# DIRECT PROOF OF THE SHIFT STABILITY OF EQUILIBRIUM IN CLASSICAL CIRCUMSTANCES

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**Abstract.** In classical models of exchange under smoothness and strict concavity assumptions that in particular only support positive quantities of goods, every equilibrium is shift-stable. This property, referring to good behavior in response to local perturbations, can be established by elementary means without resorting to techniques of differential topology. Utilization of Lagrange multipliers for the budget constraints furthermore brings to light additional features not recorded in the past.

**Key Words.** Exchange equilibrium, comparative statics, shift stability, localized uniqueness

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## 1 Statement of the Results

In economic models of the exchange of goods, the set of equilibrium price vectors for a given instance of initial holdings among the agents can sometimes be very strange, even under the strongest classical assumptions on utility. However, when the initial holdings are close enough to the equilibrium holdings, the price vector is unique. This was proved by Balasko [1] (1975) using differential topology. Through an underlying connection with regularity in that framework, his argument implies more than he brought out explicitly at the time, namely that equilibrium holdings and prices are sure to depend smoothly on nearby initial holdings. Here we formalize that property as shift stability and, in the case of concave utility functions, confirm it along with the price uniqueness while relying only on the implicit function theorem.

Let agent $i$ for $i = 1, \dots, m$ have a $\mathcal{C}^2$ utility function $u_i$ on the positive orthant $I\!R^n_{++}$ with its upper level sets $\{\, x_i \,|\, u_i(x_i) \geq c \,\}$ all closed, and let $u_i(x_i)$ be nondecreasing in $x_i$ with Hessian matrices $\nabla^2 u_i(x_i)$ that are negative definite. Then $u_i$ is concave, locally strongly, and strictly increasing in every component of $x_i$. Suppose also that the gradient $\nabla u_i(x_i)$ grows unboundedly as $x_i$ approaches any boundary point of $I\!R^n_+$, which is a property also expressible in terms of marginal utility. Then, as known in convex analysis [2, Section 26], the range of $\nabla u_i$ is an open convex set $D_i$, and $\nabla u_i$ is globally invertible with $(\nabla u_i)^{-1}$ being $\mathcal{C}^1$ from $D_i$ onto $I\!R^n_{++}$.

**Definition 1.** *An equilibrium for initial holdings $x_i^0 \in I\!R^n_{++}$ is a combination $(p, x_1, \ldots, x_m)$ in $S \times I\!R^n_+ \times \cdots \times I\!R^n_+$, where $S$ is the $n$-dimensional price simplex, such that*

$$x_i \text{ maximizes } u_i \text{ subject to } p{\cdot}x_i = p{\cdot}x_i^0 \text{ for all } i, \text{ and } \textstyle\sum_{i=1}^m x_i = \sum_{i=1}^m x_i^0. \tag{1}$$

*On the other hand, $(p, x_1, \ldots, x_m)$ is an equilibrium, in itself, if this holds with $x_i^0 = x_i$, i.e.,, if at the holdings $x_i$ the agents have no incentive to change to other holdings under the prices in $p$.*

Note that if $(p, x_1, \ldots, x_m)$ is an equilibrium for some initial holdings $(x_1^0, \ldots, x_m^0)$, then it is also an equilibrium in itself. Standard theory ensures under our assumptions that, for any initial holdings $x_i^0 \in I\!R^n_{++}$, at least one equilibrium $(p, x_1, \ldots, x_m)$ will exist, necessarily with $x_i \in I\!R^n_{++}$ and $p \in I\!R^n_{++}$. Let

$$E(x_1^0, \ldots, x_m^0) = \text{the nonempty set of all equilibria } (p, x_1, \ldots, x_m) \text{ for } (x_1^0, \ldots, x_m^0). \tag{2}$$

**Definition 2.** *An equilibrium $(\bar{p}, \bar{x}_1, \ldots, \bar{x}_m)$ is smoothly shift-stable if there are neighborhoods $N^0$ of $(\bar{x}_1, \ldots, \bar{x}_m)$ and $N$ of $(\bar{p}, \bar{x}_1, \ldots, \bar{x}_m)$ such that*

$$\text{for } (x_1^0, \ldots, x_m^0) \in N^0 \text{ there is one and only one } (p, x_1, \ldots, x_m) \in E(x_1^0, \ldots, x_m^0) \cap N, \tag{3}$$

*and this single-valued localization $(x_1^0, \ldots, x_m^0) \to (p, x_1, \ldots, x_m)$ of the equilibrium mapping $E$ is continuously differentiable.*

**Theorem 1.** *Under the given assumptions on utility, every equilibrium is smoothly shift-stable.*

Shift stability does not preclude having additional equilibria $(p, x_1, \ldots, x_m)$ in $E(x_1^0, \ldots, x_m^0)$ outside of $N$. However, it turns out here that $N$ can be dropped from (3) when $N^0$ is small enough. To formulate this insightfully, let $X$ be the set of all equilibrium holdings $(x_1, \ldots, x_m)$,

$$X = \{\, (x_1, \ldots, x_m) \,|\, \exists\, p \in S \text{ with } (p, x_1, \ldots, x_m) \in E(x_1, \ldots, x_m) \,\}. \tag{4}$$

**Theorem 2.** *Under the given assumptions on utility, the price vector $p$ for any $(x_1, \ldots, x_m) \in X$ is uniquely determined. In fact $X$ is a connected differentiable manifold of dimension $m+n-1$, and there is an open set $X^0 \supset X$ of initial holdings $(x_1^0, \ldots, x_m^0)$ in $(I\!R^n_{++})^m$ on which the equilibrium mapping $E$ is single-valued and continuously differentiable.*

These results, in the format of relative prices and under utility assumptions that force holdings to the interior of the goods orthant, complement our much more comprehensive analysis in [3], where numéraire prices are the focus, utility functions need not be strongly concave locally in all goods, and the boundary of the orthant is realistically allowed to come into play. In that setting, the equilibrium mapping is semidifferentiable and Lipschitz continuous instead of continuously differentiable in a neighborhood of the equilibrium holdings. However, this requires a methodology not yet familiar to most economists. Here, although the statements are special and much more limited in scope, the proofs are readily accessible.

An advantage of the utility functions being concave is that optimality conditions both necessary and sufficient for the maximization problem of agent $i$ can be expressed with a Lagrange multiplier $\lambda_i$ for the budget constraint, which must be positive because $u_i$ inceases (strictly). The conditions take the form

$$\lambda_i p - \nabla u_i(x_i) = 0, \qquad p{\cdot}(x_i - x_i^0) = 0. \tag{5}$$

The multiplier $\lambda_i$, if uniquely determined, gives marginal utility with respect to changes in the budget of agent $i$, and thus has some economic significance in itself. Our multiplier-based approach to Theorems 1 and 2 not only provides information about them but also leads to the following enhancement.

**Theorem 3.** *For $(x_1^0,\ldots,x_m^0)$ in the open set $X^0$ of Theorem 2, the Lagrange multipliers $\lambda_1,\ldots,\lambda_m$, are uniquely determined along with the equilibrium $(p,x_1,\ldots,x_m = E(x_1^0,\ldots,x_m^0)$, and the extended equilibrium mapping*

$$\overline{E} : (x_1^0,\ldots,x_m^0) \to (p,x_1,\ldots,x_m,\lambda_1,\ldots,\lambda_m), \tag{6}$$

*like $E$, is continuously differentiable. continuously differentiable. Moreover at an extended equilibrium*

$$(\bar{p},\bar{x}_1,\ldots,\bar{x}_m,\bar{\lambda}_1,\ldots,\bar{\lambda}_m) = \overline{E}(x_1^0,\ldots,x_m^0)$$

*the derivatives with respect to shifts of $(x_1^0,\ldots,x_m^0)$ in the direction of a vector $(x_1^0{}',\ldots,x_m^0{}')$, namely*

$$\begin{aligned}(p',x_1',\ldots,x_m',\lambda_1',\ldots,\lambda_m') &= D\overline{E}(x_1^0,\ldots,x_m^0;x_1^0{}',\ldots,x_m^0{}') \\ &= \lim_{h\to 0}\tfrac{1}{h}\Big[\overline{E}(x_1^0+hx_1^0{}',\ldots,x_m^0+hx_m^0{}') - \overline{E}(x_1^0,\ldots,x_m^0)\Big]\end{aligned} \tag{7}$$

*are given by*

$$\begin{aligned}(p',x_1',\ldots,x_m',\lambda_1',\ldots,\lambda_m') \;\;=\;\; &\text{the unique solution to the linear equations:} \\ \lambda_i'\bar{p} + \bar{\lambda}_i p' - \nabla^2 u_i(\bar{x}_i)x_i' = 0, \quad &\bar{p}{\cdot}(x_i' - x_i^0{}') = 0, \quad \textstyle\sum_{i=1}^m (x_i' - x_i^0{}') = 0, \quad e{\cdot}p' = 0.\end{aligned} \tag{8}$$

In particular, Theorem 3 furnishes the directional derivatives of the (unextended) equilbrium mapping $E$. But it should be noticed that the multipliers $\bar{\lambda}_i$ are nonetheless involved in expressing the derivatives of $E$. Those multipliers are implicit functions of $(x_1^0,\ldots,x_m^0)$, but getting direct functional expressions could be tedious in comparison to just employing specific multipliers in (8).

# 2 Proofs of the Results

For an equilibrium with $p$ in the unit simplex $S$, the optimality conditions (5) on $x_i$ with multiplier $\lambda_i > 0$ must be combined with

$$\textstyle\sum_{i=1}^m (x_i - x_i^0) = 0, \qquad e{\cdot}p = 1, \quad \text{where } e = (1,\ldots,1). \tag{9}$$

It can immediately be seen that

$$(p, x_1, \ldots, x_m) \in E(x_1^0, \ldots, x_m^0) \implies \lambda_i = e{\cdot}\nabla u_i(x_i), \quad p = [e{\cdot}\nabla u_i(x_i)]^{-1}\nabla u_i(x_i). \tag{10}$$

**Proof of Theorem 1.** Let $(\bar{x}_1^0, \ldots, \bar{x}_m^0)$ and $(\bar{p}, \bar{x}_1, \ldots, \bar{x}_m)$ along with $(\bar{\lambda}_1, \ldots, \bar{\lambda}_m)$ solve (5)–(6) with $\bar{x}_i^0 = \bar{x}_i$; in other words, let $(\bar{p}, \bar{x}_1, \ldots, \bar{x}_m)$ an equilibrium with associated multipliers $\bar{\lambda}_i$. It will suffice to show that, locally around these elements, we can solve (5)+(9) for $x_i$, $p$, $\lambda_i$, as continuously differentiable functions of $(x_1^0, \ldots, x_m^0)$. This can be confirmed by applying the usual implicit function theorem, but instead of dealing directly with matrices, we can carry it out by linearizing (5)+(9) at the reference elements. In terms of auxiliary variables $x_i'$, $p'$, $\lambda_i'$, and $x_i^{0\prime}$, the linearized equations at the elements $\bar{x}_i$, $\bar{p}$, $\bar{\lambda}_i$ and $\bar{x}_i^0 = \bar{x}_i$, are

$$\lambda_i'\bar{p} + \bar{\lambda}_i p' - \nabla^2 u_i(\bar{x}_i)x_i' = 0, \quad \bar{p}{\cdot}(x_i' - x_i^{0\prime}) = 0, \quad \textstyle\sum_{i=1}^m (x_i' - x_i^{0\prime}) = 0, \quad e{\cdot}p' = 0. \tag{11}$$

Checking the matrix rank condition in the implicit function theorem amounts to demonstrating for these linearized equations that, in the case of $x_i^{0\prime} = 0$ for all $i$, which reduces (8) to

$$\lambda_i'\bar{p} + \bar{\lambda}_i p' - \nabla^2 u_i(\bar{x}_i)x_i' = 0, \quad \bar{p}{\cdot}x_i' = 0, \quad \textstyle\sum_{i=1}^m x_i' = 0, \quad e{\cdot}p' = 0, \tag{12}$$

the only possible solution is $p' = 0$, $x_i' = 0$, and $\lambda_i' = 0$.

First we take the dot product of the initial equation in (12) by $x_i'$ and divide by $\bar{\lambda}_i$ (known to be positive), getting $\bar{\lambda}_i^{-1}\lambda_i'\bar{p}{\cdot}x_i' + p'{\cdot}x_i' - \bar{\lambda}_i^{-1}[x_i'{\cdot}\nabla^2 u_i(\bar{x}_i)x_i'] = 0$. Converting $\bar{p}{\cdot}x_i'$ to 0 through the second relation in (12), and adding over $i$ while invoking the final equation in (12), yields $-\sum_{i=1}^m \bar{\lambda}_i^{-1}[x_i'{\cdot}\nabla^2 u_i(\bar{x}_i)x_i'] = 0$. The negative definiteness of $\nabla^2 u_i(\bar{x}_i)$ tells us then that $x_i' = 0$ for all $i$. Back in the first equation of (12) we now have $\lambda_i'\bar{p} + \bar{\lambda}_i p' = 0$. Taking the dot product of this with $e = (1, \ldots, 1)$ and applying the final equation in (12), we get $\lambda_i' = 0$. Returning once more to (9) with the knowledge that $x_i' = 0$ and $\lambda_i' = 0$, we see that $p' = 0$, too, as needed.

**Proof of Theorem 2.** We begin by demonstrating that, for $(\bar{x}_1, \ldots, \bar{x}_m) \in X$ with associated price vector $\bar{p}$, there cannot be a different $(\tilde{x}_1, \ldots, \tilde{x}_m) \in X$ with associated price vector $\tilde{p}$ such that $(\tilde{p}, \tilde{x}_1, \ldots, \tilde{x}_m) \in E(\bar{x}_1, \ldots, \bar{x}_m)$. In those circumstances we would have (5)–(6) holding for $p = \bar{p}$ with $x_i^0 = x_i = \bar{x}_i$ and multipliers $\bar{\lambda}_i$ and also holding for $p = \tilde{p}$ with $x_i^0 = x_i = \tilde{x}_i$ and multipliers $\tilde{\lambda}_i$, but likewise holding for $p = \tilde{p}$ with $x_i^0 = \bar{x}_i$ and $x_i = \tilde{x}_i$. Then

$$(\bar{\lambda}_i\bar{p} - \tilde{\lambda}_i\tilde{p}){\cdot}(\bar{x}_i - \tilde{x}_i) = (\nabla u_i(\bar{x}_i) - \nabla u_i(\tilde{x}_i){\cdot}(\bar{x}_i - \tilde{x}_i) \text{ for all } i,$$

where on the one hand $\tilde{p}{\cdot}(\tilde{x}_i - \bar{x}_i) = 0$, and on the other hand, because $u_i$ is strictly concave, $(\nabla u_i(\bar{x}_i) - \nabla u_i(\tilde{x}_i){\cdot}(\bar{x}_i - \tilde{x}_i) \leq 0$ for all $i$, with this inequality being strict when $\bar{x}_i \neq \tilde{x}_i$. Therefore $\bar{p}{\cdot}(\bar{x}_i - \tilde{x}_i) \leq 0$ for all $i$ with $< 0$ for at least one $i$. But that contradicts $\sum_{i=1}^m (\bar{x}_i - \tilde{x}_i) = 0$. The initial uniqueness assertion in Theorem 2 in confirmed in this way as the case where $\tilde{x}_i = \bar{x}_i$.

To get the broader uniqueness, we consider a sequence $\{ (x_1^{0k}, \ldots, x_m^{0k} \}_{k=1}^\infty$ converging to an $(\bar{x}_1, \ldots, \bar{x}_m) \in X$ with price vector $\bar{p}$, and a sequence $\{ (p^k, x_1^k, \ldots, x_m^k) \}_{k=1}^\infty$ with $(p^k, x_1^k, \ldots, x_m^k) \in E(x_1^{0k}, \ldots, x_m^{0k})$ as characterized by (5)+(9). These sequences are bounded, because total supplies are bounded and the price vectors belong to $S$, so by passing to subsequences if necessary we can suppose that $(p^k, x_1^k, \dots, x_m^k)$ converges to some $(\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_m)$. The associated sequences of multipliers $\lambda_i^k$ in (5) must be bounded as well, for the following reason. For any $\varepsilon > 0$ we have, through the concavity of $u_i$ in combination with (5)+(9)+(10), that

$$0 \geq (\nabla u_i(x_i^k) - \nabla u_i(\varepsilon e))\cdot(x_i^k - \varepsilon e) = \lambda_i^k(p^k\cdot x_i^k - \varepsilon) - \nabla u_i(\varepsilon e)\cdot(x_i^i - \varepsilon e),$$

where $p^k\cdot x_i^k = p^k\cdot x_i^{k0} \to \bar{p}\cdot\bar{x}_i > 0$. By taking $\varepsilon < \bar{p}\cdot\bar{x}_i$ we obtain the bound

$$\lambda_i^k \leq \frac{\nabla u_i(\varepsilon e)\cdot(x_i^k - \varepsilon e)}{p^k\cdot x_i^{0k} - \varepsilon} \;\to\; \frac{\nabla u_i(\varepsilon e)\cdot(\bar{x} - \varepsilon e)}{\bar{p}\cdot\bar{x}_i - \varepsilon}.$$

Since the multipliers stay bounded, we can assume $\lambda_i^k \to \tilde{\lambda}_i$. The conditions (5)+(9) persist as $x_i^k \to \tilde{x}_i$ and $p^k \to \tilde{p}$ along with $x_i^{0k} \to \bar{x}_i$, and therefore $(\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_m) \in E(\bar{x}_1, \dots, \bar{x}_m)$. But that necessitates $(\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_m) = (\bar{p}, \bar{x}_1, \dots, \bar{x}_m)$, as just seen. This reveals, in terms of neighborhoods $N^0$ and $N$ in the shift stability of the equilibrium $(\bar{p}, \bar{x}_1, \dots, \bar{x}_m)$ in Theorem 1 that, once $(x_1^{0k}, \dots, x_m^{0k})$ is near enough to $(\bar{x}_1, \dots, \bar{x}_m)$, the entire set $E(x_1^{0k}, \dots, x_m^{0k})$ must lie in $N$ and therefore reduce to a singleton. Thus, $E$ must be single-valued in a neighborhood of $(\bar{x}_1, \dots, \bar{x}_m)$. Taking such an open neighborhood of each point of $X$ we get the set $X^0$.

The fact that $X$ is a differentiable manifold comes from recalling that $p$ and $(\lambda_i, \dots, \lambda_m)$ are continuously differentiable functions of $(x_1, \dots, x_m)$ in $X$, while on the hand, from the invertibility of the gradient mappings $\nabla u_i$, noted in the introduction, with $x_i = (\nabla u_i)^{-1}(\lambda_i p)$, we also have $(x_1, \dots, x_k)$ in $X$ being a continuously differentiable function of $p$ and $(\lambda_1, \dots, \lambda_m)$. More specifically, we have a diffeomorphism between $X$ and the elements $p$ and $(\lambda_1, \dots, \lambda_m)$ satisfying $\lambda_i p \in D_i$ for the open convex ranges $D_i$ of the mappings $\nabla u_i$. Because $p \in S$, this furnishes a global $m + n - 1$-dimensional parameterization of $X$ over a connected set of parameters.

**Proof of Theorem 3.** The linear equations in (8) are just the ones in (11). Their unique solvability and interpretation as giving derivatives were addressed in the proof of Theorem 1.

# References


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