 and D3 for a deviation measure, R1 and D1 are entirely different, in fact mutually incompatible — no functional on $\mathcal{L}^2(\Omega)$ can satisfy both R1 and D1. Despite this, there is a simple relationship between the two notions. In order to explain it clearly, however, we have to consider also a slightly different concept of risk measure.

Definition 5 (strictly expectation bounded risk measures). *By a* strictly expectation bounded risk measure *will be meant any functional* $\mathcal{R} : \mathcal{L}^2(\Omega) \to (-\infty, \infty]$ *satisfying the axioms R1, R2 and R3 (but not necessarily R4) of Definition 4, along with*

(R5) $\mathcal{R}(X) > E[-X]$ *for all nonconstant* X*.*

When all the axioms R1, R2, R3, R4 and R5 are satisfied, we speak of course of a *coherent, strictly expectation bounded risk measure.*

The strict inequality in R5 is the key, since R1 already guarantees that $\mathcal{R}(X) = E[-X]$ when X is a constant r.v. This is why we speak of "strict expectation boundedness". The version with a weak inequality in place of the strict inequality would accordingly just be "expectation boundedness."[1] Neither version was contemplated in Artzner et al. in [3], where no reference probability distribution was assigned to the state space Ω.

Theorem 2 (strictly expectation bounded risk measures). *Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations*

(a) $\mathcal{D}(X) = \mathcal{R}(X - EX)$,

(b) $\mathcal{R}(X) = E[-X] + \mathcal{D}(X)$.

Specifically, if $\mathcal{R}$ is a strictly expectation bounded risk measure and $\mathcal{D}$ is defined by (a), then $\mathcal{D}$ is a deviation measure that yields back $\mathcal{R}$ through (b). On the other hand, if $\mathcal{D}$ is any deviation measure and $\mathcal{R}$ is defined by (b), then $\mathcal{R}$ is a strictly expectation bounded risk measure that yields back $\mathcal{D}$ through (a). In this correspondence, the risk envelope associated with $\mathcal{D}$ (in the presence of lower semicontinuity) furnishes for $\mathcal{R}$ the representation

$$\mathcal{R}(X) = -\inf_{Q \in \mathcal{Q}} E[XQ]. \tag{20}$$

[1] In our working paper [23], from which some of the present paper is derived, we used this simpler term for the strict version, but have since thought better of it. The weak version has recently been called "risk relevance" in [14], and this could be a still better term, but referring to a "risk relevant risk measure" seems awkward.

Furthermore, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is lower range dominated. Thus, lower range bounded deviation measures (satisfying D1, D2, D3, D4, D5) correspond one-to-one with coherent, strictly expectation bounded risk measures (satisfying R1, R2, R3, R4, R5) under (a) and (b).

Proof In passing from $\mathcal{R}$ to $\mathcal{D}$ by way of (a), D1 is immediate (since $\mathcal{R}([X+C]-E[X+C]) = \mathcal{R}(X-EX)$; axioms D2 and D3 follow from R2 and R3, while D4 comes out of R5. Because R1 implies $\mathcal{R}(X-EX) = \mathcal{R}(X)+EX$, we also get (b). On the other hand, in passing from $\mathcal{D}$ to $\mathcal{R}$ by way of (b), the properties in R2 and R3 are immediate from D2 and D3. We have

$$\mathcal{R}(X+C) = \mathcal{D}(X+C) - E[X+C] = \mathcal{D}(X) - EX - C = \mathcal{R}(X) - C$$

via D1, so that $\mathcal{R}$ satisfies R1. That also shows that (a) will give back $\mathcal{D}$. Axiom D4 clearly corresponds to R5 through (b).

The risk envelope representation of $\mathcal{D}$ in Theorem 1 clearly translates through (a) and (b) into the representation of $\mathcal{R}$ in (20).

In the presence of R2 and R3, the monotonicity property R4 implies that $\mathcal{R}(X) \leq 0$ when $X \geq 0$ and in fact it is equivalent to that seemingly weaker property, since if $X \geq X'$ we have $X = X' + X''$ for $X'' \geq 0$ and consequently $\mathcal{R}(X) \leq \mathcal{R}(X') + \mathcal{R}(X'')$, hence $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $\mathcal{R}(X'') \leq 0$. In the (a)(b) correspondence, the condition that $\mathcal{R}(X) \leq 0$ when $X \geq 0$ comes out as requiring $\mathcal{D}(X) \leq EX$ when $X \geq 0$. That is definitely true under D5, but in turn it actually guarantees D5. Indeed, to say that $\mathcal{D}(X) \leq EX - \inf X$ is to say that $\mathcal{D}(X) \leq EX - C$ whenever $X \geq C$, and through D1 that is the same as having $\mathcal{D}(X-C) \leq E[X-C]$ whenever $X - C \geq 0$. $\square$

Under the (a)(b) correspondence in Theorem 1, $\mathcal{D}$ can be called the *deviation measure associated with* $\mathcal{R}$, whereas $\mathcal{R}$ can be called the *risk measure associated with* $\mathcal{D}$. Note that $\mathcal{R}$ is finite if and only if $\mathcal{D}$ is finite. Likewise, $\mathcal{R}$ is lower semicontinuous, or for that matter continuous, if and only if $\mathcal{D}$ has that property.

The degree to which strict expectation boundedness imposes a restriction on a risk measure $\mathcal{R}$ can readily be understood through the risk envelope characterization in Theorem 2: the associated $\mathcal{Q}$ must satisfy Q1, Q2 and Q3 of Theorem 1, which have been identified with the geometry in Proposition 3. Plain expectation boundedness, with the inequality in R5 no longer strict, would relax the "quasi-interior" requirement in Proposition 3 to simply having the constant 1 belong to $\mathcal{Q}$. In contrast, the representation of type (20) that characterizes general coherent risk measures $\mathcal{R}$ centers on sets $\mathcal{Q}$ that satisfy Q1, Q3 and Q4, with no mention of Q2; cf. [3], [10].[2] Since it is easy to come up with examples of sets $\mathcal{Q}$ satisfying Q1, Q3 and Q4 that contain 1 but not in their quasi-interior, or do not contain 1 at all,

[2] In those works, no name was given to the set appearing in this formula. The term "risk envelope" was introduced in our working paper [23].

it is evident that coherent risk measures may or may not be expectation bounded, and if so, need not, in general, be strictly expectation bounded.

For instance, the functional $\mathcal{R}(X) = E[-X]$ is a coherent risk measure which is expectation bounded but not strictly expectation bounded, whereas the functional $\mathcal{R}(X) = -E[XQ_0]$ for a fixed $Q_0 \geq 0$ with $Q \neq 1$ but $EQ_0 = 1$ is a coherent risk measure which is not even expectation bounded. On the other hand, it has been shown in [10, p. 118] that, when the probability space is atomless, $\mathcal{R}(X) \geq E[-X]$ must hold if $\mathcal{R}$ depends only on the distribution function F_X of X and has a monotone convergence property called "continuity from above."

There are echoes of the relationships in Theorem 2 in [13], [14], but those researchers focus on several examples rather than axiomatic definitions and proofs.[3] In [6], certain functionals are considered that have the form of $\mathcal{D}$ in (a) of Theorem 2 for an $\mathcal{R}$ that is not strictly expectation bounded, but on the other hand is a coherent risk measure which furthermore is required to be monotone with respect to second-order stochastic dominance. As seen in Theorem 2, such functionals $\mathcal{D}$ fail in general to be nonnegative, as demanded by our axiom D4, and thus are not deviation measures in our sense, nor even when they happen to be nonnegative, would they include all our deviation measures.

For coherent risk measures, the representation in (20) can be interpreted like the one for deviation measures in (9). The elements $Q \in \mathcal{Q}$ can be viewed as the densities with respect to P of alternative probability measures P' as in (13) and (14). The quantity $-E[XQ] = E_{P'}[-X]$ then designates the loss under that alternative, and $\mathcal{R}$ identifies the worst possible loss with respect to the specified class of alternatives; cf. [3], [10, Proposition 4.11].

We proceed now to bring out further examples of deviation measures by way of examples of coherent risk measures and some elementary operations.

Example 3 (CVaR-deviation). *For any $\alpha \in (0,1)$, the functional*

$$\mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX) \tag{21}$$

is a continuous, lower range dominated deviation measure on $\mathcal{L}^2(\Omega)$ for which the risk envelope is

$$\mathcal{Q} = \left\{ Q \,\middle|\, 0 \leq Q \leq \alpha^{-1},\, EQ = 1 \right\}. \tag{22}$$

It corresponds to the coherent, strictly expectation bounded risk measure

$$\mathcal{R}(X) = \mathrm{CVaR}_\alpha(X). \tag{23}$$

Detail. This $\mathcal{R}$ is already known to furnish a finite, coherent risk measure, cf. [22], and $\mathcal{D}$ corresponds to it as in (a) of Theorem 2. A representation for $\mathcal{R}$ of type (21) has already been developed in terms of the convex set $\mathcal{Q}$ in (22), cf. [10, Theorem 4.39]. This set $\mathcal{Q}$ enjoys Q1, Q2, Q3 and Q4 of Theorem 1, so the claims about $\mathcal{D}$ are correct in view of the facts in Theorem 2. □

[3] Their terminology has $-\mathcal{R}$ as a "safety measure" and $\mathcal{D}$ as a "risk measure."

Example 4 (mixed CVaR-deviation). *For a weighting measure λ on $(0,1)$ (nonnegative with total measure 1), the functional*

$$\mathcal{D}(X) = \int_0^1 \mathrm{CVaR}_\alpha(X - EX) d\lambda(\alpha) \tag{24}$$

is a finite, lower range bounded deviation measure, and the functional

$$\mathcal{R}(X) = \int_0^1 \mathrm{CVaR}_\alpha(X) d\lambda(\alpha) \tag{25}$$

is a finite, coherent, strictly expectation bounded risk measure. In particular, in taking λ to be comprised of atoms having weights λ_i at points α_i for $i = 1, \dots, m$, with $\lambda_i > 0$ and $\lambda_1 + \dots + \lambda_m = 1$, one gets

$$\mathcal{D}(X) = \lambda_1 \mathrm{CVaR}_{\alpha_1}(X - EX) + \dots + \lambda_m \mathrm{CVaR}_{\alpha_m}(X - EX), \tag{26}$$

$$\mathcal{R}(X) = \lambda_1 \mathrm{CVaR}_{\alpha_1}(X) + \dots + \lambda_m \mathrm{CVaR}_{\alpha_m}(X). \tag{27}$$

Detail. This is evident from Example 3 and the way that properties D1–D5 are propagated by the integration. □

Example 5 (worst-case mixed-CVaR deviation). *For any collection Λ of weighting measures λ on $(0,1)$, the functional*

$$\mathcal{D}(X) = \sup_{\lambda \in \Lambda} \int_0^1 \mathrm{CVaR}_\alpha(X - EX) d\lambda(\alpha) \tag{28}$$

is a lower range bounded deviation measure; correspondingly the functional

$$\mathcal{R}(X) = \sup_{\lambda \in \Lambda} \int_0^1 \mathrm{CVaR}_\alpha(X) d\lambda(\alpha) \tag{29}$$

is a coherent, strictly expectation bounded risk measure. As a special case, the mixed CVaR *elements in the maximization can be taken to be simple* CVaR *elements:*

$$\mathcal{D}(X) = \max\big\{ \mathrm{CVaR}_{\alpha_1}(X - EX), \dots, \mathrm{CVaR}_{\alpha_m}(X - EX) \big\}, \tag{30}$$

$$\mathcal{R}(X) = \max\big\{ \mathrm{CVaR}_{\alpha_1}(X), \dots, \mathrm{CVaR}_{\alpha_m}(X) \big\}. \tag{31}$$

Detail. Again, this follows out of Example 3 and the observation that the properties in question are preserved under the supremum operation. □

Risk measures of the special worst-case CVaR type in (31) have recently been utilized in [14].

Observe that all the measures in Examples 4, 5 and 6 are *distribution-independent*; $\mathcal{D}(X)$ and $\mathcal{R}(X)$ depend only on the distribution function F_X. For additional insight into their importance, we provide a "spectral" characterization of them which relates to the contributions of Acerbi [1].

Proposition 5 (spectral representation for mixed-CVaR). *As long as the weighting measure λ satisfies $\int_0^1 p^{-1} d\lambda(p) < \infty$, the functionals $\mathcal{D}$ and $\mathcal{R}$ in Example 4 can equivalently be expressed in the form*

$$\mathcal{D}(X) = \int_0^1 \mathrm{VaR}_\alpha(X - EX)\varphi(\alpha)d\alpha, \tag{32}$$

$$\mathcal{R}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\varphi(\alpha)d\alpha, \tag{33}$$

for the function φ defined on $(0,1)$ by

$$\varphi(\alpha) = \int_{[\alpha,1)} p^{-1} d\lambda(p). \tag{34}$$

This function φ is left-continuous and nonincreasing with $\varphi(0^+) < \infty$, $\varphi(1^-) = 0$ and $\int_0^1 \varphi(\alpha)d\alpha = 1$. Conversely, any function φ having those properties arises from a unique choice of λ as described.

Proof The formula for φ yields immediately the fact that φ is left-continuous and nonincreasing on $(0,1)$ with boundary limits $\varphi(1^-) = 0$ and $\varphi(0^+) < \infty$; the finiteness of $\varphi(0^+)$ comes from the assumption that $\int_0^1 p^{-1} d\lambda(p) < \infty$. The Radon-Stieltjes measure $d\varphi$ derived from φ relates to the weighting measure λ by $d\varphi(\alpha) = -\alpha^{-1} d\lambda(\alpha)$.

Conversely, for any function φ that is left-continuous and nonincreasing on $(0,1)$ with $\varphi(1^-) = 0$ and $\varphi(0^+) < \infty$, consider the measure λ on $(0,1)$ defined by $d\lambda(\alpha) = -\alpha d\varphi(\alpha)$. This is nonnegative with

$$\int_0^1 p^{-1} d\lambda(p) = -\int_0^1 p^{-1} p d\varphi(p) = -[\varphi(1^-) - \varphi(0^+)] = \varphi(0^+) < \infty.$$

To establish that $\int_0^1 d\lambda(p) = 1$ as well, we appeal to integration-by-parts (cf. [4, Prop. 8.5.5] for this rule in a Radon-Stieltjes framework):

$$\int_0^1 d\lambda(p) = -\int_0^1 p d\varphi(p) = -[p\varphi(p)]_{0^+}^{1^-} + \int_0^1 \varphi(p)dp,$$

where the boundary expressions both vanish (because $\varphi(0^+) < \infty$ and $\varphi(1^-) = 0$) and the final integral has been assumed to equal 1.

In preparation for confirming the formula in (33) for the risk measure $\mathcal{R}$ in (27), which is all that will be needed, we introduce $\psi(\alpha) = \alpha \mathrm{CVaR}_\alpha(X)$. On the basis of the CVaR integral formula (7), which entails the integrability of $\mathrm{VaR}_\alpha(X)$ with respect to α, this function also has the expression $\psi(\alpha) = \int_0^\alpha \mathrm{VaR}_\alpha(X)d\alpha$. That implies that ψ is continuous and nonincreasing on $(0,1)$ with $\psi(0^+) = 0$, $\psi(1^-) = E[-X]$ and $d\psi(\alpha) = \mathrm{VaR}_\alpha(X)d\alpha$. Utilizing integration-by-parts once more, along with the relation $d\lambda(\alpha) = -\alpha d\varphi(\alpha)$, we see that

$$\int_0^1 \mathrm{CVaR}_\alpha(X) d\lambda(\alpha) = -\int_0^1 [\alpha \mathrm{CVaR}_\alpha(X)] d\varphi(\alpha) = -\int_0^1 \psi(\alpha) d\varphi(\alpha)$$
$$= -\psi(1^-)\varphi(1^-) + \psi(0^+)\varphi(0^+) + \int_0^1 \varphi(\alpha) d\psi(\alpha) = \int_0^1 \varphi(\alpha) \mathrm{VaR}_\alpha(X) d\alpha.$$

Thus, the spectral formula is correct. The integrability of $\mathrm{VaR}_\alpha(X)$ with respect to α, along with fact that φ is nonnegative with finite upper bound $\varphi(0^+)$, ensures moreover that $\int_0^1 \varphi(\alpha)\mathrm{VaR}_\alpha(X) d\alpha < \infty$, and hence that $\int_0^1 \mathrm{CVaR}_\alpha(X) d\lambda(\alpha) < \infty$. $\square$

In the "spectral" representations in (32) and (33), the function φ on $(0,1)$ is said to provide the *risk profile*. The case of Example 4 where the λ is concentrated in finitely many points α_i, giving them weights λ_i, corresponds to the risk profile function $\varphi(\alpha) = \Sigma_{\alpha_i \geq \alpha} \lambda_i$. The functionals in Example 5 can be interpreted as arising from worst-case analysis with respect to a collection of different risk profiles.

Functionals $\mathcal{R}$ directly defined by the spectral formula in (33) were studied in 1987 by Yaari [28] and Roell [26] in connection with a theory of "dual utility". (In those days, before VaR assumed such importance in finance, the expression being integrated along with φ was only regarded as a form of the inverse of the distribution function F_X.) Acerbi [1] showed that the properties of φ listed in Proposition 5 are necessary for such a functional $\mathcal{R}$ to be a coherent risk measure, which he then termed a *spectral* risk measure. He identified spectral risk measures to some degree with CVaR mixtures, although not under the full generality here, where the mixture is given by a weighting measure. The existence of a general spectral characterization as in (33) was indicated in a remark of Fölmer and Schied [10, p. 190], but without a precise statement or proof.

The risk profile for an "unmixed" risk measure CVaR_α itself is of course the function φ that has the value α^{-1} on $(0,\alpha]$ but 0 on $(\alpha,1)$. This corresponds to Acerbi's basic formula (7).

Mixed CVaR can be generalized to a weighting measure λ on $[0,1]$ instead of $(0,1)$. That amounts to admitting the limit cases $\mathrm{CVaR}_0(X) = -\inf X$ and $\mathrm{CVaR}_1(X) = -EX$ in some proportions. As long as the weight is not all placed on the endpoint 1, and weight is placed on 0 only in the finite-dimensional case of $\mathcal{L}^2(\Omega)$ (corresponding to a finite space Ω), one still gets a finite, coherent risk measure $\mathcal{R}$ that is strictly expectation bounded, and a lower range bounded deviation measure $\mathcal{D}$ that is coherent.

4 Deviations from Error Expressions

Next in our project are results leading to another broad class of examples of deviation measures. For any $X \in \mathcal{L}^2(\Omega)$, both X_+ and X_-, as defined in (2), likewise belong to $\mathcal{L}^2(\Omega)$. So too then does $aX_+ + bX_-$ for any coefficients a and b. Throughout this section, we take $a \geq 0$ and $b \geq 0$ and work with the functional $\mathcal{E}_{a,b,p}$ from $\mathcal{L}^2(\Omega)$ that is defined for any $p \in [1, \infty]$ by

$$\mathcal{E}_{a,b,p}(X) = ||\, aX_+ + bX_- \,||_p. \tag{35}$$

We wish to think of $\mathcal{E}_{a,b,p}$ as a simple "error functional" that penalizes in a particular manner the extent to which the r.v. X differs from the constant r.v. 0. Observe, for instance, that

$$\mathcal{E}_{a,b,p}(X) = \begin{cases} ||X|| & \text{for } a = 1,\ b = 1,\ p = 2, \\ ||X_+|| & \text{for } a = 1,\ b = 0,\ p = 2, \\ ||X_-|| & \text{for } a = 0,\ b = 1,\ p = 2, \end{cases}$$

while on the other hand

$$\mathcal{E}_{a,b,p}(X) = \begin{cases} \sup X & \text{for } a = 1,\ b = 1,\ p = \infty, \\ \sup X_+ & \text{for } a = 1,\ b = 0,\ p = \infty, \\ \sup X_- & \text{for } a = 0,\ b = 1,\ p = \infty. \end{cases}$$

These cases hint strongly at connections with some examples of deviation measures that were mentioned earlier.

Along these lines and beyond, our plan is to use the functionals $\mathcal{E}_{a,b,p}$ in two broad schemes for generating deviation measures $\mathcal{D}$, and incidentally, by way of Theorem 2, corresponding risk measures $\mathcal{R}$.

Proposition 6 (error functional basics). *Each functional $\mathcal{E}_{a,b,p}$ is nonnegative and lower semicontinuous on $\mathcal{L}^2(\Omega)$, and furthermore is sublinear, i.e., satisfies*

$$\mathcal{E}_{a,b,p}(0) = 0, \qquad \mathcal{E}_{a,b,p}(\lambda X) = \lambda \mathcal{E}_{a,b,p}(X) \text{ when } \lambda > 0, \tag{36}$$

as well as

$$\mathcal{E}_{a,b,p}(X + X') \leq \mathcal{E}_{a,b,p}(X) + \mathcal{E}_{a,b,p}(X') \text{ for all } X,\, X'. \tag{37}$$

If both $a > 0$ and $b > 0$, then $\mathcal{E}_{a,b,p}(X) > 0$ for all $Y \neq 0$.

Proof All these properties are evident from the definition of $\mathcal{E}_{a,b,p}$ except for the lower semicontinuity. We can get that by verifying the closedness of the subsets of $\mathcal{L}^2$ having the form $\{\, Y \,|\, \mathcal{E}_{a,b,p}(X) \leq \delta \}$ for a choice of $\delta \in [0, \infty)$. That comes from the closedness of the sets $\{\, Y \,|\, ||Y||_p \leq \delta \}$ and the continuity of the mapping $X \mapsto aX_+ + bX_-$ from $\mathcal{L}^2(\Omega)$ into itself. □

It can be seen from Proposition 6 that $\mathcal{E}_{a,b,p}$ behaves like a norm on $\mathcal{L}^2(\Omega)$, except that it lacks symmetry (unless $a = b$) and can sometimes have $\mathcal{E}_{a,b,p}(X) = 0$ when $X \neq 0$ (unless $a > 0$ and $b > 0$). Also, $\mathcal{E}_{a,b,p}(X)$ might be ∞ in some situations. Nonetheless, it is possible to think of $\mathcal{E}_{a,b,p}(X - Y)$ as standing for a sort of "error distance" between the r.v.'s X and Y.

Proposition 7 (error functional duality). *There is a nonempty, closed, convex subset $\mathcal{B}_{a,b,p}$ of $\mathcal{L}^2(\Omega)$ yielding the representation*

$$\mathcal{E}_{a,b,p}(X) = \sup\Big\{ E[XY] \,\Big|\, Y \in \mathcal{B}_{a,b,p} \Big\}. \tag{38}$$

This set is uniquely determined and has the following forms. If both $a > 0$ and $b > 0$, then

$$\mathcal{B}_{a,b,p} = \Big\{ Y \,\Big|\, || \, a^{-1}Y_+ + b^{-1}Y_- ||_q \le 1 \Big\} \tag{39}$$

for $q \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ under the usual convention. If $a > 0$ but $b = 0$, then

$$\mathcal{B}_{a,0,p} = \Big\{ Y \,\Big|\, Y \ge 0,\, ||Y||_q \le a \Big\}, \tag{40}$$

whereas if $a = 0$ but $b > 0$, then

$$\mathcal{B}_{0,b,p} = \Big\{ Y \,\Big|\, Y \le 0,\, ||Y||_q \le b \Big\}. \tag{41}$$

Proof The existence of a uniquely determined set $\mathcal{B}_{a,b,p}$ which is nonempty, closed, convex, and expresses $\mathcal{E}_{a,b,p}$ as in (38) follows from the lower semicontinuity and sublinearity of $\mathcal{E}_{a,b,p}$ in Proposition 6 and the general duality principle of convex analysis that was put to work already in Theorem 1; cf. [5], [19]. The question here is how that set can be described in accordance with the particular structure of $\mathcal{E}_{a,b,p}$. We can start with the standard fact that $||U||_p = \sup\Big\{ E[UV] \,\Big|\, ||V||_q \le 1 \Big\}$, which yields for us the expression

$$\mathcal{E}_{a,b,p}(X) = \sup\Big\{ E\big[\,(aX_+ + bX_-)V\,\big] \,\Big|\, ||V||_q \le 1 \Big\}.$$

Since $||V||_q$ is unaffected by switching the sign of $V(\omega)$ for any set of ω's in Ω, this formula says equivalently that $\mathcal{E}_{a,b,p}(X)$ is the supremum of $E[XY]$ over all Y that can be obtained from functions V satisfying $||V||_q \le 1$ by setting $Y(\omega) = aV(\omega)$ when $V(\omega) \ge 0$ but $Y(\omega) = bV(\omega)$ when $V(\omega) < 0$. In the cases where $b = 0$ or $a = 0$, we immediately get from this the expressions for $\mathcal{B}_{a,b,p}$ in (40) and (41), respectively, but more work is needed in the case where both $a > 0$ and $b > 0$.

In that situation the functions Y in question are the ones such that $a^{-1}Y_+ + b^{-1}Y_-$ is a function V with $||V||_q \le 1$. Thus, in denoting the set on the right side of (39) by $\mathcal{B}'_{a,b,p}$, we do have the representation of $\mathcal{E}_{a,b,p}$ in terms of $\mathcal{B}'_{a,b,p}$ in place of $\mathcal{B}_{a,b,p}$. That tells us, at least, that $\mathcal{B}_{a,b,p}$ has to be the closed, convex hull of $\mathcal{B}'_{a,b,p}$. But $\mathcal{B}'_{a,b,p}$ is itself closed. Therefore, all that is missing still is the confirmation that $\mathcal{B}'_{a,b,p}$ is actually convex.

That will be obtained from the fact that the mapping $\mathcal{F} : Y \mapsto a^{-1}Y_+ + b^{-1}Y_-$ is sublinear: it satisfies $\mathcal{F}(0) = 0$, $\mathcal{F}(\lambda Y) = \lambda\mathcal{F}(Y)$ when $\lambda > 0$, and $\mathcal{F}(Y + Y') \le \mathcal{F}(Y) + \mathcal{F}(Y')$ in the ordering of $\mathcal{L}^2(\Omega)$. With that in hand, consider any Y_0 and Y_1 in $\mathcal{B}'_{a,b,p}$; the functions $V_0 = \mathcal{F}(Y_0)$ and

$V_1 = \mathcal{F}(Y_1)$ satisfy $||V_0||_q \leq 1$ and $||V_1||_q \leq 1$. For arbitrary $\lambda \in (0,1)$ let $Y_\lambda = (1-\lambda)Y_0 + \lambda Y_1$ and $V_\lambda = \mathcal{F}(Y_\lambda)$. The question is whether $||V_\lambda||_q \leq 1$. The sublinearity of $\mathcal{F}$ implies that $V_\lambda \leq (1-\lambda)V_0 + \lambda V_1$, where moreover

$$||(1-\lambda)V_0 + \lambda V_1||_q \leq (1-\lambda)||V_0||_q + \lambda ||V_1||_q \leq (1-\lambda) + \lambda = 1.$$

Then $||V_\lambda||_q \leq 1$ because $||V||_q \leq ||V'||_q$ when $0 \leq V \leq V'$. □

To make it simpler to state our results about deviation measures in this framework, we make use of our terminology in Section 1, that the probability space $(\Omega, \mathcal{M}, P)$ is *essentially infinite* if P takes on infinitely many different values over the sets in $\mathcal{M}$, or equivalently that there exist subsets of $\mathcal{M}$ having arbitrarily small positive probability.

Theorem 3 (deviation measures from penalties relative to expectation). *For any $p \in [1, \infty]$ and $a \geq 0$, $b \geq 0$, with $a + b > 0$, let*

$$\mathcal{D}(X) \;=\; \mathcal{E}_{a,b,p}(X - EX) \;=\; ||\, a[X - EX]_+ + b[X - EX]_- \,||_p. \qquad (42)$$

Then $\mathcal{D}$ is a lower semicontinuous deviation measure which is finite on $\mathcal{L}^2(\Omega)$ as long as $\mathcal{E}_{a,b,p}$ is finite on $\mathcal{L}^2(\Omega)$. It is lower range dominated when $a = 0$ and $b \leq 1$, or when merely $a + b \leq 1$ if $p = 1$, but not otherwise (when the probability space is essentially infinite). The risk envelope for $\mathcal{D}$ is given by

$$\mathcal{Q} = \Big\{\, Q \;\Big|\; EQ = 1 \text{ and a constant } C \text{ exists with } C - Q \in \mathcal{B}_{a,b,p} \Big\}, \qquad (43)$$

while the strictly expectation bounded risk measure associated with $\mathcal{D}$ is

$$\mathcal{R}(X) \;=\; E[-X] + ||\, a[X - EX]_+ + b[X - EX]_- \,||_p. \qquad (44)$$

Proof By its formula, $\mathcal{D}(X)$ depends only on $X - EX$, so $\mathcal{D}$ satisfies D1. The sublinearity properties of $\mathcal{E}_{a,b,p}$ in (36) of Proposition 6 turn into D2 and D3 for $\mathcal{D}$. We have D4 when $a > 0$ and $b > 0$ since in that case $\mathcal{E}_{a,b,p}(Z) > 0$ unless $Z = 0$. But even when $a > 0$ and $b = 0$ we have D4, inasmuch as then $\mathcal{E}_{a,b,p}(Z) > 0$ unless $Z \leq 0$, and it is impossible to have $X - EX \leq 0$ without X being constant. Similarly, we have D4 when $a = 0$ and $b > 0$.

To determine the risk envelope associated with $\mathcal{D}$, we apply the representation in Proposition 7 to see that

$$\mathcal{D}(X) = \sup\Big\{\, E[(X - EX)Y] \;\Big|\; Y \in \mathcal{B}_{a,b,p} \Big\},$$

where $E[(X - EX)Y] = E[X(Y - EY)]$. In terms of $Q = 1 + EY - Y$, which converts $E[X(Y - EY)]$ into $EX - E[XQ]$, we identify $\mathcal{D}(X)$ with $EX - \inf \{\, E[XQ] \,|\, Q \in \mathcal{Q} \}$ for the set $\mathcal{Q}$ consisting of all Q of the form $1 + EY - Y$ for some $Y \in \mathcal{B}_{a,b,p}$. That is identical to the set $\mathcal{Q}$ in (43).

This set inherits the convexity of $\mathcal{B}_{a,b,p}$ in Proposition 7, so if it also is closed, it will have to be the risk envelope in question by virtue of the uniqueness in Theorem 1. The closedness is immediate from (43) and the closedness of $\mathcal{B}_{a,b,p}$ in Proposition 7 in cases where $\mathcal{B}_{a,b,p}$ is sure to be a bounded subset of $\mathcal{L}^2(\Omega)$, since then, from the convexity, we have $\mathcal{B}_{a,b,p}$ weakly compact. That covers $p \in [1,2]$ ($q \in [2,\infty]$) and situations where Ω is finite (or just not essentially infinite).

Another closedness argument must anyway be brought in, however, to cover all cases without exception. The crux of the matter is showing that if $\{Q_k\}_{k=1}^{\infty}$ and $\{C_k\}_{k=1}^{\infty}$ satisfy $C_k - Q_k \in \mathcal{B}_{a,b,p}$, and Q_k converges in $\mathcal{L}^2(\Omega)$ to some Q, then there is a constant C such that $C - Q \in \mathcal{B}_{a,b,p}$. For this, it will suffice to establish that the sequence $\{C_k\}_{k=1}^{\infty}$ is bounded, since then, by passing to subsequences if necessary, we can suppose this sequence of constants has a limit, and then by taking that limit to be C, conclude that $C - Q \in \mathcal{B}_{a,b,p}$, as required.

By rescaling, we can harmlessly reduce to having $\max\{a,b\} = 1$. Then, through the fact that $||\cdot||_1 \le ||\cdot||_p \le ||\cdot||_\infty$ (because we are dealing with a probability measure P on Ω), we have $\mathcal{E}_{a,b,p}(X) \le \mathcal{E}_{1,1,\infty}(X) = ||X||_\infty$ and consequently $\mathcal{B}_{a,b,p} \subset \mathcal{B}_{1,1,\infty} = \{ Y \,|\, ||Y||_1 \le 1 \}$. Hence, from our assumption that $C_k - Q_k \in \mathcal{B}_{a,b,p}$, we have $||C_k - Q_k||_1 \le 1$, where moreover $||C_k - Q_k||_1 \ge ||C_k||_1 - ||Q_k||_1 = |C_k| - ||Q_k||_1$. Thus $|C_k| \le 1 + ||Q_k||_1 \le 1 + ||Q_k||_2$. Since Q_k converges to Q in $\mathcal{L}^2(\Omega)$, we know $||Q_k||_2 \to ||Q||_2$, and the sequence $\{ ||Q_k||_2 \}_{k=1}^{\infty}$ is thus bounded. It follows then that the sequence $\{ |C_k|| \}_{k=1}^{\infty}$ is bounded too, which is what we needed.

It remains to explore the possibilities for $\mathcal{D}$ to have the lower range dominance property D5. Recall that whenever $0 \le U \le V$ we have $||U||_p \le ||V||_p \le ||V||_1 = EV$. From this and the fact that

$$0 \;\le\; a[X - EX]_+ + b[X - EX]_- \;\le\; a[\sup X - EX] + b[EX - \inf X],$$

we see that

$$\begin{aligned} &\mathcal{D}(X) \le a[\sup X - EX] + b[EX - \inf X] \\ &\qquad\qquad \text{always, with equality holding when } p = \infty. \end{aligned} \tag{45}$$

Hence $\mathcal{D}$ is assured of being lower range dominated for any p if $a = 0$ and $b = 1$, but for $p = \infty$ this is the only guarantee. When $p = 1$, so that $\mathcal{D}(X) = aE[X - EX]_+ + bE[X - EX]_-$, we can appeal to the fact that $E[X - EX]_+ = E[X - EX]_- \le EX - \inf X$ to confirm that $\mathcal{D}(X) \le EX - \inf X$ as long as $a + b \le 1$.

To see how lower range dominance fails otherwise, for $p < \infty$ when the probability space is essentially infinite, consider a subset Ω_0 of Ω having probability $\pi \in (0,1)$, and define $X \in \mathcal{L}^2(\Omega)$ by setting $X(\omega) = 1$ for $\omega \in \Omega_0$ and $X(\omega) = 0$ for $\omega \notin \Omega_0$. This yields $\sup X = 1$, $\inf X = 0$ and $EX = \pi$, and makes the expression $a[X(\omega) - EX]_+ + b[X(\omega) - EX]_-$ take the value $a(1-\pi)$ with probability π and the value $b\pi$ with probability $1-\pi$. Then $\mathcal{D}(X) = \left[a^p(1-\pi)^p\pi + b^p\pi^p(1-\pi)\right]^{1/p}$ while $EX - \inf X = \pi$, so coherency would require that $a^p(1-\pi)^p\pi + b^p\pi^p(1-\pi) \le \pi^p$, or on dividing both sides by π^p, that $a^p(\pi^{-1}-1)^p\pi + b^p(1-\pi) \le 1$. By choosing values of π nearer and nearer to 0 (under the assumption that Ω is essentially infinite), we can produce a violation of this inequality for $p = 1$ unless $a + b \leq 1$. In the case of $p > 1$, we can likewise produce a violation unless $a = 0$ and $b \leq 1$, due to the fact that $(\pi^{-1} - 1)^p \pi$ tends to ∞ as π tends to 0 (as seen through the change of variables $s = \pi^{-1} - 1$, $\pi = (1 + s)^{-1}$, by taking the limit as s tends to ∞). Thus, without the specified restrictions on a and b, lower range dominance is impossible (for essentially infinite Ω). □

Example 6 (deviation measures of $\mathcal{L}^p$ and semi-$\mathcal{L}^p$ type). *For any $p \in [1, \infty]$, a lower semicontinuous deviation measure and its associated risk envelope are given by*

$$\begin{aligned} &\mathcal{D}(X) = \left\| X - EX \right\|_p, \\ &\mathcal{Q} = \left\{ Q \,\middle|\, EQ = 1 \text{ and a constant } C \text{ exists with } \|C - Q\|_q \leq 1 \right\}, \end{aligned} \tag{46}$$

and also by

$$\begin{aligned} &\mathcal{D}_+(X) = \left\| [X - EX]_+ \right\|_p, \\ &\mathcal{Q}_+ = \left\{ Q \,\middle|\, EQ = 1,\ \sup Q < \infty,\ \| \sup Q - Q \|_q \leq 1 \right\}, \end{aligned} \tag{47}$$

as well as by

$$\begin{aligned} &\mathcal{D}_-(X) = \left\| [X - EX]_- \right\|_p, \\ &\mathcal{Q}_- = \left\{ Q \,\middle|\, EQ = 1,\ \inf Q > -\infty,\ \| Q - \inf Q \|_q \leq 1 \right\}. \end{aligned} \tag{48}$$

Of these deviation measures, only $\mathcal{D}_-$ is lower range dominated (when the probability space is essentially infinite), except for the case of $p = 1$ (which actually has $\mathcal{D}_+ = \mathcal{D}_- = \frac{1}{2}\mathcal{D}$).

Detail. This applies Theorem 3 to the cases of $a = 1$ and $b = 1$ first, $a = 1$ and $b = 0$ second, and $a = 0$ and $b = 1$ third. The formulas in Proposition 7 are utilized in obtaining the risk envelopes. The specifics behind (47) are that the condition on C in (43) reduces for $a = 1$, $b = 0$, to having $C - Q \geq 0$ and $\|[C - Q]_+\|_q \leq 1$. Of course, $[C - Q]_+ = C - Q$ when $C - Q \geq 0$. On the other hand, having $C - Q \geq 0$ is equivalent to having $\sup Q < \infty$ and $C \geq \sup Q$. Since $\|C - Q\|_q \geq \|\sup Q - Q\|_q$ when $C \geq \sup Q$, it is clear that the existence of C satisfying $C - Q \geq 0$ and $\|C - Q\|_q \leq 1$ comes down simply to having $\sup Q < \infty$ and $\|\sup Q - Q\|_q \leq 1$. □

Specialization of Example 6 to $p = 2$ confirms that σ_- is a lower range dominated deviation measure, whereas σ and σ_+ are not (for general Ω as described). In portfolio theory, the standard deviation σ has long been central [15], although the idea of using the lower semideviation σ_- instead likewise goes back to the early times [16]. More recently, the case of $\mathcal{D}(X) = \left\| X - EX \right\|_1$, known as *mean absolute deviation*, has been investigated; cf. [12], and for the equivalent format with $\mathcal{D}_-(X) = \left\| [X - EX]_- \right\|_1$, also [9], [27], [17]. This too is a finite, lower semicontinuous (hence continuous) deviation measure on $\mathcal{L}^2(\Omega)$, and by our results it is lower range dominated.

Example 7 (coherent risk measures of semi-$\mathcal{L}^p$ type). *For any* $p \in [1, \infty]$ *and any* $\rho \in (0, \infty)$*, a coherent, strictly expectation bounded risk measure that is lower semicontinuous is given by*

$$\mathcal{R}(X) = E[-X] + \rho \big\| [X - EX]_- \big\|_p. \tag{49}$$

The coherence would fail, however (for probability spaces that are essentially infinite), if $[X - EX]_-$ *were replaced by* $[X - EX]_+$ *or by* $X - EX$*.*

Detail. In this case we combine Example 6 with the facts of Theorem 3. □

Theorem 4 (deviation measures from distances to constancy). *For any* $p \in [1, \infty]$*,* $a > 0$ *and* $b > 0$*, let*

$$\mathcal{D}(X) = \inf_C \mathcal{E}_{a,b,p}(X - C). \tag{50}$$

Then for each X *this infimum is attained by some constant* C*, and* $\mathcal{D}$ *is a lower semicontinuous deviation measure which is finite on* $\mathcal{L}^2(\Omega)$ *as long as* $\mathcal{E}_{a,b,p}$ *is finite on* $\mathcal{L}^2(\Omega)$*. It is lower range dominated when* $p = 1$ *and* $a \leq 1$*, but (for essentially infinite* Ω*) not otherwise. The risk envelope for* $\mathcal{D}$ *is*

$$\mathcal{Q} = \big\{ Q \,\big|\, 1 - Q \in \mathcal{B}_{a,b,p},\ EQ = 1 \big\}. \tag{51}$$

Proof First we show the infimum in (50) is always attained (although perhaps not uniquely). For fixed X, let $\varphi(C) = \mathcal{E}_{a,b,p}(X - C)$. Since $\mathcal{E}_{a,b,p}$ is lower semicontinuous by Proposition 6, φ is lower semicontinuous on $I\!R$. We have $\varphi(C) \geq \min\{a, b\} ||X - C||_p$ with $\min\{a, b\} > 0$, hence $\varphi(C) \to \infty$ as $|C| \to \infty$. All sets of the form $\big\{ C \,\big|\, \varphi(C) \leq d \big\}$, $d \in I\!R$, are therefore bounded in $I\!R$, so the existence of a minimizing value of C is assured.

The verification of axioms D1, D2 and D3 is elementary on the basis of the properties of $\mathcal{E}_{a,b,p}$ in Proposition 6. Through the attainment just established, D4 comes out as well: we cannot have $\mathcal{D}(X) = 0$ without a constant C for which $\mathcal{E}_{a,b,p}(X - C) = 0$, but that holds only when $X - C = 0$.

For the proof that $\mathcal{D}$ is lower semicontinuous, we distinguish two cases: $p \in [1, 2]$ and $p > 2$. When $p \in [1, 2]$, we have

$$\mathcal{D}(X) \leq \mathcal{E}_{a,b,p}(X) \leq \max\{a, b\} ||X||_p \leq \max\{a, b\} ||X||,$$

so $\mathcal{D}$ is finite and bounded from above in a neighborhood of the origin. Any finite convex functional on that is bounded from above in a neighborhood of some point is everywhere continuous; cf. [5], [19]. For the case of $p > 2$, we utilize instead the attainment of the infimum to see that, for any $c \in I\!R$,

$$\big\{ X \,\big|\, \mathcal{D}(X) \leq c \big\} = \big\{ X' + C \,\big|\, C \text{ constant}, \mathcal{E}_{a,b,p}(X') \leq c \big\}.$$

This says that the level set $\big\{ X \,\big|\, \mathcal{D}(X) \leq c \big\}$, whose closedness we must demonstrate in order to conclude the lower semicontinuity of $\mathcal{D}$, is the sum of the one-dimensional "constant subspace" of $\mathcal{L}^2(\Omega)$ and the level set $\big\{ X' \,\big|\, \mathcal{E}_{a,b,p}(X') \leq c \big\}$. That second level set is known from Proposition

6 to be closed and convex. If we can ascertain that it is also bounded in $\mathcal{L}^2(\Omega)$, it will follow that it is weakly compact, and then the sum in question is sure to be closed, as needed. To see this boundedness we note that $\mathcal{E}_{a,b,p}(X') \geq \min\{a,b\}||X'||_p \geq \min\{a,b\}||X'||$ with $\min\{a,b\} > 0$, so

$$\left\{ X' \,\middle|\, \mathcal{E}_{a,b,p}(X') \leq c \right\} \subset \left\{ X' \,\middle|\, ||X'|| \leq c/\min\{a,b\} \right\}.$$

Next we target the risk envelope representation. According to (10) of Theorem 1 and the definition of $\mathcal{D}$, we have $Q \in \mathcal{Q}$ if and only if $\mathcal{E}_{a,b,p}(X - C) \geq EX - E[XQ]$ for all X and C. Therefore

$$\mathcal{Q} = \left\{ Q \,\middle|\, \mathcal{E}_{a,b,p}(X') \geq E[(X' + C)(1 - Q)] \text{ for all } X',\, C \right\}.$$

This reveals that $Q \in \mathcal{Q}$ if and only if $E[C(1-Q)] = 0$ for all C and, on the other hand $\mathcal{E}_{a,b,p}(X') \geq E[X'(1 - Q)]$ for all X'. The former is equivalent to $E(1 - Q) = 0$, whereas the latter means through Proposition 7 that $1 - Q \in \mathcal{B}_{a,b,p}$. Hence the formula claimed for $\mathcal{Q}$ in (51) is correct.

Because the lower range dominance of $\mathcal{D}$ corresponds by Theorem 1 to the elements of $\mathcal{Q}$ being nonnegative, we see it holds if and only if the conditions $1 - Q \in \mathcal{B}_{a,b,p}$ and $EQ = 1$ necessitate $Q \geq 0$, or in other words, when the conditions $Y \in \mathcal{B}_{a,b,p}$ and $EY = 0$ necessitate $Y \leq 1$. In view of the description of $\mathcal{B}_{a,b,p}$ in Proposition 7, that is true (for essentially infinite probability spaces) only when $p = 1$ (so $q = \infty$) and $a \leq 1$. □

Example 8 (CVaR-deviation reinterpreted). *For $\alpha \in (0,1)$, the deviation measure $\mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX)$ corresponds to the case of Theorem 4 where $p = 1$, $a = 1$, $b = \alpha^{-1} - 1$. Its risk envelope $\mathcal{Q}$, expressed in Example 3, thus can be derived alternatively from the prescription in Theorem 4.*

Detail. From (5) we have $\mathcal{D}(X) = \min_C \left\{ C + \alpha^{-1}E[X - EX + C]_- \right\}$. By shifting the variable from C to $C' = EX - C$, we can rewrite this as $\mathcal{D}(X) = \min_{C'} \left\{ EX - C' + \alpha^{-1}E[X - C']_- \right\}$. Since $EX - C' = E[X - C']$ with $X - C' = [X - C']_+ - [X - C']_-$, this is the same as

$$\mathcal{D}(X) = \min_{C'} E\left\{ [X - C']_+ + (\alpha^{-1} - 1)[X - C']_- \right\}, \tag{52}$$

which corresponds to the indicated case of Theorem 4. □

The minimum distance expression in (52) relates closely to one that has been utilized by Koenker and Basset in quantile regression [11].

The classical $\mathcal{D}(X) = \sigma(X)$ fits as a special case of Theorem 4 as well, namely with $a = 1$, $b = 1$, and $p = 2$, which makes $\mathcal{E}_{a,b,p}(X) = ||X||$.

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