OPTIMALITY CONDITIONS IN PORTFOLIO ANALYSIS WITH GENERALIZED DEVIATION MEASURES

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Abstract

Optimality conditions are derived for problems of minimizing a generalized measure of deviation of a random variable, with special attention to situations where the random variable could be the rate of return from a portfolio of financial instruments. Generalized measures of deviation go beyond standard deviation in satisfying axioms that do not demand symmetry between ups and downs. The optimality conditions are applied to characterize the generalized "master funds" which elsewhere have been developed in extending classical portfolio theory beyond the case of standard deviation. The consequences are worked out for deviation based on conditional value-at-risk and its variants, in particular.

Keywords: *generalized deviation measures, portfolio analysis, generalized master funds, CAPM-like relations, optimality conditions, risk envelopes, risk identifiers, conditional value-at-risk, risk management, stochastic optimization.*

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1 Introduction

For a wide range of applications in finance and engineering, it is desirable to make decisions which minimize "risk" within the circumstances that are faced. There are many different ways of looking at risk and quantifying it, however. Risk can be identified either with the extent to which an outcome can deviate from expectations or with some assessment of the potential for absolute loss. The first of these two approaches to risk has long been central to much of portfolio analysis, especially in relying on the historical expectations and standard deviations of securities to put together "master funds" that may summarize the tendencies in a financial market [12]. The second approach, on the other hand, brings in notions like value-at-risk, conditional value-at-risk, and the "coherent measures of risk" of Artzner, Delbaen, Eber and Heath [3]. It is seen also in models that introduce penalties for failure to meet performance targets and aim at minimizing the expected values of those penalties.

In our recent paper [20], we worked at extending the first approach. We showed that most of the fundamental results of portfolio theory remain valid, in appropriate formulation, when standard deviation is replaced by a more general, nonstandard "measure of deviation," which does not have to be symmetric about the mean. We ascertained the existence of generalized "master funds" and determined their behavior in relation to the risk-free rate of return. Although other researchers had already ventured in that direction in special cases, we were able to proceed much further, avoiding restrictive assumptions. In [19], we delved more deeply into deviation measures and their partial connection with coherent risk measures, providing axiomatic foundations and numerous examples. These included measures derived from conditional value-at-risk and others coming directly or one-sidedly from $\mathcal{L}^p$ norms.

Here, we continue in that vein by determining conditions for optimality in problems where general deviation measures are minimized. In the setting we work with, "portfolios" are construed as combinations $x_1 r_1 + \cdots + x_n r_n$ of given random variables r_i, which could stand for the uncertain rates of return of some collection of securities. The task is choose the coefficients x_i, subject to constraints on expectation or other indicators, that optimize $x_1 r_1 + \cdots + x_n r_n$ with respect to some specification of the deviation measure. Having characterized the desired coefficients in this general setting, we go on to apply the results to the more basic portfolio model we studied in [20]. In that way we arrive in particular at conditions that describe the generalized master funds developed in [20].

In classical portfolio theory, where standard deviation is minimized, the conditions describing a master fund furnish what is called the capital asset pricing model (CAPM). Those conditions, in terms of expected returns and covariances, are believed to offer guidance on predicting the market behavior of financial instruments [25, 24]. An overview of CAPM results is available in [8]. Extensions to account for higher moments than variance were made in [23].

The classical CAPM relations are widely used in factor analysis, but what should be the status of the corresponding relations that come from different, nonstandard deviation measures, yielding a whole array of different master funds? This question has no simple answer. Whether certain master funds beyond the classical may turn out to be equally, or more, valuable in factor analysis, individually or collectively, is an issue outside the scope of this paper. To facilitate comparisons, however, we anyway present our CAPM-like results with their generalized β coefficients in a format similar to the classical, from which the results of other researchers on particular examples of nonstandard deviation measures can readily be derived. Covered in that manner are results associated with mean-lower partial moments [4, 11], conditional value-at-risk [26, 5, 2], and mean absolute deviation [9], in particular.

In contrast though to such earlier work, our CAPM-like results have far fewer restrictions, even in those special cases. They do not rely on the existence of density functions for the distributions that arise, or on the absence of probability atoms (corresponding to jumps in the distribution functions), which would preclude applications to discrete random variables or even to financial instruments involving options (as we demonstrated in [20]). They do not require the differentiability of the deviation with respect to the parameters specifying the relative weights of the instruments in the portfolio.

After reviewing the fundamentals of deviation measures in the remainder of this section, we present in Section 2 a number of key examples of deviation measures and their associated risk envelopes. New information is provided about the "risk identifers" that can be extracted from these risk envelopes in assessing a specific random variable. We go on then, in Section 3, to the characterization of minimum deviation, where we concentrate on a system of linear constraints on the coefficients x_i in a portfolio random variable $x_1 r_1 + \cdots + x_n r_n$. We develop necessary and sufficient conditions for optimality in terms of risk identifiers. These risk identifiers relate closely to subgradients in the sense of convex analysis but furnish probabilistic interpretations which are neatly tailored to deviation and are especially attractive in this setting.

This work on optimality draws on unpublished material in our 2001 working paper [18]. Recently, Ruszczynski and Shapiro [22] have produced optimality conditions of subgradient type for "convex risk functionals" which extend beyond the coherent risk measures of [3] and, through a certain transformation [18, 19], could be viewed as encompassing deviation measures. They analyzed optimality for composite functionals connected to nonlinear as well as linear constraint set-ups for their risk functionals and made other innovations besides. They did not, however, get into the specifics of what their results would say if translated to deviation measures and risk identifiers, nor did they turn an eye to the specifics of portfolios and the development of CAPM-like results such as we give in Sections 4 and 5.

We proceed now with some background in notation and definitions. For the purposes of this paper, a random variable (r.v.) will be an element of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, P)$, where Ω denotes the designated space of future states ω, $\mathcal{M}$ is a field of sets in Ω, and P is a probability measure on $(\Omega, \mathcal{M})$. The inner product between two elements X and Y in $\mathcal{L}^2(\Omega)$ is

$$\langle X, Y \rangle = E[XY] = \int_\Omega X(\omega) Y(\omega) dP(\omega).$$

We consistently use C to stand equally for a real number or the corresponding constant r.v. An inequality like $X \geq C$ or $X \leq Y$ is to be interpreted in an almost everywhere sense.

We let $\sup X$ and $\inf X$ stand for the *essential* supremum and infimum of X. (The first might be ∞, whereas the second might be $-\infty$.) We adopt the notation that

$$X = X_+ - X_- \text{ with } X_+ = \max\{0, X\},\ X_- = \max\{0, -X\}$$

By allowing ∞ to be a value of

$$||X||_p = \begin{cases} (E[X^p])^{1/p} & \text{when } 1 \leq p < \infty, \\ \sup |X| & \text{when } p = \infty, \end{cases}$$

we are able to utilize these $\mathcal{L}^p$ norm expressions for all p, even though we work only in $\mathcal{L}^2(\Omega)$.

In line with our earlier efforts in [20], [19] (and [18]), we take a *deviation measure* to be a functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ satisfying the axioms

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C,
(D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,
(D3) $\mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(X')$ for all X and X',
(D4) $\mathcal{D}(X) > 0$ for all nonconstant X, whereas $\mathcal{D}(X) = 0$ for constant X.

These properties are modeled after those of standard deviation, but they do not require symmetry: perhaps $\mathcal{D}(-X) \neq \mathcal{D}(X)$. Indeed, a major motivation for moving to nonstandard deviations is the need for flexibility in treating outcomes with $X(\omega) < EX$ different from ones with $X(\omega) > EX$. They imply in particular that $\mathcal{D}$ is a convex functional. On the other hand, they allow $\mathcal{D}(X)$ to be ∞ in some situations. The import of axioms D1–D4 is thoroughly discussed in [19] along with other ways of constituting them.

A deviation measure $\mathcal{D}$ is called *lower semicontinuous* if every subset of $\mathcal{L}^2(\Omega)$ having the form $\{ X \,|\, \mathcal{D}(X) \le c\}$ for some $c \in I\!R$ is closed. (These sets are convex, so closedness in the weak topology is the same as closedness in the strong topology.) When a lower semicontinuous deviation measure $\mathcal{D}$ is *finite*, i.e. has $\mathcal{D}(X) < \infty$ for all $X \in \mathcal{L}^2(\Omega)$, it must actually be continuous on $\mathcal{L}^2(\Omega)$; cf. [19, Proposition 2]. Of course, when Ω is a finite set, so that $\mathcal{L}^2(\Omega)$ is finite-dimensional, finiteness already implies continuity.

Lower semicontinuous deviation measures $\mathcal{D}$ on $\mathcal{L}^2(\Omega)$ have an important dualization in terms of subsets $\mathcal{Q}$ of $\mathcal{L}^2(\Omega)$, called *risk envelopes*, which satisfy

(Q1) $\mathcal{Q}$ is a nonempty, closed and convex,
(Q2) for every nonconstant X there is some $Q \in \mathcal{Q}$ having $E[XQ] < EX$,
(Q3) $EQ = 1$ for all $Q \in \mathcal{Q}$.

We established in [19, Theorem 1] that there is a one-to-one correspondence between such $\mathcal{D}$ and $\mathcal{Q}$ in which

$$\mathcal{D}(X) = EX - \inf_{Q \in \mathcal{Q}} E[XQ], \tag{1.1}$$

and on the other hand,

$$\mathcal{Q} = \left\{ Q \,\middle|\, \mathcal{D}(X) \ge EX - E[XQ] \text{ for all } X \right\}. \tag{1.2}$$

Note that $EX - E[XQ]$ can equivalently be written at $E[(EX - X)Q]$ under Q3. A variety of examples of $\mathcal{D} \leftrightarrow \mathcal{Q}$ pairs will be recalled in the next section.

The relationship between a deviation measure $\mathcal{D}$ and its risk envelope $\mathcal{Q}$ can best be understood in the case where $\mathcal{D}$ has the additional property of being *lower range dominated*, which means

(D5) $\mathcal{D}(X) \le EX - \inf X$ for all X.

The significance of this property stems from our result in [19, Theorem 1] that, on top of the other axioms, one has

$$Q \ge 0 \text{ for all } Q \in \mathcal{Q} \iff \mathcal{D} \text{ satisfies D5}. \tag{1.3}$$

Then each $Q \in \mathcal{Q}$ can be viewed as the density dP'/dP of some probability measure P' which may be an alternative to P, and $E[XQ]$ comes out as the expectation of X with respect to P' instead of P. In that setting, $\mathcal{Q}$ stands for a set of alternative probability measures forming a kind of neighborhood of P, and formula (1.1) expresses $\mathcal{D}(X)$ as giving the greatest amount by which the expectation under P' can fall short of the expectation under P.

A sharper understanding of the relationship between $\mathcal{D}$ and $\mathcal{Q}$ in (1.1) and (1.2) can be achieved through study of an additional concept which we bring to center stage here for the first time.

Definition 1 (risk identifiers)**.** *For each $X \in \mathcal{L}^2(\Omega)$, the elements Q of the set*

$$\mathcal{Q}(X) = \operatorname*{argmin}_{Q \in \mathcal{Q}} E[XQ] = \operatorname*{argmax}_{Q \in \mathcal{Q}} E[(EX - X)Q] = \left\{ Q \in \mathcal{Q} \,\middle|\, \mathcal{D}(X) = EX - E[XQ] \right\}, \tag{1.4}$$

will be called the risk identifiers for X with respect to $\mathcal{D}$.

In the framework of coherency with its probabilistic interpretation, the risk identifiers for X are the densities Q of the probability measures P', among the admitted alternatives to P in association with $\mathcal{D}$, that bring out the worst in X. Other insight into how to think about risk identifiers can be gained by noting that the r.v. $EX - X$ stands for the *downside* of X and observing that

$$E[\,(EX - X)Q\,] = \text{covar}\,(EX - X, Q) = \text{covar}\,(-\,X, Q) \tag{1.5}$$

(through the rule that $E[YZ] = \text{covar}(Y, Z)$ when $EY = 0$ or $EZ = 0$). The elements $Q \in \mathcal{Q}(X)$ are thus the ones that track the downside of X as closely as possible.

Risk identifiers will have a crucial role for us in descriptions of optimality. It will be important, therefore, to know what they are for given $\mathcal{D}$ and X. Trivially,

$$\mathcal{Q}(X) = \mathcal{Q} \text{ when } X \text{ is constant,} \tag{1.6}$$

so effort needs mainly to be directed toward the determination of $\mathcal{Q}(X)$ for nonconstant X.

The relationship between deviation measures $\mathcal{D}$ and the coherent risk measures of Artzner et al. in [3] has been explained fully in our paper [19]. However, a brief overview may help in placing our current efforts in a broader context. In the definition we adopt, recast from [3] with minor modifications, a *coherent risk measure* is a functional $\mathcal{R} : \mathcal{L}^2(\Omega) \to (-\infty, \infty]$ that satisfies

(R1) $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C,
(R2) $\mathcal{R}(0) = 0$ and $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ for all X and all $\lambda > 0$,
(R3) $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ for all X and X',
(R4) $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \geq X'$.

On the other hand, a *strictly expectation bounded risk measure* in our parlance is a functional $\mathcal{R}$ that satisfies R1, R2, R3 and

(R5) $\mathcal{R}(X) > E[-X]$ for all nonconstant X, whereas $\mathcal{R}(X) = E[-X]$ for constant X.

As laid out in [19], there is a one-to-one correspondence between deviation measures $\mathcal{D}$ and strictly expectation bounded risk measures $\mathcal{R}$ in which

$$\mathcal{D}(X) = \mathcal{R}(X - EX), \qquad \mathcal{R}(X) = E[-X] + \mathcal{D}(X). \tag{1.7}$$

In this correspondence, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is lower range dominated as in D5. Lower semicontinuity, or continuity, carries over from one functional to the other, of course.

In general, a risk measure $\mathcal{R}$ can be coherent without being strictly expectation bounded, or vice versa, but the risk measures of greatest interest have both properties. This will be clear from the examples below in which $\mathcal{D}$ is identified as being lower range dominated.

Despite the similarities in (1.7), minimizing a deviation measure is inherently different from minimizing a risk measure, in which thresholds of risk acceptance can lead to the optimal value being $-\infty$ and the nonexistence therefore of an optimal solution. We leave the investigation of such phenomena for later work.

2 Key Deviation Examples With Their Risk Identifiers

Some of the potentially most interesting deviation measures $\mathcal{D}$ and their associated risk envelopes $\mathcal{Q}$ will now be listed. These examples are taken from our paper [19], where even more such pairings are developed and their properties are elaborated. However, the description of the corresponding risk identifiers $Q \in \mathcal{Q}(X)$ in each case is offered here as a fresh contribution. Because of the general rule for constant X in (1.6), only the case of nonconstant X requires attention.

Example 1 (risk identifiers for standard deviation and semideviations)**.** *An instance of a finite, continuous deviation measure on $\mathcal{L}^2(\Omega)$ with its associated risk envelope is furnished by standard deviation:*

$$\mathcal{D}(X) = \sigma(X) = ||X - EX||_2, \qquad \mathcal{Q} = \Big\{ Q \,\Big|\, EQ = 1,\ \sigma(Q) \le 1 \Big\}. \tag{2.1}$$

The risk identifier set for any nonconstant X is then the singleton

$$\mathcal{Q}(X) = \Big\{ 1 - \sigma(X)^{-1}[X - EX] \Big\}. \tag{2.2}$$

Another such pair is furnished by standard lower semideviation,

$$\mathcal{D}_-(X) = \sigma_-(X) = ||\,[X - EX]_-\,||_2, \quad \mathcal{Q}_- = \Big\{ Q \,\Big|\, EQ = 1,\ \inf Q > -\infty,\ ||Q - \inf Q||_2 \le 1 \Big\}, \tag{2.3}$$

where the risk identifier set for any nonconstant X is the singleton

$$\mathcal{Q}_-(X) = \Big\{ 1 - \sigma_-(X)^{-1}\Big(E[X - EX]_- - [X - EX]_- \Big) \Big\}. \tag{2.4}$$

Yet another such pair is furnished by standard upper semideviation,

$$\mathcal{D}_+(X) = \sigma_+(X) = ||\,[X - EX]_+\,||_2, \quad \mathcal{Q}_+ = \Big\{ Q \,\Big|\, EQ = 1,\ \sup Q < \infty,\ ||\sup Q - Q||_2 \le 1 \Big\}, \tag{2.5}$$

where the risk identifier set for any nonconstant X is the singleton

$$\mathcal{Q}_+(X) = \Big\{ 1 - \sigma_+(X)^{-1}\Big([X - EX]_+ - E[X - EX]_+ \Big) \Big\}. \tag{2.6}$$

Of these three deviation measures, only $\mathcal{D}_-$ is lower range dominated, in general.

Detail. These deviation measures and risk envelopes fit into a category covered by [19, Example 7]. The risk envelope in (2.1) has the equivalent description that $Q \in \mathcal{Q}$ if and only if $EQ = 1$ and there is a constant C for which $||C - Q||_2 \le 1$. In terms of $Z = C - Q$, where having $EQ = 1$ corresponds to having $C = 1 + EZ$, we can think of $\mathcal{Q}$ as consisting of all Q of the form $1 + EZ - Z$ with $||Z||_2 \le 1$. Maximizing $E[(EX - X)Q]$ over $Q \in \mathcal{Q}$ for arbitrary X, as we wish to do in order to determine $\mathcal{Q}(X)$, can be translated to maximizing $E[(EX - X)(1 + EZ - Z)]$ subject to $||Z||_2 \le 1$. Since $E[(EX - X)(1 + EZ - Z)] = E[(X - EX)Z]$, this is the same as choosing Z to maximize $E[(X - EX)Z]$ subject to $||Z||_2 \le 1$. For nonconstant X, the r.v. $X - EX$ is not the zero r.v. and the maximum is uniquely attained when $Z = (X - EX)/||X - EX||_2 = \sigma(X)^{-1}(X - EX)$. Correspondingly, then, the unique $Q \in \mathcal{Q}(X)$ is $Q = 1 + EZ - Z = 1 - \sigma(X)^{-1}[X - EX]$.

The pattern of argument for the other deviation measures is similar. The risk envelope $\mathcal{Q}_-$ in (2.3) can be described equivalently as the set of Q for which $EQ = 1$ and there is a constant $C \le Q$ such that $||Q - C||_2 \le 1$. The same change of variables turns this into the set of Q having the form $1 + EZ - Z$ for some $Z \le 0$ with $||Z||_2 \le 1$. The maximization of $E[(EX - X)Q]$ over $Q \in \mathcal{Q}$ thereby translates to the maximization of $E[(X - EX)Z]$ subject to $Z \le 0$ and $||Z||_2 \le 1$. For nonconstant X, the unique solution is $Z = -[X - EX]_-/||[X - EX]_-||_2$, which yields the singleton $\mathcal{Q}_-(X)$ in (2.4).

For $\mathcal{D}_+$, the only difference is that we instead maximize $E[(X - EX)Z]$ subject to $Z \ge 0$ and $||Z||_2 \le 1$. For nonconstant X, the maximum is attained uniquely by $Z = [X - EX]_+/||[X - EX]_+||$, and we get $\mathcal{Q}_+(X)$ as in (2.6). □

In order to express the risk identifiers in the next example succinctly, we introduce the notion of the *sign* of an r.v. Y by:

$$\operatorname{sign} Y = \text{ set of all } Z \in \mathcal{L}^2(\Omega) \text{ such that } \begin{cases} Z(\omega) = 1 & \text{when } Y(\omega) > 0, \\ Z(\omega) = -1 & \text{when } Y(\omega) < 0, \\ Z(\omega) \in [-1, 1] & \text{when } Y(\omega) = 0 \end{cases} \tag{2.7}$$

(almost surely). Note that if $Y(\omega) = 0$ with probability 0, $\operatorname{sign} Y$ consists of a Z that is essentially unique (i.e., almost surely), taking on only 1 or -1 according to the positivity or negativity of Y.

Example 2 (risk identifiers for mean absolute deviation and semideviations)**.** *An instance of a finite, continuous deviation measure on $\mathcal{L}^2(\Omega)$ with its associated risk envelope is furnished by mean absolute deviation,*

$$\mathcal{D}(X) = E|X - EX| = ||X - EX||_1, \qquad \mathcal{Q} = \left\{ Q \,\middle|\, EQ = 1,\ \sup Q - \inf Q \leq 2 \right\}. \tag{2.8}$$

The risk identifer set for any nonconstant X is given then in the notation (2.7) by

$$\mathcal{Q}(X) = \left\{ Q = 1 + EZ - Z \,\middle|\, Z \in \operatorname{sign}[X - EX] \right\}. \tag{2.9}$$

The same holds also for lower and upper semideviations

$$\mathcal{D}_-(X) = ||[X - EX]_-||_1, \qquad \mathcal{D}_+(X) = ||[X - EX]_+||_1, \tag{2.10}$$

but actually $||[X - EX]_-||_1 = ||[X - EX]_+||_1 = \frac{1}{2}||X - EX||_1$, so that $\mathcal{D}_-$ and $\mathcal{D}_+$ offer little that is different, except that they are lower range dominated, whereas $\mathcal{D}$ is not. Their risk envelopes $\mathcal{Q}_-$ and $\mathcal{Q}_+$ come out in the rescaled form

$$\mathcal{Q}_- = \mathcal{Q}_+ = \left\{ Q \,\middle|\, EQ = 1,\ \sup Q - \inf Q \leq 1 \right\}, \tag{2.11}$$

where the conditions entail $Q \geq 0$. Their risk identifiers similarly come out as

$$\mathcal{Q}_-(X) = \mathcal{Q}_+(X) = \left\{ Q = 1 + \tfrac{1}{2}(EZ - Z) \,\middle|\, Z \in \operatorname{sign}[X - EX] \right\}. \tag{2.12}$$

Detail. Again, these deviation measures and risk envelopes fit into the broader statement in [19, Example 7], where the lower range dominance is addressed as well. The lower range dominance can also be confirmed through the risk envelope characterization (1.3) for that property by noting that if $EQ = 1$ then necessarily $\inf Q \leq 1 \leq \sup Q$, so the inequality in (2.11) entails $\inf Q \geq 0$, which is the same as $Q \geq 0$.

We concentrate now on the characterization of the risk identifiers. In (2.8), having $\sup Q - \inf Q \leq 2$ corresponds to having $\min_C ||C - Q||_\infty \leq 1$, inasmuch as that minimum equals $\frac{1}{2}(\sup Q - \inf Q)$. As in Example 1, we approach the maximization in the definition of $\mathcal{Q}(X)$ by using the change of variables $Q = 1 + EZ - Z$ to translate it into the maximization of $E[(EX - X)Z]$ subject to $||Z||_\infty \leq 1$, i.e., $-1 \leq Z \leq 1$. This maximum is achieved by Z if and only if $Z(\omega) = 1$ almost surely when $X(\omega) - EX > 0$, whereas $Z(\omega) = -1$ almost surely when $X(\omega) - EX < 0$ (and both cases must indeed occur with positive probability for nonconstant X). This means $Z \in \operatorname{sign}[X - EX]$. The corresponding elements $Q \in \mathcal{Q}$ are then the ones described in (2.9).

The very definition of EX entails having $E[X - EX]_+ = E[X - EX]_-$, and since $E|X - EX| = E([X - EX]_+ + [X - EX]_-)$, it follows that $||\,[X - EX_+\,||_1 = ||\,[X - EX_-\,||_1 = \frac{1}{2}||X - EX||_1$. The assertions about risk envelopes and risk identifiers for these semideviations are supported by the principle that when a deviation measure $\mathcal{D}$ is rescaled to $\mathcal{D}' = \lambda\mathcal{D}$ for some $\lambda > 0$, its risk envelope $\mathcal{Q}$ is replaced by the set of elements $Q' = (1 - \lambda) + \lambda Q$ as Q ranges over $\mathcal{Q}$; cf. [19, Proposition 4(c)]. □

For the next example we will make use of the generalized *median* of a random variable X, this being defined by

$$\operatorname{med} X = \operatorname*{argmin}_C E|X - C| = \operatorname*{argmin}_C ||X - C||_1, \tag{2.13}$$

which is the same as

$$\operatorname{med} X = \Big\{ C \,\Big|\, \operatorname{prob}[X \le C] \ge \operatorname{prob}[X > C],\ \operatorname{prob}[X \ge C] \ge \operatorname{prob}[X < C] \Big\}.$$

Because $||X - C||_1$ thus has the same value for every $C \in \operatorname{med} X$ (when $\operatorname{med} X$ is not just a singleton), we denote that common value simply by $||X - \operatorname{med} X||_1$ so that

$$||X - \operatorname{med} X||_1 = \min_C ||X - C||_1 \text{ by definition.}$$

Example 3 (risk identifiers for worst-case deviation and semideviations). *An instance of a lower semicontinuous (but not necessarily finite) deviation measure on $\mathcal{L}^2(\Omega)$ with its risk envelope is furnished by worst-case deviation,*

$$\mathcal{D}(X) = \sup |X - EX| = ||X - EX||_\infty, \qquad \mathcal{Q} = \Big\{ Q \,\Big|\, EQ = 1,\ ||Q - \operatorname{med} Q||_1 \le 1 \Big\}, \tag{2.14}$$

The corresponding risk identifier set for nonconstant X is characterized by

$$Q \in \mathcal{Q}(X) \iff Q = 1 + EZ - Z \text{ with } \begin{cases} E|Z| = 1, \\ Z_+(\omega) = 0 \text{ when } X(\omega) - EX < ||X - EX||_\infty, \\ Z_-(\omega) = 0 \text{ when } EX - X(\omega) < ||X - EX||_\infty \end{cases} \tag{2.15}$$

(almost surely). Another instance of such a pair is furnished by lower worst-case deviation,

$$\mathcal{D}_-(X) = EX - \inf X = ||\,[X - EX]_-\,||_\infty, \qquad \mathcal{Q}_- = \Big\{ Q \,\Big|\, EQ = 1,\ Q \ge 0 \Big\}, \tag{2.16}$$

where the risk identifier set for any nonconstant X is characterized by

$$Q \in \mathcal{Q}_-(X) \iff \begin{cases} Q \ge 0,\ EQ = 1, \\ Q(\omega) = 0 \text{ when } X(\omega) > \inf X. \end{cases} \tag{2.17}$$

Yet another instance of such a pair is furnished by upper worst-case deviation,

$$\mathcal{D}_+(X) = \sup X - EX = ||\,[X - EX]_+\,||_\infty, \qquad \mathcal{Q}_+ = \Big\{ Q \,\Big|\, EQ = 1,\ \sup Q \le 2 \Big\}, \tag{2.18}$$

where the risk identifier set for any nonconstant X is characterized by

$$Q \in \mathcal{Q}_+(X) \iff \begin{cases} Q \le 2,\ EQ = 1, \\ Q(\omega) = 2 \text{ when } X(\omega) < \sup X. \end{cases} \tag{2.19}$$

Of these three deviation measures, $\mathcal{D}_-$ is lower range dominated but the others are generally not.

Proof. Once more, these deviation measures and their risk envelopes emerge from [19, Example 7]. The condition $||Q - \operatorname{med} Q||_1 \le 1$ in the formula for $\mathcal{Q}$ in (2.14) corresponds to $\min_C ||C - Q||_1 \le 1$, as seen through the definition of $\operatorname{med} Q$ above. To get the risk identifier formula in (2.15), we use this to view the maximization of $E[(EX - X)Q]$ over $Q \in \mathcal{Q}$ as referring to the maximization over all Q for which $EQ = 1$ and there is a constant C such that $||C - Q||_1 \le 1$. We approach that as in Examples 1 and 2 by changing the maximization variable to Z under $Q = 1 + EZ - Z$, where having $||Z||_1 \le 1$ corresponds to having $Q \in \mathcal{Q}$. Under this change of variables, $E[(EX - X)Q]$ becomes $E[(X - EX)Z]$, which we have to maximize subject only to $||Z||_1 \le 1$. Since X is not constant, cannot furnish the maximum without having $||Z||_1 \le 1$, i.e., $E|Z| = 1$, so that $|Z|$ is a probability density. Attainment corresponds to having this density concentrated (almost surely) in the subset of Ω where $|X(\omega) - EX|$

equals $\sup|X - EX|$, with the sign of $Z(\omega)$ conforming to the sign of $X(\omega) - EX$. This prescription for Z is captured by the conditions in (2.15), and the corresponding elements $Q = 1 + EZ - Z$ then make up $\mathcal{Q}(X)$.

The case of $\mathcal{Q}_-$ in (2.16) is simpler. We can directly appeal to the fact that the elements Q of $\mathcal{Q}_-(X)$ minimize $E[XQ]$ subject to $Q \geq 0$, $EQ = 1$, and see that they have the form in (2.17).

The condition $\sup Q \leq 2$ in the formula for $\mathcal{Q}_+$ in (2.18) is equivalent in the presence of $EQ = 1$ to $\sup Q - EQ \leq 1$ and therefore to the existence of a constant $C \geq Q$ with $||C - Q||_1 \leq 1$. In this case, the maximization of $E[(EX - X)Q]$ over $Q \in \mathcal{Q}$ converts under the change of variables $Q = 1 + EZ - Z$ to the maximization of $E[(X - EX)Z]$ subject to $Z \geq 0$ and $EZ \leq 1$. Since $X - EX$ must take on positive values with nonzero probability when X is not constant, the maximum is attained by the nonnegative elements that have $EZ = 1$ and are concentrated in $\{\,\omega \mid X(\omega) = \sup X\}$. The corresponding elements Q are given then by (2.19). □

Of course, when the state space Ω is finite, the deviation measures in Example 3 really are finite on $\mathcal{L}^2(\Omega)$ and therefore also continuous.

Our next examples draw on the concept of the conditional value-at-risk of X at a level $\alpha \in (0,1)$, which has several equivalent expressions, reviewed now briefly. First, in terms of the cumulative distribution function F_X on $I\!R$, recall that the *value-at-risk* of X at level α is defined by

$$\mathrm{VaR}_\alpha(X) = -\inf\{\, z \mid F_X(z) > \alpha\}. \tag{2.20}$$

The corresponding *conditional value-at-risk*, $\mathrm{CVaR}_\alpha(X)$ equals the conditional expectation of $-X$ subject to $-X > \mathrm{VaR}_\alpha(X)$ when F_X is continuous at $z = -\mathrm{VaR}_\alpha(X)$. But in general, to account properly for a possible discontinuity, it needs to be construed in expectation terms as $-\int_{-\infty}^{\infty} z dF_X^\alpha(z)$, where F_X^α gives the lower α-tail distribution associated with X; namely $F_X^\alpha(z) = \alpha^{-1}F_X(z)$ when $z < \mathrm{VaR}_\alpha(X)$, but $F_X^\alpha(z) = 1$ when $z \geq \mathrm{VaR}_\alpha(X)$. For the details, see [17], where the theoretical foundations for conditional value-at-risk were first laid out in full generality; the term "conditinal value-at-risk" itself originated in our earlier paper [16]. Alternatively there is the expression

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_p(X)dp, \tag{2.21}$$

which, before it was identified as representing $\mathrm{CVaR}_\alpha(X)$, was used by Acerbi [1] to define what he called "shortfall." Furthermore there is the minimization formula we brought to light in [16, 17],

$$\mathrm{CVaR}_\alpha(X) = \min_C \Big\{\, C + \alpha^{-1}E[X + C]_- \Big\}, \tag{2.22}$$

which promotes to major simplifications in computing solutions to optimization problems involving CVaR expressions. It is notable that in (2.22) the argmin set is a C interval having $\mathrm{VaR}_\alpha(X)$ as its right endpoint; cf. [16, 17]. Still more on conditional value-at-risk can be found in [13] and the book of Föllmer and Schied [7] (who altered the name from "conditional" to "average").

Example 4 (risk identifiers for deviations from conditional value-at-risk)**.** *For any $\alpha \in (0,1)$, a finite, continuous, lower range dominated deviation measure is furnished with its risk envelope by*

$$\mathcal{D}_\alpha(X) = \mathrm{CVaR}_\alpha(X - EX), \qquad \mathcal{Q}_\alpha = \Big\{\, Q \,\Big|\, EQ = 1,\ 0 \leq Q \leq \alpha^{-1} \Big\}. \tag{2.23}$$

The risk identifier set for any nonconstant X is characterized then by

$$Q \in \mathcal{Q}_\alpha(X) \iff \begin{cases} 0 \leq Q \leq \alpha^{-1},\ EQ = 1, \\ Q(\omega) = 0 \text{ when } X(\omega) > -\mathrm{VaR}_\alpha(X), \\ Q(\omega) = \alpha^{-1} \text{ when } X(\omega) < -\mathrm{VaR}_\alpha(X) \end{cases} \tag{2.24}$$

(almost surely). More generally, for any levels α_k in $(0,1)$ and weights $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$, a finite, continuous, lower range dominated deviation measure $\mathcal{D}$ and its risk envelope $\mathcal{Q}$ are given in terms of deviation measures $\mathcal{D}_{\alpha_k}$ and their risk envelopes $\mathcal{Q}_{\alpha_k}$ in the preceding notation by

$$\mathcal{D}(X) = \sum_{k=1}^m \lambda_k \mathcal{D}_{\alpha_k}(X), \qquad \mathcal{Q} = \Big\{ Q \,\Big|\, Q = \sum_{k=1}^m \lambda_k Q_k \text{ with } Q_k \in \mathcal{Q}_{\alpha_k} \Big\}. \tag{2.25}$$

The risk identifier sets are given then by

$$Q \in \mathcal{Q}(X) \iff Q = \sum_{k=1}^m \lambda_k Q_k \text{ for some choice of } Q_k \in \mathcal{Q}_{\alpha_k}(X). \tag{2.26}$$

Detail. The risk envelope $\mathcal{Q}_\alpha$ for $\mathcal{D}_\alpha$ was already been noted in [19, Example 4]. In determining $\mathcal{Q}_\alpha(X)$ for nonconstant X from the minimization or $E[XQ]$ subject to $0 \le Q \le \alpha^{-1}$ and $EQ = 1$, we recall that, by the definition of $\mathrm{VaR}_\alpha(X)$, we have

$$\mathrm{prob}\Big\{ \omega \,\Big|\, X(\omega) < -\mathrm{VaR}_\alpha(X) \Big\} \le \alpha, \qquad \mathrm{prob}\Big\{ \omega \,\Big|\, X(\omega) \le -\mathrm{VaR}_\alpha(X) \Big\} \ge \alpha.$$

It is clear, therefore, that the minimum in question is achieved by Q if and only if, in addition to satisfying $0 \le Q \le \alpha^{-1}$ and $EQ = 1$, it takes on the highest possible value α^{-1} almost surely when $X(\omega) < -\mathrm{VaR}_\alpha(X)$, but on the other hand vanishes almost surely when $X(\omega) > -\mathrm{VaR}_\alpha(X)$.

The pair in (2.25) specializes [19, Proposition 4]. To get the risk identifier set for X, we only have to observe that the minimum of $E[X(\lambda_1 Q_1 + \cdots + \lambda_r Q_r)]$ over all $Q_k \in \mathcal{Q}_k$, $k = 1, \ldots, r$, is achieved by separately minimizing $E[XQ_k]$ over $Q_k \in \mathcal{Q}_k$ for $k = 1, \ldots, r$. □

The deviation measure in (2.23) of Example 4 has previously been investigated by Bertsimas, Lauprete and Samarov [5], but only relative to special finite-dimensional subspaces of $\mathcal{L}^2(\Omega)$. The distribution functions of the nonconstant elements X in these subspaces must be continuous and (strictly) increasing between 0 and 1. No such restrictions are imposed here. Our result is applicable to discretely distributed random variables too, for instance.

3 Conditions for Minimum Deviation

We now take up the study of optimality in problems where a deviation measure $\mathcal{D}$ is minimized over a subset $\mathcal{X}$ of $\mathcal{L}^2(\Omega)$. With a view towards applications in finance in particular, we focus on the case where $\mathcal{X}$ is comprised of linear combinations of a given collection or r.v.'s,

$$r_i \in \mathcal{L}^2(\Omega), \quad i = 1, \ldots, n,$$

which may give the rates of return of various financial instruments indexed by i. Thus, we take

$$\mathcal{X} = \Big\{ X = x_1 r_1 + \cdots + x_n r_n \,\Big|\, (x_1, \ldots, x_n) \in S \Big\}, \tag{3.1}$$

where S is in turn a subset or $\mathbb{R}^n$ giving the admissible coefficient vectors $(x_1, \ldots, x_n)$. Moreover we suppose here that S is described by linear constraints on the coefficients:

$$x = (x_1, \ldots, x_n) \in S \iff \sum_{i=1}^n a_{ki} x_i \begin{cases} \ge b_k & \text{for } k = 1, \ldots, s, \\ = b_k & \text{for } k = s+1, \ldots, m. \end{cases} \tag{3.2}$$

The optimization problem we aim to study can be posed then as one in the variables x_i:

$(\mathcal{P})$ minimize $\mathcal{D}(x_1 r_1 + \cdots + x_n r_n)$ over all $x = (x_1, \ldots, x_n)$ in S.

In the financial setting, each choice of a vector $x = (x_1, \ldots, x_n)$ in S corresponds to an admissible *portfolio* formed from the available instruments $i = 1, \ldots, n$. The uncertain rate of return of this x-portfolio is given by the r.v. $X = x_1 r_1 + \cdots + x_n r_n$. The constraints (3.2) could involve expectations or requirements like $x_i \geq 0$ (for instruments that cannot be shorted). They could also enforce diversification requirements.

Theorem 1 (existence of deviation-minimizing portfolios)**.** *In problem $(\mathcal{P})$ with $\mathcal{D}$ lower semicontinuous and such that $\mathcal{D}$ is finite somewhere on $\mathcal{X}$, an optimal solution x^* always exists.*

Proof. Let $\mathcal{S}$ be the finite-dimensional subspace of $\mathcal{L}^2(\Omega)$ generated by the constant functions along with the r.v.'s r_i. We can identify $(\mathcal{P})$ with minimizing over $\mathcal{X}$ (which is a polyhedral convex subset of $\mathcal{S}$) the restriction of $\mathcal{D}$ to $\mathcal{S}$ (which is a lower semicontinuous convex function on $\mathcal{S}$ that is finite somewhere on $\mathcal{X}$). In that finite-dimensional setting, we can apply the following criterion in [14, Cor. 27.3.1] for guaranteeing the existence of an $X^* \in \mathcal{X}$ at which the minimum of $\mathcal{D}$ over $\mathcal{X}$ is attained: if an element $Y \in \mathcal{S}$ has the property that

$$\mathcal{D}(X+Y) \leq \mathcal{D}(X) \text{ for all } X \in \mathcal{S}, \tag{3.3}$$

then $-Y$ also has that property, i.e.,

$$\mathcal{D}(X-Y) \leq \mathcal{D}(X) \text{ for all } X \in \mathcal{S}. \tag{3.4}$$

In our setting, (3.3) entails $\mathcal{D}(Y) \leq \mathcal{D}(0) = 0$ because $\mathcal{S}$ contains the zero r.v. in particular, so that Y must be a constant C by D4. Then, however, (3.4) holds as well because $\mathcal{S}$ contains the constant r.v.'s C, and $\mathcal{D}(X \pm C) = \mathcal{D}(X)$ by D1. □

Although problem $(\mathcal{P})$ has a familiar appearance, with its system of m linear constraints on n variables, the difficulty to be kept in mind is that the objective function

$$f_{\mathcal{D}}(x) = f_{\mathcal{D}}(x_1, \ldots, x_n) = \mathcal{D}(x_1 r_1 + \cdots + x_n r_n) \tag{3.5}$$

cannot be counted on as being differentiable. This phenomenon can be traced to the risk envelope representation of $\mathcal{D}$ in (1.1) in recalling that $\mathcal{Q}$ can even be a polyhedral convex set when Ω is a finite set, and then only the finitely many extreme points of $\mathcal{Q}$ really need to enter in the maximization. However, another source for a lack of differentiability of $f_{\mathcal{D}}$ in finance has been demonstrated out at the end of our paper [20]. The trouble there arises because of options. The underlying r_i's come from derivative instruments, which can create portfolios having discontinuously distributed returns.

The convexity of $f_{\mathcal{D}}$, which follows from axioms D2 and D3 on $\mathcal{D}$, does at least entail $f_{\mathcal{D}}$ being differentiable almost everywhere when it is finite on $\mathbb{R}^n$ [14, Theorem 25.1]. But that is of little help, since there is no way to guarantee that a solution x^* lies at one of the points of differentiability.

Rather than getting into the properties of $f_{\mathcal{D}}$ associated with its expression in terms of $\mathcal{D}$, our tactic for handling this difficulty will be to work directly and more simply with $\mathcal{D}$ itself. Convex analysis provides a robust substitute for the differentiability of $\mathcal{D}$ in the notion of a subgradient. For $\mathcal{D}$, like any other convex functional on $\mathcal{L}^2(\Omega)$, a *subgradient* at X is by definition an element Y of $\mathcal{L}^2(\Omega)$ such that

$$\mathcal{D}(X') \geq \mathcal{D}(X) + E[(X'-X)Y] \text{ for all } X'. \tag{3.6}$$

The set of such subgradients Y at X is denoted by $\partial\mathcal{D}(X)$. It is always a closed, convex set, possibly reducing to a single element, but in some situations it could be empty, as in particular when $\mathcal{D}(X) = \infty$. The representation of $\mathcal{D}$ in (1.1) leads through the basics of convex analysis to a convenient description of subgradients by way of the risk envelope $\mathcal{Q}$, or more specifically, the various risk identifier sets $\mathcal{Q}(X)$.

Proposition 1 (subgradients of deviation measures)**.** *For a lower semicontinuous deviation measure $\mathcal{D}$, the set of subgradients of $\mathcal{D}$ at X is expressible through the risk identifier set $\mathcal{Q}(X)$ as*

$$\partial\mathcal{D}(X) \,=\, 1-\mathcal{Q}(X) \,=\, \Big\{\, Y \,\Big|\, Y = 1-Q \text{ for some } Q \in \mathcal{Q}(X) \Big\}. \tag{3.7}$$

In particular, therefore,

$$\partial\mathcal{D}(X) \,=\, 1-\mathcal{Q} \,=\, \Big\{\, Y \,\Big|\, Y = 1-Q \;\text{ for some } Q \in \mathcal{Q} \Big\} \text{ when } X \text{ is constant.} \tag{3.8}$$

Proof. Under the change of variables $Q = 1-Y$ and $X' = X+Z$, the subgradient inequality (3.6) for Y becomes the condition on Q that

$$\mathcal{D}(X+Z) \geq \mathcal{D}(X) + EZ - E[ZQ] \;\text{ for all } Z. \tag{3.9}$$

Our task is to demonstrate that this is equivalent to having $Q \in \mathcal{Q}(X)$. First, suppose $Q \in \mathcal{Q}(X)$. Then by (1.1) and (1.4) we have $\mathcal{D}(X) = E[(EX-X)Q]$ while, for all Z,

$$\mathcal{D}(X+Z) \geq E[(E(X+Z)-(X+Z))Q] = E[(EX-X)Q] + EZ - E[ZQ].$$

This gives (3.9). Second, suppose instead that Q satisfies (3.9). From the special case of (3.9) where $Z=-X$, so $\mathcal{D}(X+Z)=0$ by D2, we have

$$\mathcal{D}(X) \leq EX - E[XQ] < \infty. \tag{3.10}$$

On the other hand, because $\mathcal{D}(X+Z) \leq \mathcal{D}(X)+\mathcal{D}(Z)$ by D3, we see from (3.9) that $\mathcal{D}(X)+\mathcal{D}(Z) \geq \mathcal{D}(X) + EZ - E[ZQ]$, where the finiteness of $\mathcal{D}(X)$ guaranteed by (3.10) allows cancellation of $\mathcal{D}(X)$ from both sides. This being true for all Z, we have

$$\mathcal{D}(Z) \geq EZ - E[ZQ] \text{ for all } Z, \tag{3.11}$$

hence $Q \in \mathcal{Q}$ by (1.2). Moreover by combining the inequality in (3.11) for $Z = X$ with the one in (3.10) we get $\mathcal{D}(X) = EX - E[XQ]$, so actually $Q \in \mathcal{Q}(X)$ by (1.4). The special case in (3.8) reflects (1.6). □

Proposition 2 (subgradient condition for optimality)**.** *In the minimization of a lower semicontinuous deviation measure $\mathcal{D}$ over $\mathcal{X}$, a subset of $\mathcal{L}^2(\Omega)$ that is convex and closed, a sufficient condition for an element X^* to be optimal is that*

$$X^* \in \mathcal{X} \text{ and there exists } Y^* \in \partial\mathcal{D}(X^*) \text{ such that } E[XY^*] \geq E[X^*Y^*] \text{ for all } X \in \mathcal{X}. \tag{3.12}$$

This condition is necessary for X^* to be optimal when $\mathcal{D}$ is not just lower semicontinuous but also finite (hence continuous).

Proof. This general form of optimality result would be valid for minimizing any lower semicontinuous convex functional, aside from its expression adapted to the expectation inner product we are utilizing here in $\mathcal{L}^2(\Omega)$. The sufficiency is elementary: by combining the subgradient inequality behind having $Y^* \in \partial\mathcal{D}(X^*)$, namely that $\mathcal{D}(X) \geq \mathcal{D}(X^*) + E[(X-X^*)Y^*]$ for all $X \in \mathcal{L}^2(\Omega)$, together with the inequality in (3.12) that $E[(X-X^*)Y^*] \geq 0$ for all $X \in \mathcal{X}$, we see that $\mathcal{D}(X) \geq \mathcal{D}(X^*)$ for all $X \in \mathcal{X}$. The necessity is not so immediate but is likewise well known; see [15, Example 1, p. 57]. □

Theorem 2 (portfolios that minimize deviation). *Let $\mathcal{D}$ be lower semicontinuous. Suppose that $x^* = (x_1^*, \ldots, x_n^*)$ satisfies the constraints in (3.2), and let $X^* = x_1^* r_1 + \cdots + x_n^* r_n$. A sufficient condition then for x^* to be optimal in problem $(\mathcal{P})$ is the existence of a risk identifier*

$$Q^* \in \mathcal{Q}(X^*) = \mathcal{Q}(x_1^* r_1 + \cdots + x_n^* r_n) \tag{3.13}$$

together with multipliers $\lambda_1, \ldots, \lambda_m$ such that

$$E\Big[(Er_i - r_i)Q^*\Big] = \sum\nolimits_{k=1}^m \lambda_k a_{ki} \ \text{ for } i = 1, \ldots, n \tag{3.14}$$

with λ_k arbitrary in sign for $k \in \{s+1, \ldots, m\}$ but

$$\lambda_k \begin{cases} \geq 0 & \text{for } k \in \{1, \ldots, s\} \text{ such that } \sum_{i=1}^n a_{ki} x_i^* = b_k, \\ = 0 & \text{for } k \in \{1, \ldots, s\} \text{ such that } \sum_{i=1}^n a_{ki} x_i^* > b_k. \end{cases} \tag{3.15}$$

This condition, which entails having

$$\mathcal{D}(X^*) = \mathcal{D}(x_1^* r_1 + \cdots + x_n^* r_n) = \sum\nolimits_{k=1}^m \lambda_k b_k, \tag{3.16}$$

is moreover necessary for optimality when $\mathcal{D}$ is everywhere finite and continuous.

Proof. The set $\mathcal{X}$ in (3.1)–(3.2) is convex and closed in $\mathcal{L}^2(\Omega)$. For the minimization of $\mathcal{D}$ over $\mathcal{X}$, Propositions 1 and 2 combine to give us an optimality condition in terms of the existence of $Q^* \in \mathcal{Q}(X^*)$ (hence with $EQ^* = 1$) such that

$$E[X(1 - Q^*)] \geq E[X^*(1 - Q^*)] \ \text{ for all } \ X \in \mathcal{X}, \tag{3.17}$$

this condition being sufficient in general and necessary when $\mathcal{D}$ is finite and continuous. Our job is to translate the inequality in (3.17) into the multiplier condition in (3.14)–(3.15).

First, we rewrite the inequality in (3.17) in terms of $X = x_1 r_1 + \cdots + x_n r_n$ and $X^* = x_1^* r_1 + \cdots + x_n^* r_n$ as saying that

$$(x_1^*, \ldots, x_n^*) \ \text{ minimizes } \ x_1 E[r_1(1 - Q^*)] + \cdots + x_n E[r_n(1 - Q^*)] \ \text{ over } \ (x_1, \ldots, x_n) \in S,$$

or in other words, minimizes over the constraints in (3.2). This minimization is an instance of linear programming in the variables x_i for which optimality is captured by Lagrange multipliers satisfying the sign conditions in (3.15) along with

$$E[r_i(1 - Q^*)] = \sum\nolimits_{k=1}^m \lambda_k a_{ki} \ \text{ for } \ i = 1, \ldots, n.$$

It remains then only to observe that, since $EQ^* = 1$, we have $E[r_i(1 - Q^*)] = E[(Er_i - r_i)Q^*]$.

The theorem's final claim in (3.16) is justified from the fact that $\mathcal{D}(X^*) = E[(EX^* - X^*)Q^*]$ by (1.5), whereas $E[(EX^* - X^*)Q^*] = \sum_{i=1}^n x_i^* E[(Er_i - r_i)Q^*]$. On the basis of (3.14), this sum equals

$$\sum\nolimits_{i=1}^n x_i^* \Big(\sum\nolimits_{k=1}^m \lambda_k a_{ki} \Big) = \sum\nolimits_{k=1}^m \lambda_k \Big(\sum\nolimits_{i=1}^n a_{ki} x_i^* \Big),$$

which in turn comes out as $\sum_{k=1}^m \lambda_k b_k$ by virtue of the constraints satisfied by the coefficients x_i^* and the sign conditions on the multipliers λ_k. □

Theorem 2 can be applied to a wide range of deviation measures by drawing on the examples in the preceding section and in particular the formulas provided there for the risk identifiers, which enter through (3.13). An especially interesting feature of the optimality condition in Theorem 2 is the way that the expressions

$$E[(Er_i - r_i)Q^*] = \mathrm{covar}(Er_i - r_i, Q^*), \tag{3.18}$$

involving the *downsides* of the r.v.'s r_i, invite comparison with the corresponding expression for the portfolio r.v. X^*, which in fact yields the optimal deviation: since $Q^* \in \mathcal{Q}(X^*)$, we have from (1.4) that

$$\mathcal{D}(X^*) = E[(EX^* - X^*)Q^*] = \mathrm{covar}\,(EX^* - X^*, Q^*). \tag{3.19}$$

Theorem 2 leads to an interesting duality involving not only multipliers λ_k but also elements of the risk envelope $\mathcal{Q}$. We can consider, as dual to $(\mathcal{P})$, the problem

$$(\mathcal{P}^d) \qquad \begin{cases} \text{choose } Q \in \mathcal{Q} \text{ and } \lambda_1, \ldots, \lambda_m \text{ with } \lambda_k \geq 0,\ k = 1, \ldots, s, \text{ to maximize} \\ \sum_{k=1}^m \lambda_k b_k \text{ subject to } E[(Er_i - r_i)Q] - \sum_{k=1}^m \lambda_k a_{ki} = 0,\ i = 1, \ldots, n. \end{cases}$$

This is suggested by the tight connection that emerges between the *optimal values* in $(\mathcal{P})$ and $(\mathcal{P}^d)$, i.e., between the minimum deviation in $(\mathcal{P})$ and the maximum in $(\mathcal{P}^d)$, along with other relationships.

Theorem 3 (dual problem in risk envelope format)**.** *Let $\mathcal{D}$ be everywhere finite and continuous, and let x^* furnish a solution to problem $(\mathcal{P})$. Then the elements Q^* and $\lambda_1, \ldots, \lambda_m$ that furnish a solution to problem $(\mathcal{P}^d)$ are the ones which satisfy, with x^*, the optimality condition in Theorem 2. Furthermore,*

$$[\text{ optimal value in } (\mathcal{P}) \text{ }] = [\text{ optimal value in } (\mathcal{P}^d) \text{ }]. \tag{3.20}$$

Proof. First consider any x satisfying the constraints in $(\mathcal{P})$ and any Q, $\lambda_1, \ldots, \lambda_m$ satisfying the constraints in $(\mathcal{P}^d)$. Let $X = x_1 r_1 + \cdots + x_n r_n$. We have from (1.1) that

$$\mathcal{D}(X) \geq E[(EX - X)Q] = \sum\nolimits_{i=1}^n x_i E[(Er_i - r_i)Q]$$

where the equations in $(\mathcal{P}^d)$ turn this sum into

$$\sum\nolimits_{i=1}^n x_i \left[\sum\nolimits_{k=1}^m \lambda_k a_{ki}\right] = \sum\nolimits_{k=1}^m \lambda_k \left[\sum\nolimits_{i=1}^n a_{ki} x_i\right] \geq \sum\nolimits_{k=1}^m \lambda_k b_k.$$

The outer inequality in this chain, $\mathcal{D}(X) \geq \sum_{k=1}^m \lambda_k b_k$, reveals that the minimum in $(\mathcal{P})$ cannot be less than the maximum in $(\mathcal{P}^d)$. The optimality condition in Theorem 2, however, turns the inequality into an equation. Hence the minimum in $(\mathcal{P})$ must equal the maximum in $(\mathcal{P}^d)$, and the elements satisfying the optimality condition achieve these extremes. □

Theorem 3 could be of particular interest when $(\mathcal{P}^d)$ is a problem of linear programming, which occurs whenever the future state space Ω is finite and the risk envelope $\mathcal{Q}$ is polyhedral, it being specified by additional linear constraints. This would hold in Examples 2, 3 and 4, for instance.

4 Applications to Basic Portfolio Theory

The applications of Theorem 2 that we especially wish to pursue concern the classical paradigm for balancing the expected return in a portfolio against the uncertainty in that return as captured by its standard deviation. What happens when standard deviation is replaced by some other measure of deviation?

In [20], we studied this question in depth and provided answers revolving around a generalized "one-fund theorem" and the existence of special portfolios, called "master funds," which can serve the interests of all the investors who regard uncertainty as suitably captured by the same deviation measure $\mathcal{D}$. We did not, however, have the tools in [20] to characterize the master funds by conditions resembling those in the standard "capital asset pricing model" (CAPM). Now we do have the right tools, and we want to use them to complete the picture of master funds by providing generalized CAPM-like conditions tailored to any choice of $\mathcal{D}$.

In the classical situation we wish to address, the r.v.'s r_i for $i = 1, \dots, n$ give the rates of return of financial instruments, as already suggested, but there is also a special instrument having a riskless rate of return r_0, which can be interpreted as a constant r.v. It is assumed here that, in contrast, the other instruments are risky, and indeed that no positive risk-free return can be constructed from them. Specifically,

$$X = x_1 r_1 + \cdots + x_n r_n \text{ is assumed to be nonconstant for every nonzero } x = (x_1, \dots, x_n). \tag{4.1}$$

The basic problem in these circumstances, with respect to a parameter value $\Delta > 0$, is to choose $x_0, x_1, \dots, x_n$ to

$$\mathcal{P}(\Delta) \qquad \text{minimize } \mathcal{D}(x_0 r_0 + x_1 r_1 + \cdots + x_n r_n) \text{ subject to } \begin{cases} x_0 + x_1 + \cdots + x_n = 1, \\ x_0 r_0 + x_1 E r_1 + \cdots + x_n E r_n \geq r_0 + \Delta. \end{cases}$$

The first constraint requires that the x_i's, all together, must constitute a "unit investment." The second constraint requires the expected rate of return of the x-portfolio to exceed the risk-free rate r_0 by at least the given amount $\Delta > 0$. There are no sign constraints or bounds on the x_i's in this formulation. (A negative x_i refers to a "short position" in the instrument in question.)

Problem $\mathcal{P}(\Delta)$ can be reconstituted in a simpler form which facilitates the analysis. First, of course, we have

$$\mathcal{D}(x_0 r_0 + x_1 r_1 + \cdots + x_n r_n) = \mathcal{D}(x_1 r_1 + \cdots + x_n r_n) \tag{4.2}$$

in view of axiom D1. On the other hand, the unit investment equation in $\mathcal{P}(\Delta)$ makes the x_0 variable redundant:

$$x_0 = 1 - x_1 - \cdots - x_n. \tag{4.3}$$

This expression for x_0 can be substituted into the constraint on rate of return in $\mathcal{P}(\Delta)$. The resulting problem is then one in $x_1, \dots, x_n$ alone:

$$\mathcal{P}_0(\Delta) \qquad \text{minimize } \mathcal{D}(x_1 r_1 + \cdots + x_n r_n) \text{ subject to } x_1(E r_1 - r_0) + \cdots + x_n(E r_n - r_0) \geq \Delta.$$

This problem fits into the format of (3.2) as concerned with a single linear inequality constraint. The way is then open to applying Theorem 2.

Theorem 4 (optimality rule for classical portfolios). *In problem $\mathcal{P}_0(\Delta)$ for a lower semicontinuous deviation measure $\mathcal{D}$ and any $\Delta > 0$, a sufficient condition for the optimality of an x^*-portfolio for which $X^* = x_1^* r_1 + \cdots + x_n^* r_n$ satisfies the constraint in $\mathcal{P}_0(\Delta)$ is the existence of a risk identifier*

$$Q^* \in \mathcal{Q}(X^*) = \mathcal{Q}(x_1^* r_1 + \cdots + x_n^* r_n) \tag{4.4}$$

together with a multiplier $\lambda > 0$ such that

$$E\Big[(E r_i - r_i) Q^*\Big] = \lambda [E r_i - r_0] \text{ for } i = 1, \dots, n. \tag{4.5}$$

In fact the multiplier then has to be

$$\lambda = \mathcal{D}(X^*)/\Delta > 0, \tag{4.6}$$

and the constraint in $\mathcal{P}_0(\Delta)$ must actually be satisfied as an equation. When $\mathcal{D}$ is finite and continuous, this sufficient condition is also necessary for optimality.

Proof. In expressing the constraint in $\mathcal{P}_0(\Delta)$ in the pattern of (3.2), we take $a_{1i} = (Er_i - r_0)$ and $b_1 = \Delta$. For x^* fulfilling this constraint, optimality corresponds to the existence of $Q^* \in \mathcal{Q}(X^*)$ and a multiplier λ such that

$$E\Big[(Er_i - r_i)Q^*\Big] = \lambda(Er_i - r_0) \text{ for } i = 1, \ldots, n, \tag{4.7}$$

where $\lambda \geq 0$, but if the constraint is slack then necessarily $\lambda = 0$. Theorem 2 also notes that this implies $\mathcal{D}(X^*) = \lambda\Delta$, hence $\lambda = \mathcal{D}(X^*)/\Delta$. The constraint in $\mathcal{P}_0(\Delta)$ could not be satisfied with $x^* = 0$, inasmuch as $\Delta > 0$, hence necessarily $\mathcal{D}(X^*) > 0$ by (4.1). Therefore $\lambda > 0$, and the constraint must hold as an equation. □

For problem $\mathcal{P}_0(\Delta)$ with $\Delta \leq 0$, incidently, a solution is obtained by taking $x^* = 0$. This is evident, but note also that this is the only solution, due to (4.1).

Our "one-fund" result in [20] confirms that if x^* solves $\mathcal{P}_0(\Delta)$ for some $\Delta > 0$, then cx^* solves $\mathcal{P}_0(c\Delta)$ for any scaling factor $c > 0$. This suggests a normalization in which the focus is on Δ^* being such that $\mathcal{P}_0(\Delta^*)$ has a solution x^* satisfying $x_1^* + \cdots + x_n^* = 1$. Then for any $\Delta > 0$ a solution x to $\mathcal{P}_0(\Delta)$ can be obtained by taking $x = cx^*$ for $c = 1/(x_1^* + \cdots + x_n^*)$. The trouble is, though, that there is no guarantee of the existence of $\Delta^* > 0$ having the targeted property (although this seems to be taken for granted in textbooks on the subject).

It is possible for a solution x to $\mathcal{P}_0(\Delta)$ to fail to have $x_1 + \cdots + x_n > 0$ and hence not be representable through such rescaling. Moreover, as we laid out in detail in [20], instances where solutions have $x_1 + \cdots + x_n = 0$ or $x_1 + \cdots + x_n < 0$ are unavoidable when a full range of possibilities for the risk-free interest rate r_0 is allowed. That analysis gave rise to the following definition, relative to the positive and negative cases, to which we now add a term covering the case where the sum is 0.

Definition 2 (master funds)**.** *The x^*-portfolio for $x^* = (x_1^*, \ldots, x_n^*)$ provides a master fund of positive type with respect $\mathcal{D}$ if x^* solves $\mathcal{P}_0(\Delta^*)$ for some $\Delta^* > 0$, and $x_1^* + \cdots + x_n^* = 1$. In contrast, it provides a master fund of negative type if instead it has $x_1^* + \cdots + x_n^* = -1$, whereas it provides a master fund of threshold type if instead it has $x_1^* + \cdots + x_n^* = 0$.*

Whether a master fund of one of these types exists depends on the size of r_0. When r_0 is below a certain level, a master fund of positive type exists, but no other type. When r_0 is above a certain level, a master fund of negative type exists, but no other type. In the transition between these cases, master funds of threshold type (not previously given a name) can exist and sometimes other types as well. The facts of the matter have thoroughly been brought out in our paper [20] and need not be restated here.

Rather, we move directly to characterizing the master funds from the standpoint of Theorem 4. In order put the result on a close footing with the classical one for $\mathcal{D} = \sigma$, we introduce coefficients which reduce to the classical "betas" in that case but can serve more widely.

Definition 3 (generalized betas)**.** *For any nonzero $x^* = (x_1^*, \ldots, x_n^*)$, let $\mathcal{B}(x^*) = \mathcal{B}(x_1^*, \ldots, x_n^*)$ denote the set of vectors $\beta = (\beta_1, \ldots, \beta_n)$ that are obtainable by taking a risk identifier $Q^* \in \mathcal{Q}(X^*)$, where $X^* = x_1^* r_1 + \cdots + x_n^* r_n$, and setting*

$$\beta_i = \frac{E[(Er_i - r_i)Q^*]}{E[(EX^* - X^*)Q^*]} = \frac{\operatorname{covar}(-r_i, Q^*)}{\mathcal{D}(X^*)} \quad \text{for } i = 1, \ldots, n. \tag{4.8}$$

The two ways of expressing β_i agree, and make sense, because $\text{covar}(EX^* - X^*, Q^*) = \mathcal{D}(X^*)$ by (1.4), and $\mathcal{D}(X^*) > 0$ under (4.1) and our assumption that $x^* \neq 0$. Observe that

$$\beta_1 x_1^* + \cdots + \beta_n x_n^* = 1 \quad \text{when } \beta \in \mathcal{B}(x^*). \tag{4.9}$$

This can be seen by multiplying the equations in (4.8) by x_i^* and then adding them up, since

$$x_1^*(Er_1 - r_1) + \cdots + x_n^*(Er_n - r_n) = EX^* - X^*.$$

Theorem 5 (characterization of master funds)**.** *Let the deviation measure $\mathcal{D}$ be finite and continuous.*

(a) *An x^*-portfolio with $x_1^* + \cdots + x_n^* = 1$ furnishes a master fund of positive type if and only if $EX^* > r_0$ for $X^* = x_1^* r_1 + \cdots + x_n^* r_n$ and there is a vector $\beta \in \mathcal{B}(x^*)$ such that*

$$Er_i - r_0 \;=\; \beta_i \left[EX^* - r_0 \right] \;\; \textit{for } i = 1, \ldots, n. \tag{4.10}$$

(b) *An x^*-portfolio with $x_1^* + \cdots + x_n^* = -1$ furnishes a master fund of negative type if and only if $EX^* > -r_0$ for $X^* = x_1^* r_1 + \cdots + x_n^* r_n$ and there is a vector $\beta \in \mathcal{B}(x^*)$ such that*

$$Er_i - r_0 \;=\; \beta_i \left[EX^* + r_0 \right] \;\; \textit{for } i = 1, \ldots, n. \tag{4.11}$$

(c) *An x^*-portfolio with $x_1^* + \cdots + x_n^* = 0$ furnishes a master fund of threshold type if and only if $EX^* > 0$ for $X^* = x_1^* r_1 + \cdots + x_n^* r_n$ and there is a vector $\beta \in \mathcal{B}(x^*)$ such that*

$$Er_i - r_0 \;=\; \beta_i \left[EX^* \right] \;\; \textit{for } i = 1, \ldots, n. \tag{4.12}$$

Proof. In the light of Definition 2, the issue in (a) is whether, in combination with $x_1^* + \cdots + x_n^* = 1$ and $EX^* > r_0$, the existence of $\beta \in \mathcal{B}(x^*)$ satisfying (4.10) is equivalent to the existence of $\Delta^* > 0$ such that X^* solves $\mathcal{P}_0(\Delta^*)$. We approach this through the optimality condition for $\mathcal{P}_0(\Delta^*)$ that emerges from Theorem 4, which is both necessary and sufficient by our assumption that $\mathcal{D}$ is finite and continuous.

When the optimality condition in Theorem 4 is fulfilled with respect to x^* for some $\Delta^* > 0$, we can use the fact that the multiplier has to be $\lambda = \mathcal{D}(X^*)/\Delta^* > 0$ to rewrite the equations (4.5) in the form

$$\left[\frac{\Delta^*}{\mathcal{D}(X^*)} \right] E\Big[(Er_i - r_i) Q^* \Big] = Er_i - r_0 \quad \text{for } i = 1, \ldots, n. \tag{4.13}$$

We also know from Theorem 4 that the optimality condition entails having the constraint in $\mathcal{P}_0(\Delta^*)$ hold with equality: $x_1^*(Er_1 - r_0) + \cdots + x_n^*(Er_n - r_0) = \Delta^*$. Since

$$x_1^*(Er_1 - r_0) + \cdots + x_n^*(Er_n - r_0) = EX^* - r_0 \quad \text{when } x_1^* + \cdots + x_n^* = 1, \tag{4.14}$$

it follows that $EX^* - r_0 = \Delta^*$. Hence $EX^* - r_0 > 0$, and we can put $EX^* - r_0$ in place of Δ^* in (4.13). The equations in (4.13) then become the desired ones in (4.10) by virtue of definition of the coefficients β_i. Thus, we have the properties in (a).

Conversely, suppose the properties in (a) hold for x^*. By taking Δ^* to be $EX^* - r_0$, we have $\Delta^* > 0$, and the constraint in $\mathcal{P}_0(\Delta^*)$ is fulfilled with equality, due to (4.14). By setting $\lambda = \mathcal{D}(X^*)/\Delta^* > 0$, we can next rewrite the equations in (4.10) in the form of the multiplier equations in Theorem 4. The conclusion then from Theorem 4 is that x^* yields optimality in $\mathcal{P}_0(\Delta^*)$.

In (b), we instead need to work with the fact that $x_1^*(Er_1 - r_0) + \cdots + x_n^*(Er_n - r_0) = EX^* + r_0$ when $x_1^* + \cdots + x_n^* = -1$, but otherwise the argument is virtually identical. That is the case similarly in (c), where $x_1^*(Er_1 - r_0) + \cdots + x_n^*(Er_n - r_0) = EX^*$ when $x_1^* + \cdots + x_n^* = 0$. □

The generality of Theorems 4 and 5 deserves emphasis. Other researchers studying nonstandard deviations have dealt with special classes of measures and generally have narrowed the scope of their results by supposing the uniqueness of the optimal portfolio in question, or additionally in the case of a master fund, that it is of positive type. They have also relied on the function $f_{\mathcal{D}}$ in (3.5) being differentiable. Theorems 4 and 5, in contrast, do not have these limitations. In particular, the characterization of master funds in Theorem 5 is both necessary and sufficient, in contrast to other work (focused anyway just on master funds of positive type) in which only the necessity is developed, under various special assumptions.

5 Examples of CAPM-like Relations

We now explore the CAPM-like characterization of master funds with respect to a variety of choices of the deviation measure $\mathcal{D}$. We retain the notation that

$$X^* = x_1^* r_1 + \cdots + x_n^* r_n \text{ for } x^* = (x_1^*, \ldots, x_n^*)$$

and work toward insights into various realizations of β-equations in (4.10) and (4.11). The main issue in our examples for each of the choices of $\mathcal{D}$ is the form of the β coefficients of Definition 3 that appear in the equations of Theorem 5. Although we hold back here from any pricing interpretation of these CAPM-like equations, we note comparisons in some cases with the "beta" formulas derived by other researchers who may have been motivated by such an interpretation.

Example 5 (master funds for standard deviation and semideviations)**.** *When $\mathcal{D} = \sigma$, the β coefficients in Definition 3 and in the master fund characterization in Theorem 5 are uniquely determined and take the form*

$$\beta_i = \frac{\operatorname{covar}(r_i, X^*)}{\sigma^2(X^*)}. \tag{5.1}$$

For the standard lower semideviation $\mathcal{D}_- = \sigma_-$, the form is instead

$$\beta_i = \frac{\operatorname{covar}(-r_i, [X^* - EX^*]_-)}{\sigma_-^2(X^*)}, \tag{5.2}$$

whereas for the standard upper semideviation $\mathcal{D}_+ = \sigma_+$ it is

$$\beta_i = \frac{\operatorname{covar}(r_i, [X^* - EX^*]_+)}{\sigma_+^2(X^*)}. \tag{5.3}$$

Detail. This comes from invoking the risk identifiers of Example 1 in the β coefficients of Definition 1. For $\mathcal{D} = \sigma$, the unique element of $\mathcal{Q}(X^*)$ according to (2.2) is $Q^* = 1 - \sigma(X^*)^{-1}[X^* - EX^*]$, which makes $\operatorname{covar}(-r_i, Q^*) = \operatorname{covar}(r_i, \sigma(X^*)^{-1}[X^* - EX^*])$. Then $\beta_i = \sigma(X^*)^{-2}\operatorname{covar}(r_i, X^* - EX^*)$, where the EX^* can be dropped as making no difference.

For $\mathcal{D}_- = \sigma_-$, the only $Q^* \in \mathcal{Q}(X^*)$ is $Q^* = 1 - \sigma_-(X^*)^{-1}(E[X^* - EX^*] - [X^* - EX^*]_-)$ by (2.4). Then $\operatorname{covar}(-r_i, Q^*) = \operatorname{covar}(-r_i, \sigma_-(X^*)^{-1}[X^* - EX^*]_-)$, so $\beta_i = \sigma_-(X^*)^{-2}\operatorname{covar}(-r_i, [X^* - EX^*]_-)$. This time the EX^* cannot be dropped. For $\mathcal{D}_+ = \sigma_+$ the pattern is the similar. □

In this framework of σ, σ_- and σ_+, a unique master fund of positive type exists only as long as the risk-free rate r_0 is not too high, i.e., not above threshold in [20, Theorem 6]. Beyond that threshold, a unique master fund of negative type exists instead.[4]

For standard deviation σ itself, the coefficients β_i in (5.1) turn the relations in (4.10) into the *standard CAPM equations* in terms of the expected rate of return of this master fund (or so-called "market portfolio"). These classical covariance relations have been interpreted as furnishing a one-factor predictive model in the form

$$r_i - r_0 \approx \beta_i[X^* - r_0] \text{ for } i = 1, \dots, n. \tag{5.4}$$

This is based conceptually on a supposition that all investors seek essentially to minimize standard deviation when they put together a portfolio at a specified level of expected gain.

It is tempting to think that the CAPM-like equations in (4.10) of Theorem 5 might be able to take on such a role as in (5.4) more widely, for other deviation measures. For instance, why not view the version with β_i as in (5.2), or the version from (5.3), in this same light? One must be careful not to jump directly to such an interpretation. We are operating here from a distinctly different standpoint, where the investors employing any particular deviation measure $\mathcal{D}$ are viewed only as a subgroup of all the investors, perhaps just a small subgroup. There is little basis for believing that the actions of such a subgroup ought to have a determining influence on market behavior as a whole.

Another issue which must not be ignored, in the general picture of our CAPM-like equations (4.10), is that the coefficients β_i might not be uniquely determined by $\mathcal{D}$ and $r_0, r_1, \dots, r_n$. This might happen because, in contrast to the case for standard deviation, the set $\mathcal{B}(x^*)$ from which the β vector is selected need not be a singleton, and more than one candidate β could conceivably work for x^*. It could also happen, perhaps, because x^* itself is not uniquely determined through the criterion of optimality. Yet another possibility is that x^* might be the unique solution to $\mathcal{P}_0(\Delta^*)$ for a particular Δ^*, but another portfolio, corresponding to a Δ different from Δ^*, might furnish a different master fund. This could arise from a "flat spot" on the efficient frontier; cf. [20, Figures 3 and 3a].

Of course, it is conceivable nonetheless that, through statistical analysis, the CAPM-like relations with respect to one or maybe several alternative deviation measures in combination, may lead to interesting predictive models of type (5.4) with advantages over the classical CAPM. (The underlying assumption of the classical model is anyway not beyond controversy.) That is not a topic to be taken up in this paper, however.

It may be noted, though, that the use of σ_- in place of σ was advocated early in the development of portfolio theory by its star founder, Markowitz, in [12]. Perhaps, therefore, the generalized CAPM relations for σ_-, with the coefficients β_i as in (5.2), might be regarded as covered by the classical market argument. The generalized β's for σ_- in Example 5 have not, prior to this, been identified.

Standard lower semideviation σ_- was among the measures covered by Malevergne and Sornette [11]. Those authors, although concerned especially with "moments," based their results on axioms aimed at covering a wide class of measures of "deviation" type. They did not require convexity or continuity, or invoke either of those properties anywhere, so the underpinnings to their assertions of the existence and uniqueness of optimal portfolios appear to be without foundation. The same is true of their claims of having determined master funds of positive type without making any restriction on the risk-free rate.

[4] Little, if any, attention has been paid in the classical context to the potential nonexistence of a positive master fund. The availability of such a fund is generally just taken for granted, and this is reinforced by finance textbooks aimed at presenting basic material without going through rigorous arguments. For instance, in Luenberger's derivation in [10, p. 168], a master fund of positive type is produced by tacitly assuming that a certain nonconvex function attains a minimum, and that the vanishing of first partial derivatives is adequate for determining where that minimum occurs. The threshold phenomenon with the risk-free rate is thereby missed entirely.

Example 6 (master funds for mean absolute deviation)**.** *In the case of the deviation measure* $\mathcal{D}(X) = ||X - EX||_1 = 2||[X - EX]_-||_1 = 2||[X - EX]_+||_1$, *the* β *coefficients in Definition 3 and the master fund characterization in Theorem 5 take the form, in the sign notation of (2.7), that*

$$\beta_i = \frac{\operatorname{covar}(r_i, Z)}{||X^* - EX^*||_1} \quad \text{for some } Z \in \operatorname{sign}[X^* - EX^*]. \tag{5.5}$$

This entails having

$$\operatorname{covar}(r_i, Z) = \begin{cases} E\Big[r_i - Er_i \Big| X^* - EX^* > 0\Big] \cdot \operatorname{prob}\{ X^* - EX^* > 0\} \\ +E\Big[r_i - Er_i \Big| X^* - EX^* < 0\Big] \cdot \operatorname{prob}\{ X^* - EX^* < 0\} \\ +E\Big[(r_i - Er_i)Z \Big| X^* - EX^* = 0\Big] \cdot \operatorname{prob}\{ X^* - EX^* = 0\}, \end{cases} \tag{5.6}$$

with the values of Z *on* $\{\,\omega \mid X^*(\omega) - EX^* = 0\}$ *being restricted only by the requirement that* $|Z| \leq 1$.

Detail. Here we specialize β_i in (4.8) by taking Q^* to belong to the set in (2.9): $Q^* = 1 + EZ - Z$ for some $Z \in \operatorname{sign}[X^* - EX^*]$. Then $\operatorname{covar}(-r_i, Q^*)$ comes out simply as $\operatorname{covar}(r_i, Z)$. □

Konno [9] investigated mean absolute deviation under the extra assumption that the r_i's have a multivariate distribution given by a density function on $\mathbb{R}^n$. With that he obtained similar β_i's without confronting the possibility of the third term in (5.6). Such a nonzero term could lead more generally to nonuniqueness of the β_i's, but anyway, even in the setting of [9], the uniqueness of a master fund is not clearly assured, since mean absolute deviation lacks the kind of strict convexity that could guarantee that.

Another special feature in [9] which complicates comparisons with our broader contribution in Example 6 is that short positions are said to be excluded; in other words, the constraints $x_i \geq 0$ are imposed on the portfolio weights, whereas we do not impose them in our adoption of $\mathcal{P}(\Delta)$ and $\mathcal{P}_0(\Delta)$ as the basic models. But these constraints against shorting are anyhow suppressed in the developments of [9] by assuming that the Lagrange multipliers associated with them can merely be taken to be 0, which amounts to a claim that nonnegativity will automatically follow from optimality. No support for such a claim is offered. In the absence of shorting, of course, the potential need for a master fund of negative type does not come into view, either.

Example 7 (master funds for worst-case deviation and semideviations)**.** *Let the state space* Ω *be finite, so that* $\inf X$ *and* $\sup X$ *are finite for all* $X \in \mathcal{L}^2(\Omega)$. *Then for* $\mathcal{D}(X) = ||X - EX||_\infty$, *which is in this case is finite and continuous, the* β *coefficients in Definition 3 and the master fund characterization in Theorem 5 take the form*

$$\beta_i = \frac{\operatorname{covar}(r_i, Z)}{||X^* - EX^*||_\infty} \quad \text{for some } Z \text{ with } \begin{cases} E|Z| = 1, \\ Z_+(\omega) = 0 \text{ when } X(\omega) - EX < ||X - EX||_\infty, \\ Z_-(\omega) = 0 \text{ when } EX - X(\omega) < ||X - EX||_\infty \end{cases} \tag{5.7}$$

(almost surely). For the corresponding lower semideviation, $\mathcal{D}_-(X) = EX - \inf X$, *they take the form*

$$\beta_i = \frac{\operatorname{covar}(-r_i, Q^*)}{EX^* - \inf X^*} \quad \text{for some } Q^* \text{ with } \begin{cases} Q^* \geq 0, \ EQ^* = 1, \\ Q^*(\omega) = 0 \text{ when } X^*(\omega) > \inf X^*. \end{cases} \tag{5.8}$$

For the corresponding upper semideviation, $\mathcal{D}_+(X) = \sup X - EX$, *they are instead*

$$\beta_i = \frac{\operatorname{covar}(-r_i, Q^*)}{\sup X^* - EX^*} \quad \text{for some } Q^* \text{ with } \begin{cases} Q^* \leq 2, \ EQ^* = 1, \\ Q^*(\omega) = 2 \text{ when } X^*(\omega) < \sup X^*. \end{cases} \tag{5.9}$$

Detail. Drawing on Example 3, we note in obtaining (5.7) that when $Q^* = 1 + EZ - Z$ we have $\mathrm{covar}(-r_i, Q^*) = \mathrm{covar}(r_i, Z)$. For the semideviations, the risk identifiers in Example 3 are utilized directly. □

Example 8 (master funds for CVaR deviations)**.** *For $\mathcal{D}_\alpha(X) = \mathrm{CVaR}_\alpha(X - EX)$, obtained from any choice of $\alpha \in (0,1)$, the β coefficients in Definition 3 and the master fund characterization in Theorem 5 take the form*

$$\beta_i = \frac{E[(Er_i - r_i)Q^*]}{\mathrm{CVaR}_\alpha(X^* - EX^*)} \quad \textit{for some } Q^* \textit{ with } \begin{cases} 0 \le Q^*(\omega) \le \alpha^{-1}, \ EQ^* = 1, \\ Q^*(\omega) = 0 \textit{ when } X^*(\omega) > -\mathrm{VaR}_\alpha(X^*), \\ Q^*(\omega) = \alpha^{-1} \textit{ when } X^*(\omega) < -\mathrm{VaR}_\alpha(X^*). \end{cases} \tag{5.10}$$

When $\mathrm{prob}\{\, X^* = -\mathrm{VaR}_\alpha(X^*) \,\} = 0$, *this reduces to a ratio of conditional probabilities:*

$$\beta_i = \frac{E\Big[\, Er_i - r_i \,\Big|\, X^* \le -\mathrm{VaR}_\alpha(X^*) \,\Big]}{E\Big[\, EX^* - X^* \,\Big|\, X^* \le -\mathrm{VaR}_\alpha(X^*) \,\Big]}. \tag{5.11}$$

Detail. In (5.10) we directly invoke the risk identifiers in Example 4. When there is zero probability of X^* taking on the value $-\mathrm{VaR}_\alpha(X^*)$, we know that $\mathrm{CVaR}_\alpha(X^* - EX^*)$ is simply the conditional expectation of $EX^* - X^*$ subject to $X^* - EX^* < -\mathrm{VaR}_\alpha(X^* - EX^*)$, which is the same as $X^* < -\mathrm{VaR}_\alpha(X^*)$, so the denominator in (5.10) can be written as in (5.11). Then too, Q is uniquely determined by the prescription in (5.10) (in the almost sure sense), with α being the probability that $X^* < -\mathrm{VaR}_\alpha(X^*)$. The numerator in (5.10) comes out in that case as just the conditional expectation of $Er_i - r_i$ subject to $X^* < -\mathrm{VaR}_\alpha(X^*)$. □

Previous work on CVaR-based master funds in [26, 5, 2] has avoided the issue of discontinuities coming up in the probability distributions and has moreover needed the differentiability of the composite function $f_{\mathcal{D}}$ in (3.5). Such differentiability is lacking when X^* can equal $-\mathrm{VaR}_\alpha(X^*)$ with positive probability. The need for a threshold assumption on the risk-free rate r_0, in order to be assured of the existence of a master fund of positive type (the only type considered), did not get addressed, nor did the issue of nonuniqueness of such a fund, even when it exists.

References

[1] C. Acerbi, Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion, *Journal of Banking and Finance* 26 (2002), 1505–1518.

[2] C. Acerbi, P. Simonetti, Portfolio Optimization with Spectral Measures of Risk, preprint 2002.

[3] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent Measures of Risk, *Mathematical Finance* 9 (1999), 203–227.

[4] V.S. Bawa, E.B. Lindenberg, Capital Market Equilibrium in a Mean-Lower Partial Moment Framework, *Journal of Financial Economics* 5 (1977), 189–200.

[5] D. Bertsimas, G.J. Lauprete, A. Samarov, Shortfall as a Risk Measure: Properties Optimization and Applications, *Journal of Economic Dynamics and Control* 28 (2004), 1227–1480.

[6] C.D. Feinstein, M.N. Thapa, A Reformulation of a Mean-Absolute Deviation Portfolio Optimization Model, *Management Science* 39 (1993), 1552–1553.

[7] H. Föllmer, A. Schied, *Stochastic Finance*, deGruyter, 2002.

[8] R.R. Grauer, Introduction to Asset Pricing Theory and Tests, in *The International Library of Critical Writings in Financial Economics* (R. Roll, ed.), Edward Elgar Publishing Inc., 2001.

[9] H. Konno, H. Shirakawa, Equilibrium Relations in the Mean-Absolute Deviation Capital Market, *Asia-Pacific Financial Markets* 1 (1994), 21–35.

[10] D.G. Luenberger, *Investment Science*, Oxford University Press, Oxford, New York, 1998.

[11] Y. Malevergne, D. Sornette, Multi-Moments Method for Portfolio Management: Generalized Capital Asset Pricing Model in Homogeneous and Heterogeneous markets, preprint 2002.

[12] H.M. Markowitz, *Portfolio Selection, Efficient Diversification of Investments*, Wiley, New York, 1959.

[13] G.C. Pflug, Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk, in *Probabilistic Constrained Optimization: Methodology and Applications* (S.P. Uryasev, ed.), Kluwer, 2000, 278–287.

[14] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[15] R.T. Rockafellar, *Conjugate Duality and Optimization*, monograph No. 16 in Conference Board of Math. Sciences Series, SIAM Publications, 1974.

[16] R.T. Rockafellar, S.P. Uryasev, Optimization of Conditional Value-at-Risk, *Journal of Risk* 2 (2000), 21–42.

[17] R.T. Rockafellar, S.P. Uryasev, Conditional Value-at-Risk for General Loss Distributions, *Journal of Banking and Finance* 26 (2002), 1443–1471.

[18] R.T. Rockafellar, S. Uryasev, M. Zabarankin, Deviation Measures in Risk Analysis and Optimization, *Research Report 2002–7,* Department of Industrial and Systems Engineering, University of Florida, 2002.

[19] R.T. Rockafellar, S. Uryasev, M. Zabarankin, Generalized Deviations in Risk Analysis, *Research Report 2004–4,* Department of Industrial and Systems Engineering, University of Florida, 2004.

[20] R.T. Rockafellar, S. Uryasev, M. Zabarankin, Master Funds in Portfolio Analysis with General Deviation Measures, *Journal of Banking and Finance*, to appear.

[21] R. T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[22] A. Ruszczynski, A. Shapiro, Optimization of Convex Risk Functions, preprint 2004.

[23] P.A. Samuelson, The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments, *The Review of Economic Studies* 37 (1970), 537–542.

[24] W.F. Sharpe, Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk, *The Journal of Finance* 19 (1964), 425–442.

[25] W.F. Sharpe, Capital Asset Prices With and Without Negative Holdings, *The Journal of Finance* 46 (1991), 489–509.

[26] D. Tasche, Modern Portfolio Theory with Homogeneous Risk Measures, Zentrum Mathematik, Technische Universitat, Munich, preprint 2001.