

CALCULUS OF PROX-REGULARITY

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Abstract. In this paper we show that the operations of composition and addition, under appropriate conditions, preserve prox-regularity. The class of prox-regular functions covers all l.s.c., proper, convex functions, lower- \mathcal{C}^2 functions, strongly amenable functions (i.e. convexly composite functions), and pln functions, hence a large core of functions of interest in variational analysis and optimization. These functions, despite being in general nonconvex, possess many of the properties that one would expect only to find in convex or near convex (lower- \mathcal{C}^2) functions e.g. the Moreau-envelopes are \mathcal{C}^{1+} , a localization of the subgradient mapping is hypomonotone, etc... In this paper, we add to this list of convex-like properties by showing, under suitable conditions, that locally the subdifferential of the sum of prox-regular functions is equal to the sum of subdifferentials.

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1. Introduction

Let f be a proper, l.s.c. function on \mathbb{R}^n , we denote by $\partial f(x)$ the set of limiting proximal subgradients of f at any point $x \in \text{dom } f$.

Definition 1.1. *The function f is prox-regular at \bar{x} relative to \bar{v} if $\bar{x} \in \text{dom } f$, $\bar{v} \in \partial f(\bar{x})$, and there exist $\varepsilon > 0$ and $r > 0$ such that*

$$f(x') > f(x) + \langle v, x' - x \rangle - \frac{r}{2} |x' - x|^2 \quad (1.1)$$

whenever $|x' - \bar{x}| < \varepsilon$ and $|x - \bar{x}| < \varepsilon$, with $x' \neq x$ and $|f(x) - f(\bar{x})| < \varepsilon$, while $|v - \bar{v}| < \varepsilon$ with $v \in \partial f(x)$.

Prox-regular functions were first introduced in [14]. Additional results were provided in [15] and [17]. For a comprehensive survey, see [17]. Extensions to non-finite dimensional spaces were given in [2]–[4]. Other applications and results are given in [5]–[9], [11], and [12].

First note that it is obvious that if f is convex, it is prox-regular at \bar{x} for every subgradient $\bar{v} \in \partial f(\bar{x})$. The same is true for lower- \mathcal{C}^2 functions, primal-lower-nice (pln) functions, and *strongly amenable* functions, cf. [17]. A function f is strongly amenable at \bar{x} if it can be written as $f = g \circ F$ in a neighborhood of \bar{x} for a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of class \mathcal{C}^2 and a proper, l.s.c., convex function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ satisfying at \bar{x} with respect to the convex set $D = \text{dom } g$ the basic constraint qualification that

$$\text{there is no vector } y \neq 0 \text{ in } N_D(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0. \quad (1.2)$$

(Here $N_D(F(\bar{x}))$ refers to the normal cone to D at $F(\bar{x})$.) Amenability itself merely requires F to be of class \mathcal{C}^1 , whereas *full* amenability is the subcase of strong amenability where g is also piecewise linear-quadratic. Full amenability already covers most applications that arise in the framework of nonlinear programming and its extensions. For more on amenability, see [17].

The initial goal in the development of prox-regularity was to demonstrate that many of the important properties of convex and convex-like functions can be found in a large class of nonconvex functions. In this paper we show in Section 2, that the prox-regularity is preserved under natural operations: composition and addition. In the remainder of the introduction we recall some of the convex-like features of prox-regular functions. In Section 2 we add to the list of convex-like features by showing, under suitable conditions, that locally, the subdifferential of the sum of prox-regular functions is equal to the sum of

subdifferentials. This is somewhat of a surprise since prox-regular functions are in general non “regular”; see [17, Cor. 10.9] for the calculus concerning the sum of regular functions.

Let’s start with the subgradient mapping. The fact that the subgradient mapping of a convex function is monotone plays a vital role in many applications. Something similar is true for prox-regular functions in that a selection or localization of the subgradient mapping is r -monotone i.e. a subset of the subgradient mapping plus rI is monotone, where $r > 0$ and I is the identity mapping, is monotone. To be more precise, for any $\varepsilon > 0$, the f -attentive ε -localization of ∂f at (\bar{x}, \bar{v}) is the mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$T(x) = \begin{cases} \{v \in \partial f(x) \mid |v - \bar{v}| < \varepsilon\} & \text{when } |x - \bar{x}| < \varepsilon \text{ and } |f(x) - f(\bar{x})| < \varepsilon, \\ \emptyset & \text{otherwise.} \end{cases} \quad (1.3)$$

The graph of T is the intersection of $\text{gph } \partial f$ with the product of an f -attentive neighborhood of \bar{x} and an ordinary neighborhood of \bar{v} . (The f -attentive topology on \mathbb{R}^n is the weakest topology in which f is continuous.)

Theorem 1.2 [14, Thm. 3.2]. *The function f is prox-regular at a point \bar{x} relative to \bar{v} if and only if the vector \bar{v} is a proximal subgradient of f at \bar{x} and there exist $\varepsilon > 0$ and $r > 0$ such that, for the f -attentive ε -localization T of ∂f , the mapping $T + rI$ is monotone, i.e., one has $\langle v_1 - v_0, x_1 - x_0 \rangle \geq -r|x_1 - x_0|^2$ whenever $|x_i - \bar{x}| < \varepsilon$, $|f(x_i) - f(\bar{x})| < \varepsilon$, and $|v_i - \bar{v}| < \varepsilon$ with $v_i \in \partial f(x_i)$, $i = 0, 1$.*

To illustrate further that prox-regular functions have many convex-like properties, we recall some of the results of [14] dealing with the properties of Moreau-envelopes e_λ and proximal mappings P_λ . Recall that

$$e_\lambda(x) := \min_{x'} \left\{ f(x') + \frac{1}{2\lambda} |x' - x|^2 \right\},$$

$$P_\lambda(x) := \operatorname{argmin}_{x'} \left\{ f(x') + \frac{1}{2\lambda} |x' - x|^2 \right\}.$$

Theorem 1.3 [14, Thms. 4.4, 4.6, 5.2]. *Suppose that f is prox-regular at $\bar{x} \in \operatorname{argmin} f$ for $\bar{v} = 0$ with respect to ε and r , and let T be the f -attentive ε -localization of ∂f at $(\bar{x}, 0)$. Then for any $\lambda \in (0, 1/r)$ there is a convex neighborhood X_λ of \bar{x} such that*

(a) *the mapping P_λ is single-valued and Lipschitz continuous on X_λ with*

$$P_\lambda = (I + \lambda T)^{-1}, \quad |P_\lambda(x') - P_\lambda(x)| \leq \frac{1}{1 - \lambda r} |x' - x|, \quad P_\lambda(\bar{x}) = \bar{x},$$

(b) the function e_λ is C^{1+} and lower- C^2 on X_λ with

$$e_\lambda + \frac{r}{2(1-\lambda r)}|\cdot|^2 \text{ convex}, \quad \nabla e_\lambda = \lambda^{-1}[I - P_\lambda] = [\lambda I + T^{-1}]^{-1}.$$

Proximal regularity can also be defined for sets as well as for functions. A set $C \subset \mathbb{R}^n$ is prox-regular at a point $\bar{x} \in C$ for a vector \bar{v} if and only if its indicator function δ_C is prox-regular at \bar{x} for \bar{v} . A function is prox-regular at \bar{x} for the subgradient \bar{v} if and only if its epigraph is prox-regular at $(\bar{x}, f(\bar{x}))$ for the normal vector $(\bar{v}, -1)$; see [1] and [14]. Convex sets, “weakly convex” sets (see [19]), “proximally smooth” sets (see [6]), and sets with the “Shapiro” property (see ([18])) are all examples of prox-regular sets. Strong amenability provides further examples: a set $C \subset \mathbb{R}^n$ is *strongly amenable* at one of its points \bar{x} if its indicator function δ_C is strongly amenable at \bar{x} , or equivalently, there is an open neighborhood U of \bar{x} and a C^2 mapping $F : U \rightarrow \mathbb{R}^m$ along with a closed, convex set $D \subset \mathbb{R}^m$ such that

$$C \cap U = \{x \in U \mid F(x) \in D\},$$

and the constraint qualification is satisfied that no nonzero vector $y \in N_D(F(\bar{x}))$ has $\nabla F(\bar{x})^* y = 0$. The following theorem summarizes some of the results of [16] concerning the local differentiability of the distance functions to prox-regular sets. In particular a set is prox-regular if and only if the projection mapping is locally single-valued.

Theorem 1.4 [16, Thm. 1.3]. *For a closed subset C of a Hilbert space H , and any point $\bar{x} \in C$, the following properties are equivalent:*

- (a) C is prox-regular at \bar{x} ;
- (b) d_C is continuously differentiable on $O \setminus C$ for some open neighborhood O of \bar{x} ;
- (c) d_C is Fréchet differentiable on $O \setminus C$ for some open neighborhood O of \bar{x} ;
- (d) d_C is Gâteaux differentiable on $O \setminus C$ for some open neighborhood O of \bar{x} , and P_C is nonempty-valued on O ;
- (e) d_C^2 is C^{1+} on an open neighborhood O of \bar{x} ; i.e., Fréchet differentiable on O with the derivative mapping $D(d_C^2)(x) : H \rightrightarrows H$ depending Lipschitz continuously on x ;
- (f) P_C is single-valued and strongly-weakly continuous (i.e., from the strong topology in the domain to the weak topology in the range) on a neighborhood of \bar{x} ;
- (g) C has the Shapiro property at \bar{x} .

Then there is a neighborhood O of \bar{x} on which P_C is single-valued, monotone and Lipschitz continuous with $P_C = (I + N_C^r)^{-1}$ on O for some $r > 0$, whereas $D(d_C) = [I - P_C]/d_C$ on $O \setminus C$. Here $I : H \rightarrow H$ denotes the identity mapping.

If the set C is weakly closed relative to a (strong) neighborhood of \bar{x} (which is always the case when the space H is finite-dimensional), then one can add the following to the set of equivalent properties:

(h) P_C is single-valued around \bar{x} .

2. Calculus of Prox-Regularity

Consider a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a proper function $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$. We say that the constraint qualification for F and g is satisfied at \bar{x} if $F(\bar{x}) \in \text{dom } g$ and

$$\text{there is no vector } y \neq 0 \text{ in } \partial^\infty g(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0. \quad (2.1)$$

Here $\partial^\infty g(F(\bar{x}))$ refers to the set of horizon subgradients (a vector v is a horizon subgradient for f at \bar{x} if there exist $v_n \in \partial f(x_n)$ and $\lambda_n \searrow 0$ with $\lambda_n v_n \rightarrow v$, $x_n \rightarrow \bar{x}$ and $f(x_n) \rightarrow f(\bar{x})$). This constraint qualification guarantees that $\partial f(x) \subset \nabla F(x)^* \partial g(F(x))$ in an f -attentive neighborhood of \bar{x} , cf. [17, Thm. 10.6].

Theorem 2.1. *Let $f(x) = g(F(x))$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class \mathcal{C}^2 , $g : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is l.s.c. and proper, and suppose that the constraint qualification (2.1) is satisfied at \bar{x} (which lies in $\text{dom } f$). Assume further that $\bar{v} \in \partial f(\bar{x})$ is a vector such that the function g is prox-regular at $F(\bar{x})$ for every $y \in \partial g(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = \bar{v}$. Then f is prox-regular at \bar{x} for \bar{v} .*

Proof. As already mentioned, (2.1) guarantees the existence of $\varepsilon_0 > 0$ such that

$$\partial f(x) \subset \nabla F(x)^* \partial g(F(x)) \quad \text{when } |x - \bar{x}| < \varepsilon_0, \quad |f(x) - f(\bar{x})| < \varepsilon_0. \quad (2.2)$$

The constraint qualification further ensures that the mapping

$$S : (x, v) \rightarrow \{y \in \partial g(F(x)) \mid \nabla F(x)^* y = v\}$$

has closed graph and is locally bounded at (\bar{x}, \bar{v}) with respect to the f -attentive topology. In particular, $S(\bar{x}, \bar{v})$ is a compact set. By our assumptions, for each $y \in S(\bar{x}, \bar{v})$ the function g is prox-regular at $F(\bar{x})$ for y with constants (say) r_y and ε_y . The compactness of $S(\bar{x}, \bar{v})$ enables us to make this uniform: there exist $\varepsilon_1 > 0$ and $\bar{r} > 0$ such that

$$\left. \begin{aligned} \langle u_1 - u_0, y_1 - y_0 \rangle &\geq -\bar{r} |u_1 - u_0|^2 \\ \text{whenever } y_i &\in \partial g(u_i), |u_i - F(\bar{x})| < \varepsilon_1, |g(u_i) - g(F(\bar{x}))| < \varepsilon_1, \\ \text{dist}(y_i, S(\bar{x}, \bar{v})) &< \varepsilon_1. \end{aligned} \right\} \quad (2.3)$$

Thus in combining (2.2) and (2.3) with the established properties of S we obtain that there exist $\varepsilon > 0$, $\bar{r} > 0$, and Y a compact set such that

$$\left. \begin{array}{l} |x_i - \bar{x}| < \varepsilon, \\ |f(x_i) - f(\bar{x})| < \varepsilon \\ v \in \partial f(x_i), |v_i - \bar{v}| < \varepsilon \end{array} \right\} \implies \left\{ \begin{array}{l} \exists y_i \in \partial g(F(x_i)) \cap Y \text{ with} \\ \nabla F(x_i)^* y_i = v_i \text{ and} \\ \langle F(x_1) - F(x_0), y_1 - y_0 \rangle \geq -\bar{r} |F(x_1) - F(x_0)|^2 \end{array} \right. \quad (2.4)$$

Note that when $v_i = \nabla F(x_i)^* y_i$ then

$$\begin{aligned} \langle x_1 - x_0, v_1 - v_0 \rangle &= \langle x_1 - x_0, \nabla F(x_1)^* y_1 - \nabla F(x_0)^* y_1 + \nabla F(x_0)^* y_1 - \nabla F(x_0)^* y_0 \rangle \\ &= \langle x_1 - x_0, [\nabla F(x_1)^* - \nabla F(x_0)^*] y_1 \rangle + \langle x_1 - x_0, \nabla F(x_0)^* (y_1 - y_0) \rangle. \end{aligned} \quad (2.5)$$

Because Y is compact and F is of class \mathcal{C}^2 we know there exists $r_1 > 0$ such that

$$|[\nabla F(x_1)^* - \nabla F(x_0)^*] y| \leq r_1 |x_1 - x_0| \quad \forall y \in Y.$$

Thus

$$\langle x_1 - x_0, [\nabla F(x_1)^* - \nabla F(x_0)^*] y \rangle \geq -r_1 |x_1 - x_0|^2. \quad (2.6)$$

On the other hand, we can write the final term in (2.5) as $\langle \nabla \varphi(x_0), x_1 - x_0 \rangle$ for the \mathcal{C}^2 function $\varphi(x) = \langle y_1 - y_0, F(x) \rangle$. Let r_2 be an upper bound for the eigenvalues of the Hessian of the mapping $x \rightarrow \langle \eta, F(x) \rangle$ where x ranges over $\bar{x} + \varepsilon \bar{B}$ and η ranges over the compact set $Y - Y$. Then in particular $\varphi(x) \leq \varphi(x_0) + \langle \nabla \varphi(x_0), x - x_0 \rangle + r_2 |x - x_0|^2$ when $|x - \bar{x}| \leq \varepsilon$. It follows that

$$\begin{aligned} \langle \nabla \varphi(x_0), x - x_0 \rangle &\geq -r_2 |x_1 - x_0|^2 + \varphi(x_1) - \varphi(x_0) \\ &= -r_2 |x_1 - x_0|^2 + \langle y_1 - y_0, F(x_1) - F(x_0) \rangle \\ &\geq -r_2 |x_1 - x_0|^2 - \bar{r} |F(x_1) - F(x_0)|^2. \end{aligned} \quad (2.7)$$

But there is also a constant λ such that $|F(x_1) - F(x_0)| \leq \lambda |x_1 - x_0|$ when $|x_i - \bar{x}| < \varepsilon$. Using this fact with the estimates (2.6) and (2.7) for the two terms at the end of (2.5) we obtain

$$\langle x_1 - x_0, v_1 - v_0 \rangle \geq -r_1 |x_1 - x_0|^2 - r_2 |x_1 - x_0|^2 - \bar{r} \lambda^2 |x_1 - x_0|^2$$

Thus for $r = r_1 + r_2 + \bar{r} \lambda^2$ and the same ε as in (2.4) we have

$$\left. \begin{array}{l} |x_i - \bar{x}| < \varepsilon, |f(x_i) - f(\bar{x})| < \varepsilon \\ v_i \in \partial f(x_i), |v_i - \bar{v}| < \varepsilon \end{array} \right\} \implies \langle v_1 - v_0, x_1 - x_0 \rangle \geq -r |x_1 - x_0|^2.$$

This means that f is prox-regular at \bar{x} for \bar{v} . \square

Theorem 2.2. *Let f_i , $i = 1, 2$ be extended real-valued functions on \mathbb{R}^n . Consider $\bar{x} \in [\text{dom } f_1 \cap \text{dom } f_2]$, and assume that*

$$\text{the only choice of } v_i \in \partial f_i^\infty(\bar{x}) \text{ with } v_1 + v_2 = 0 \text{ is } v_1 = v_2 = 0. \quad (2.8)$$

Let $\bar{v} \in \partial f(\bar{x})$, where $f(x) = f_1(x) + f_2(x)$. Assume further that for each $v_i \in \partial f_i(\bar{x})$ with $v_1 + v_2 = \bar{v}$, the function f_i is prox-regular at \bar{x} for v_i . Then f is prox-regular at \bar{x} for \bar{v} and there exist $\varepsilon > 0$ such that

$$[\partial f_1(x) + \partial f_2(x)] \cap B(\bar{v}, \varepsilon) = \partial f(x) \cap B(\bar{v}, \varepsilon) \quad (2.9)$$

whenever $|x - \bar{x}| < \varepsilon$ with $|f(x) - f(\bar{x})| < \varepsilon$.

Proof. Let $g(u_1, u_2) := f_1(u_1) + f_2(u_2)$ and $F(x) := (x, x)$ then all conditions in the previous theorem are satisfied. This gives that f is prox-regular at \bar{x} for \bar{v} . The fact that $\partial f(x) \cap B(\bar{v}, \varepsilon) \subset [\partial f_1(x) + \partial f_2(x)] \cap B(\bar{v}, \varepsilon)$ follows from the constraint qualification (2.8) and [17, Cor. 10.9]. Now suppose that the other inclusion in (2.9) is not verified. If so, there exist x_n converging to \bar{x} with $f(x_n) \rightarrow f(\bar{x})$ and $v_n^i \in \partial f_i(x_n)$ with $v_n^1 + v_n^2$ converging to \bar{v} and such that $v_n^1 + v_n^2$ is not a subgradient to f at x_n . It follows from (2.8) that for $i = 1, 2$, v_n^i can not be unbounded (if say v_n^1 is unbounded while v_n^2 is bounded, then we have $v_n^1 / \|v_n^1\| \rightarrow 0$, this contradicts (2.8)). Therefore we may assume that $v_n^i \rightarrow v_i$ with $v_i \in \partial f_i(\bar{x})$ and $v_1 + v_2 = \bar{v}$. From our assumptions f_i is prox-regular at \bar{x} for v_i . Therefore for $i = 1, 2$, there exist $r_i > 0$ and $\varepsilon_i > 0$ such that

$$f_i(y) \geq f_i(x) + \langle v, y - x \rangle - (r_i/2)|y - x|^2 \quad (2.10)$$

whenever $|y - \bar{x}| < \varepsilon_i$, $|x - \bar{x}| < \varepsilon_i$, $|f_i(x) - f_i(\bar{x})| < \varepsilon_i$ while $|v - v_i| < \varepsilon_i$ with $v \in \partial f_i(x)$. It therefore follows that eventually v_n^i is a proximal subgradient to f at x_n . From this we conclude that $v_n^1 + v_n^2$ is a proximal subgradient to f at x_n (to see this just add (2.10) in the case of $i = 1$ to (2.10) in the case of $i = 2$). This contradicts our earlier assumption and completes the proof. \square

These results can be applied to indicator functions to obtain rules for generating new prox-regular sets from known ones.

Corollary 2.3. *Let $C = \{x \mid F(x) \in D\}$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class \mathcal{C}^2 and $D \subset \mathbb{R}^m$ is closed. Let $\bar{x} \in C$, and suppose that the following constraint qualification is satisfied:*

$$\text{there is no vector } y \neq 0 \text{ in } N_D(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0.$$

Assume further that $\bar{v} \in N_C(\bar{x})$ is a vector such that the set D is prox-regular at $F(\bar{x})$ for every $y \in N_D(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = \bar{v}$. Then C is prox-regular at \bar{x} for \bar{v} .

Corollary 2.4. Let C_i , $i = 1, 2$, be closed subsets of \mathbb{R}^n , and let $\bar{x} \in C = C_1 \cap C_2$. Suppose that

$$\text{the only choice of } v_i \in N_{C_i}(\bar{x}) \text{ with } v_1 + v_2 = 0 \text{ is } v_1 = v_2 = 0.$$

Let $\bar{v} \in N_C(\bar{x})$, and assume that for each choice of $v_i \in N_{C_i}(\bar{x})$ with $v_1 + v_2 = \bar{v}$, the set C_i is prox-regular at \bar{x} for v_i . Then C is prox-regular at \bar{x} for \bar{v} and there exist $\varepsilon > 0$ such that

$$[N_{C_1}(\bar{x}) + N_{C_2}(\bar{x})] \cap B(\bar{v}, \varepsilon) = N_C(\bar{x}) \cap B(\bar{v}, \varepsilon)$$

whenever $|x - \bar{x}| < \varepsilon$.

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