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ON THE STABILITY AND EVOLUTION OF ECONOMIC EQUILIBRIUM

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Abstract. In an economic model of exchange of goods, an equilibrium of market prices and resultant holdings of the agents can, in response to small additions and subtractions of goods, uniquely reconstitute itself with slightly adjusted prices and holdings. That stability property of equilibrium is shown to persist even when individual agents are only interested in some of the goods and prefer zero quantities of others, as long as some good is indispensable to all agents and able to serve then effectively as money. Assumptions about marginal utility that entail concavity rather than just quasi-concavity of utility functions assist in establishing this and lead to a new vision of equilibrium where prices and holdings are not static. Instead, they evolve continuously in time according to a utility-based law in the form of a one-sided differential equation. Broad possibilities are opened up for dynamic modeling in which the changes in holdings that require ongoing readjustments could be driven by consumption, production, taxation or subsidy, among other influences. The goods can then be more than commodities destined for immediate disposal. Because the equilibrium at any moment has a past and a future, money, in particular, can carry value as a good and naturally enter that way into preferences.

In these developments, tools of convex analysis, and variational analysis beyond, are employed to extend and reorient stability results in the theory of economic equilibrium that previously had to rely on differential analysis alone.

Key Words. Walrasian economic equilibrium, market stability, evolutionary dynamics, evolution equation, concave utility functions, monotone mappings, variational inequalities, variational analysis

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[1]Deserving co-authorship of this paper, although not a writer of it, for his valuable influence at early stages through long discussions of how a market equilibrium should be approached and what it should stand for.

1 Introduction

Equilibrium in market models of exchange of goods among agents describes a situation in which the participants are satisfied with their holdings and have no wish to buy or sell. The main focus of discussion for economists has been on identifying prices under which agents, having quantities of goods they would prefer to change, can reach such an equilibrium all at once by individually optimized market transactions. When dynamics are brought up, the context is how the prices might be found through a negotiation process with an "auctioneer," called tâtonnement. But there is a different side to dynamics that deserves attention as well.

A market may be in equilibrium with respect to current prices and holdings, but those holdings could be affected by incremental additions and subtractions induced from outside the market. It can be imagined that the balance of supply and demand could naturally and uniquely then be restored under slightly adjusted prices. This might be repeated over and over in small intervals of time. In passing to an ideal limit as those intervals shrink to zero, the picture emerges of an equilibrium evolving in continuous time in response to external influences.

Could there be a kind of differential equation to describe this evolution in its dependence on the agents' utilities? That is the question we take up in this paper and *answer in the affirmative*, moreover in a setting where agents are not obliged to hold positive quantities of all goods, either initially or in equilibrium.[2] The specific mathematical results will be presented in Section 2 and proved in Section 3. The remainder of this introductory Section 1 is devoted to explaining the novelty of our contribution from perspectives of economics and methodology.

To achieve our results, we use convex analysis and tools of modern variational analysis for the first time in extending stability results for Walrasian equilibrium and tâtonnement that were previously obtained through standard differential analysis. However, we take the utility functions to be concave, in contrast to the usual assumption in equilbrium studies that they are quasi-concave, and this may be viewed by economists as a shortcoming. Whether our conclusions about stability and evolution of equilibrium might be reached with just quasi-concavity remains to be seen and could be addressed in the future. However, concavity is possibly essential, and then the issue perhaps is whether something is lacking in the classical utility axioms which lead only to quasi-concavity. The missing ingredient could be *marginal* utility. Shouldn't an agent, in contemplating a change to existing holdings in some direction, be able to quantify the rate of change of utility in that direction? And shouldn't that rate be subject to a law of diminishing returns? If so, then concavity would be guaranteed.[3]

The reason that no theory of evolving equilibrium has been articulated up to now may go back to the original Walras model. It loads market interactions with a heavy burden of artificiality by presenting them as having no past or future. The focus is on a single grand adjustment from original holdings of agents, however out of kilter they may be with preferences, to optimized holdings under somehow-identified prices that balance supply with demand. The thought that

[2]Technical insistence on positive quantities is typical in the literature on economic equilibrium.

[3]Anyway, preferences based on quasi-concave utility might be approximated arbitrarily closely by based on a concave utility; see Kannai [31] (1977). And, from an econometric point of view, finite demand data cannot test the difference; see Brown and Shannon [13] (2008).

an existing equilibrium might be pushed slightly out of balance by external influences and then undergo readjustment never comes up.

The traditional terms of discussion in the economics literature may play into the static mindset too. In the commonly used notation where vectors representing all the holdings of all the agents are denoted by ω and vectors of prices by p, one can think of a (generally) set-valued mapping that leads from initial holdings ω^0 to pairs (p, ω)[4] in which the agents are content with what they have. Then (p, ω) constitutes an equilibrium of prices and holdings in our sense that is *associated* with ω^0, and with ω as well, because $(p, \omega) \in \mathcal{E}(\omega)$. But economists instead speak of the pair (p, ω^0) as a Walras equilibrium. Depicted as (p, ω^0), with ω^0 fixed as giving the "endowments" of the agents, equilibrium can hardly be seen to evolve. Even stability, in referring to some continuity and single-valueness aspect of the mapping $\mathcal{E}$, is elusive, because examples are known in which (with quasi-concave utility, at least) the set $\mathcal{E}(\omega^0)$ can itself be utterly complex [37], [42]. How then could an incremental shift in ω^0 have a predictable outcome?

In a simple illustration of this fundamental issue, furnished by Dontchev and Rockafellar in [18], the same pair (p, ω) is associated with two different instances of initial holdings, ω_1^0 and ω_2^0, with ω_2^0 farther from ω than ω_1^0. Small perturbations of ω_1^0 lead to small, locally unique, perturbations of (p, ω), but small perturbations of ω_2^0 cause wild reactions. Those reactions do not, however, signify "instability of equilibrium" when equlibrium refers to (p, ω). *Stability of equilibrium should not be tied conceptually to a particular initial condition from which the equilibrium might be reached.*[5]

With all this in mind we concentrate on the equilibrium mapping $\mathcal{E}$ and its range, the equilibrium pairs (p, ω), instead of on the set of pairs (p, ω^0) widely studied by others as the "equilibrium manifold." We show that, in a neighborhood of the set Ω of equilibrium holdings, the ω components of the equilibrium pairs in our sense, $\mathcal{E}$ has a *single-valued localization* with properties of *semidifferentiability*. Actual differentiability as revealed by Balasko [6] in his classical context can't generally be expected in our context where orthant boundaries can have an active role. Continual perturbations in initial holdings may cause the continually adjusted equilibrium holdings to hit some part of the boundary, bounce off and then hit again. Such effects have to be taken into account in the potential dynamics of evolution of equilibrium, and this is where variational analysis from [44], [17] and eventually [18] will be critical.

Results of Balasko [6] and Sattinger [47] from 1975 did in fact indicate that, under certain assumptions, the mapping $\mathcal{E}$ has a single-valued localization when applied to holdings ω^0 that are close enough to the set Ω of equilibrium holdings.[6] In particular, then, for each $\omega \in \Omega$ there is a unique price vector p such that (p, ω) is an equilibrium pair. Balasko's stability results were extended by Keenan [32] in 1982, but reference to them can hardly be found elsewhere in equilibrium literature beyond later works of Balasko himself, e.g., [7, 8, 9]. They are not cited by Hirota [23] (1981), where an example indicating such localization influence was offered. Yun

[4]This is preliminary symbolism that later will be replaced by other notation anchored by rigorous definitions.

[5]That would conflict with the usage of the word in physics, game theory and elsewhere, in which a configuration deemed to give an "equilibrium" is kept separate from whatever process may have led to it.

[6]In Balasko's classical setting, Ω is a differentiable manifold, but in our extended setting with an active boundary, nonsmoothness of Ω has to be confronted.

[52] (1979) likewise omits Balasko, although he does cite Sattinger [47]. More recently the paper of Brown and Shannon [13] (2008), containing insights into the extent that "rationalization" of an economy from finite data can be carried out so as to promote tâtonnement, presents the convergence issue in terms of prices only; no localization in goods enters the picture.

Anyway, no one up to now has considered dynamics of the kind targeted here. Balasko in [8, 9] applied his results only to dynamics in the side scheme of tâtonnement.

In this paper[7] we re-examine the properties explored by Balasko and others, but are able to deal with resources on the boundary of the "goods orthant" by appealing to advances in variational analysis [44]. The boundary comes in because the utility functions we work with, although generally twice continuously differentiable as usual in stability studies, allow preferences for goods among the agents to differ sharply. Some goods can have no influence at all on a particular agent's utility while others are indispensable. Some can be attractive without being indispensable and may be present initially and at equilibrium in either positive or zero amounts. Although we require concavity, that anyway covers categories such as the econometrically important Cobb-Douglas utilities and even Scarf utilities, which are notorious for making tâtonnement fail to converge [48, p. 168].[8]

With preferences coming from utility functions as general as just described, we have to develop a convenient but broader criterion for existence of an associated equilibrium than the one ordinarily relied upon, in which a positive amount of every good must be present in the initial holdings of every agent. This is crucial, or course, to dynamics in which an evolving equilibrium might have some goods for some agents repeatedly switching between positive and zero.

Along with establishing that the equilibrium correspondence is locally stable in the sense of adjustments being well behaved, which we call *shift stability*, we demonstrate local stability with respect to Walrasian tâtonnement for the equilibrium pairs (p, ω), a property which we call *tâtonnement stability*. Due to boundary effects again, our tâtonnement price trajectories $p(t)$, in contrast to those of Balasko [6], have second derivatives that may only be one-sided because the right side of the differential equation is only semidifferentiable. The theoretical interest in tâtonnement stability lies in desiring to confirm that an economically plausible mechanism exists to restore equilibrium after a small perturbation. Walrasian tâtonnement is not, however, a *trading* mechanism. It refers rather to a fictional negotiation process in "pseudo-time" with artificially specified parameters. Early analysis of its stability utilized the differentiability of the right side of the differential equation, which is not available to us here. That was undertaken via the matrix in the linearization of the right side at an equilibrium. This approach, seen early in Hicks [21] (1939) and Metzler [40] (1945), led to the investigation of various matrix conditions with special meaning for economics, cf. Arrow and McManus [5] (1958). Other research, as in Arrow and Hahn [2] (1971), explored non-Walrasian adjustment processes which go beyond the

[7] An earlier version of this paper, with virtually same results but a different introduction and somewhat different packaging, has been available since October 2013 under the title "The robust stability of every equilibrium in economic models of exchange even under relaxed standard conditions" and can found on-line at SSRN: https://ssrn.com/abstract=2462975 or http://dx.doi.org/10.2139/ssrn.2462975.

[8] That lack of convergence stems in fact from initial holdings being too distant from equilibrium holdings, which is irrelevant for our study of small adjustments.

elicitation of supply-demand information and enter into iterative trading; cf. Keisler [33] (1996) and its references. The complicated history of interpreting the original ideas of Walras [51] (1874) is discussed in that article as well. See also Walker [50] (1987).

When we come to evolution of equilibrium being described by trajectories $(p(t), \omega(t))$, even the first derivatives in time may only be one-sided because of the boundary effects. Nonetheless we are able to derive a *one-sided differential equation* which describes how equilibrium must progress in time in response to exogenous "stresses" on the agents' holdings $\omega(t) \in \Omega$. On the interior of the orthant, this reduces to an ordinary differential equation in standard form.

The variational analysis texts of Rockafellar and Wets [44] and Dontchev and Rockafellar [17] provide background for the methodology employed here, but an article of Dontchev and Rockafellar [18], already invoking those tools in economics, is especially important to us in that respect. It is from there that we distill the broader existence result we need for evolution trajectories that might interact with the boundary of the goods orthant, and also some key properties of the equilibrium mapping. The approach to those issues in [18] depended on formulating the Walrasian model as a *variational inequality* problem which, along with prices and holdings as variables, also brought in Lagrange multipliers for budget constraints. Concavity of utility functions had a big influence in that, hence our attachment to concavity here. In particular it promotes duality in the analysis of the utility maximization problems of the agents. Variational inequality models of economic equilibrium have likewise been central to some our our work in previous papers: [24, 25, 26, 27, 28].

2 Statement of assumptions and the main results

Proceeding toward a precise formulation, we take the nonnegative orthant $I\!R^{n+1}_+$ as the space of goods[9] and suppose that the agents, indexed by $i = 1, \ldots, r$, have preferences on it which are given by utility functions u_i. To fully appreciate the equilibrium context we are aiming at, it is important to keep in mind that the goods can be very general, not just commodities destined for consumption. They can be anything physical, or perhaps even "rights," that an agent might wish to acquire and are available for trading in fixed supply. The question of what an agent might do with them is separate and will be revisited later.

Assumption A1 (utility fundamentals). *Each utility function u_i on $I\!R^{n+1}_+$ is nondecreasing, concave*[10] *and upper semicontinuous. It may take on $-\infty$, but if so, only at points on the boundary of $I\!R^{n+1}_+$. Relative to the set where it is finite, u_i is continuous.*[11]

An important provision will depend on classifying goods according to the interest that an agent has in them. A good will be called *attractive* for agent i if every increase in that good

[9] Having $n + 1$ instead of n will shortly be seen to help in the presentation when money enters.

[10] The distinction between concave and quasi-concave is vital here. For results about approximating quasi-concave utility by concave utility while taking into account various properties of strictness and differentiability, such as enter below, see Kannai [31], Mas-Colell [38, Chapter 2], and Connell and Rasmusen [14].

[11] Continuity on the interior of the orthant, where u_i is surely finite, is automatic from concavity, so this technical provision refers only to boundary behavior.

leads to a higher value of u_i. It will be called *indispensable* for agent i if it is attractive and, at any point in which the quantity of that good (but not every attractive good) is zero, either u_i takes on $-\infty$ or u_i is finite but the marginal utility of the good is $+\infty$.[12] [13]

Assumption A2 (indispensability). *There is a good that is indispensable to all agents. Every good is indispensable to at least one agent.*

The presence of a good that is indispensable to all agents will have a central role in what follows. It implies insatiability of all the utility functions, but it will have other major consequences as well. Classically standard assumptions, requiring the iso-surfaces of utility to curve away from orthant boundaries, actually force *every* good to be indispensable to *every* agent. We need just one such good and will work with it as a numéraire.

In our picture, some goods, far from being indispensable, can fail to be attractive at all to agent i. Other goods can be attractive without being indispensable. Our next condition sharpens the distinction.

Assumption A3 (unattractiveness). *If a good is not attractive to agent i, then it has no effect on the utility function u_i.*

In this condition we forgo the possibility of goods that might have positive marginal utility up to some level but zero marginal utility thereafter.

Assumption A4 (partial strong concavity). *With respect to the suborthants of the goods space that are defined by*

$$O_i = \{\, \textit{vectors in } \mathbb{R}^{n+1}_{+} \textit{ having positive components for goods indispensable to agent } i \,\}, \quad (1)$$

the utility functions u_i are twice continuously differentiable.[14] *Furthermore, the Hessian matrices, formed by the second partial derivatives, are negative definite with respect to the goods that are attractive to agent i.*[15]

In contrast to A4, it would classically be standard — with concave instead of just quasi-concave utility — to insist on the entire Hessian being negative definite. This would be combined with limiting attention to the interior of the goods space; the orthant boundary would not be allowed to complicate the analysis. But for economic theory to be more realistic, it ought to be permitted to do so.

With these assumptions at our disposal, we can move to more specific symbolism and formulations of stability. We concentrate on a particular good that is indispensable to all agents, as

[12] Marginal utility refers here to the one-sided directional derivative with respect to an increase in the good in question. That derivative exists from the concavity.

[13] This provision encompasses Cobb-Douglas utilities, for example, which are "infintely steep" along the boundary of the goods orthant.

[14] By this we mean that first and second partial derivatives not only exist continuously on the interior of O_i, namely the positive orthant itself, but also that these derivatives can be extended continuously, through limits, to the boundary points that belong to O_i.

[15] This refers to the submatrix of the Hessian obtained by excluding the goods that are not attractive. Under A3, such goods only yield zero derivatives anyway.

guaranteed by A2, calling it *money* for short.[16] [17] We designate quantities of money in the hands of agent i by m_i, and vectors giving quantities of the other goods by x_i, so that the elements of the goods space have the form (m_i, x_i) with $m_i \in \mathbb{R}_+$ and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n_+$. Initial holdings will have the notation (m^0_i, x^0_i).

Prices will always be denominated in money. Since money has price 1 with respect to itself, we will only need to be concerned with price vectors $p \in \mathbb{R}^n_+$ for the remaining goods.

Utility maximization problems. *The goal of agent i with respect to a price vector p and initial holdings $(m^0_i, x^0_i) \in O_i$ is to maximize the utility $u_i(m_i, x_i)$ over all goods vectors (m_i, x_i) satisfying the budget constraint*

$$m_i + p{\cdot}x_i = m^0_i + p{\cdot}x^0_i. \tag{2}$$

In dealing often with agents collectively, it will be expedient to use the "supervector" notation

$$(m, x) \text{ for } m = (\ldots, m_i, \ldots),\ x = (\ldots, x_i, \ldots),$$

and similarly (m^0, x^0) in the case of initial holdings, and so forth.[18]

Definition of equilibrium. *An equilibrium is a triple $(\bar{p}, \bar{m}, \bar{x})$ such that each $(\bar{m}_i, \bar{x}_i)$ component solves the utility maximization problem of agent i relative to $\bar{p}$ when $(m^0_i, x^0_i) = (\bar{m}_i, \bar{x}_i)$. More generally, $(\bar{p}, \bar{m}, \bar{x})$ is an equilibrium associated with initial holdings (m^0, x^0) possibly different from $(\bar{m}, \bar{x})$, written*

$$(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0),$$

if each pair $(\bar{m}_i, \bar{x}_i)$ solves the utility maximization problem of agent i relative to $\bar{p}$ and (m^0_i, x^0_i), and moreover

$$\textstyle\sum_{i=1}^r \bar{x}_i = \sum_{i=1}^r x^0_i. \tag{3}$$

The goods equation (3) requires supply to equal demand in all the goods other than money. Money can be left out because the corresponding condition

$$\textstyle\sum_{i=1}^r \bar{m}_i = \sum_{i=1}^r m^0_i \tag{4}$$

follows at once from (3) and the budget constraints (2). Apart from the money feature, this definition is fairly ordinary but depicts what others have called a "no-trade" equilibrium. The

[16] Our assumptions will guarantee that this good can serve as a numéraire for prices. Tradionalists may balk at our referring to it as "money," even as a convenient shortcut. However we have a different attitude and have argued strongly in our paper on financial equilibrium [28] for the importance of allowing money, indeed fiat money, to serve as a "good" in our generalized sense. The ideas of Keynes speak eloquently toward that, so we hope the reader will indulge us in this. Anyway, it is hard to think of a special good used to demoninate the prices of all other goods as anything but a form of money — as in the historic examples of gold and even seashells.

[17] In their 1988 textbook [22], Hildenbrand and Kirman in treating tâtonnement likewise appeal to numéraire prices, but they do not speak of the numéraire as "money"; the same for MacKenzie [36] (2002). This is reflected also in the earlier work of Uzawa [49]. In the more recent paper of Kitti [35] (2010), the reduction to numéraire prices is called price "normalization." For us, however, money does more than normalize. It will enter significantly into survivability conditions for existence and other assumptions.

[18] The notation (p, ω) invoked in the introduction will thus be replaced in what follows by (p, m, x), with p being of one less dimension than usual.

"no-trade" label has been dropped here for reasons explained in the introduction; no other kind of configuration of goods and prices is an "equilibrium" in our setting aimed at uncovering the dynamics of evolution.

Some observations can immediately be made which will simplify the discussions to come. First,

$$\text{in any equilibrium, all prices must be positive.} \tag{5}$$

This follows from A2 through the fact that if the price of some good were zero, then the maximization problem for an agent considering that good to be indispensable, or even just attractive, could not have a solution. Next,[19]

$$\begin{array}{l}\text{for } p > 0 \text{ the problem of an agent } i \text{ has a unique solution,}\\ \text{and it has to belong to the suborthant } O_i \text{ defined by (1).}\end{array} \tag{6}$$

Indeed, the budget constraint defines a compact set of goods vectors which meets the interior of $I\!R^{n+1}$, where utility is surely finite. The upper semicontinuity of u_i in A1 guarantees then that the maximum is finitely attained. Quantities of goods that are not attractive are pushed to zero, since otherwise they would drag down the budget available for attractive goods, but goods with infinite marginal utility at zero are forced to be positive. The partial strict concavity guaranteed by A4 then provides the uniqueness.

On the platform of (5) and (6) we can introduce, with respect to price vectors $p > 0$, the demand mappings X_i defined by

$$X_i(p; m_i^0, x_i^0) = \text{ the corresponding unique optimal } x_i \text{ for agent } i, \tag{7}$$

observing that the associated optimal money amount will then be

$$m_i = M_i(p; m_i^0, x_i^0) = m_i^0 + p{\cdot}[x_i^0 - X_i(p; m_i^0, x_i^0)], \tag{8}$$

while the excess demand mapping Z for the non-money goods will be given by

$$Z(p; m^0, x^0) = \textstyle\sum_{i=1}^r \left[X_i(p; m_i^0, x_i^0) - x_i^0\right]. \tag{9}$$

In this notation we can say for the correspondence E in the definition of equilibrium that, with respect to $\bar{x}_i = X_i(p; m_i^0, x_i^0)$ and $\bar{m}_i = M_i(p; m_i^0, x_i^0)$,

$$(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0) \iff \bar{p} > 0,\ Z(\bar{p}; m^0, x^0) = 0. \tag{10}$$

In particular, $(\bar{p}, \bar{m}, \bar{x})$ is an equilibrium when this holds with $(m^0, x^0) = (\bar{m}, \bar{x})$.

The following result about the existence of an equilibrium, the proof of which will come later, provides an important underpinning for our investigations of stability.

[19]For us, a strong vector inequality refers to a strict inequality in each component.

Ample Existence Theorem. *Under assumptions A1, A2, A3, A4, there exists for every instance of initial holdings* (m^0, x^0) *such that* $(m_i^0, x_i^0) \in O_i$ *for all agents* i *at least one equilibrium* $(\bar{p}, \bar{m}, \bar{x}) \in E(m^0, x^0)$. *Every equilibrium* $(\bar{p}, \bar{m}, \bar{x})$ *likewise has* $(\bar{m}_i, \bar{x}_i) \in O_i$ *for all agents* i.

An existence result that bears closely on this one was obtained in [18] in terms the initial holdings being "amply survivable." That condition, couched in money terms, is far weaker than the customary assumption that initial goods belong to survival set interiors.[20] It will be shown to be satisfied in the present setting by all the instances of initial holdings under consideration.

Two kinds of stability of an equilibrium will be considered in this framework. The first has to do with perturbations and the second with tâtonnement. The neighborhoods appearing in the definitions can be regarded as closed balls with respect to the Euclidean norm $||\cdot||$.

Definition of shift stability. *An equilibrium* $(\bar{p}, \bar{m}, \bar{x})$ *is* shift-stable *if there are neighborhoods* N_0 *of* $(\bar{m}, \bar{x})$ *and* N_1 *of* $(\bar{p}, \bar{m}, \bar{x})$ *such that*

for $(m^0, x^0) \in N_0$ *there is one and only one associated equilibrium* (p, m, x) *in* N_1,

and the corresponding localized equilibrium mapping

$$E : (m^0, x^0) \in N_0 \mapsto E(m^0, x^0) = (p, m, x) \in N_1, \ \text{ having } E(\bar{m}, \bar{x}) = (\bar{p}, \bar{m}, \bar{x}), \tag{11}$$

is Lipschitz continuous. The equilibrium is semidifferentiably *shift stable if* E *is not only Lipschitz continuous but also possesses, with respect to all choices of* $(m^{0\prime}, x^{0\prime})$, *the one-sided directional derivative*[21]

$$DE(m^0, x^0; m^{0\prime}, x^{0\prime}) = \lim_{h \to 0^+} \frac{1}{h}\Big[E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) - E(m^0, x^0)\Big]. \tag{12}$$

The one-sided limit in (12) with $h \to 0^+$ refers to h tending to 0 only from above; the classical two-sided directional derivative would have $h \to 0$ with no such restriction. The companion derivative expression with the opposite one-sided limit is tacitly covered by this as well, because

$$\lim_{h \to 0^-} \frac{1}{h}\Big[E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) - E(m^0, x^0)\Big] = -DE(m^0, x^0; -m^{0\prime}, -x^{0\prime}).$$

By virtue of the Lipschitz continuity in the definition of shift stability, we get a sort of Taylor expansion of the localized equilibrium mapping:

$$E(m^0 + hm^{0\prime}, x^0 + hx^{0\prime}) = E(m^0, x^0) + hDE(m^0, x^0; m^{0\prime}, x^{0\prime}) + o(h).$$

This corresponds to the differentiability of E at (m^0, x^0) if and only if $DE(m^0, x^0; m^{0\prime}, x^{0\prime})$ is *linearly* dependent on $(m^{0\prime}, x^{0\prime})$. Without that linearity it still signals *semidifferentiability,*[22]

[20] Apart from that strict positivity assumption there are other, more subtle equilibrium-supporting conditions in [1] and later in [19], [20], involving "irreducibility." However, these are all rather unwieldy in comparison with "ample survivability."

[21] The primes here do not, themselves, refer to derivatives.

[22] See [44, Chapter 7] for more on semidifferentiability.

which serves as the tool enabling us to handle the one-sided effects on equilibrium distributions of goods that may be caused by the orthant boundary coming into play.

Note that the possibility of there being more than one equilibrium associated with initial holdings (m^0, x^0) chosen from the neighborhood N_0 of $(\bar{m}, \bar{x})$ is not ruled out by shift stability. The requirement is only that there cannot be a second equilibrium within the specified neighborhood N_1 of $(\bar{p}, \bar{m}, \bar{x})$. For $(m^0, x^0) \notin N_0$ there might be multiple equilibria in N_1, but none could be an equilibrium associated with $(\bar{m}, \bar{x})$.

In turning next to tâtonnement, backed by shift stability, we follow Arrow and Hurwicz [3] in posing it in terms of an ordinary differential equation; see also [4]. Versions in could similarly be laid out in discrete (pseudo-)time along the lines in [22] and [49], but in our opinion the case of continuous time puts the ideas in sharper focus.[23]

Definition of tâtonnement stability. *An equilibrium $(\bar{p}, \bar{m}, \bar{x})$ is tâtonnement-stable if there is a neighborhood N of $\bar{p}$ and a neighborhood N_0 of $(\bar{m}, \bar{x})$ such that, for all $p^0 \in N$ and all $(m^0, x^0) \in N_0$ having $E(m^0, x^0) = (\bar{p}, \bar{m}, \bar{x})$,[24] the differential equation of tâtonnement, namely*[25]

$$\dot{p}(t) = Z(p(t); m^0, x^0) \ \text{ for } t \geq 0 \text{ with } p(0) = p^0, \tag{13}$$

has a unique solution $p(t)$ converging to $\bar{p}$, while the associated demands $x_i(t) = X_i(p(t); m^0, x^0)$ and $m_i(t) = M_i(p(t); m^0, x^0)$ converge then to $\bar{x}_i$ and $\bar{m}_i$. It is strongly tâtonnement stable if, for such neighborhoods, there is a constant $\mu > 0$ such that

$$(p' - p)\cdot(Z(p'; m^0, x^0) - Z(p; m^0, x^0)) \leq -\mu||p' - p||^2 \ \text{ for all } p', p \in N, \ (m^0, x^0) \in N_0. \tag{14}$$

According to (13), the price of a good rises at a rate equal to the current excess demand for that good (which amounts to falling if that is negative).[26] An equilibrium price vector $\bar{p}$, in having $Z(\bar{p}; m^0, x^0) = 0$, furnishes a stationary point for the differential equation (13): starting from $p^0 = \bar{p}$ one would get $p(t) \equiv \bar{p}$ as a solution.

The property in (14), which in the weaker form of having

$$(p' - p)\cdot(Z(p'; m^0, x^0) - Z(p, m^0, x^0)) < 0 \ \text{ when } p' \neq p, \tag{15}$$

is known as "monotonicity" among economists,[27] has already been recognized as guaranteeing

[23] Anyway, the issue here is a conceptual property of stability of an equilibrium. It deserves to be seen in basic terms without getting into a myriad possible variants, which perhaps would not add much in overall understanding to stability theory. See Kitti [35] for a comprehensive discussion of the efforts that have been put into the discrete-time case and the latest accomplishments in that direction.

[24] In the sense of (11); this N_0 shrinks the one there, if necessary.

[25] We employ $\dot{p}(t)$ for the derivative of $p(t)$ in order to preserve primed symbols like p' for other uses.

[26] It is easy to make the rates depend instead on different proportionality coefficients for different goods. However, this requires no additional mathematics because it really amounts only to changing the units of measurement for the goods other than money, e.g., prices per pound becoming prices per kilo. Arrow and Hurwicz observed this already in [3]. Other formulations in which the right side of (13) depends in extra ways on $p(t)$ have been explored by MacKenzie [36].

[27] There is an unfortunate conflict with long-established terminology in mathematics, according to which (15) is the *strict* monotonicity of $-Z(\cdot; m^0, x^0)$, not $Z(\cdot; m^0, x^0)$. Plain monotonicity would have $\leq$ in place of < 0, whereas (14) is the *strong* monotonicity of $-Z(\cdot; m^0, x^0)$. For an introduction to the remarkable theory of such monotonicity, which comprises a valuable and much applied a branch of convex analysis, see [43] and in greater detail [44, Chapter 12].

the convergence of $p(t)$ to $\bar{p}$, which is indeed easy to prove,[28] as for instance in the textbook of Hildenbrand and Kirman [22, page 237]. (The convergence of the demands $x_i(t)$ then follows from the continuity of the mappings X_i, which will come out later.) From the stronger property in (14) an additional conclusion readily follows about the convergence rate:

$$||p(t) - \bar{p}|| \leq e^{-\mu t}||p^0 - \bar{p}||. \tag{16}$$

Indeed, (14) implies even that two price trajectories $p_1(t)$ and $p_2(t)$ starting from different initial states $\bar{p}_1^0$ and $\bar{p}_2^0$ have $||p_1(t) - p_2(t)|| \leq e^{-\mu t}||p_1^0 - p_2^0||$.

The question of what aspects of utility might induce the "monotonicity" of excess demand has received attention from a number of researchers over the years; see the article of Quah [41] (2000) and its references. However, the results in that literature are not applicable in our context. They concern an excess demand mapping which differs from the one in (9) by being defined in terms of a wealth parameter that suppresses the role of initial holdings and their proximity to equilibrium holdings.

It has to be emphasized that *tâtonnement, as formulated here, does not represent a process in which distributions of goods get adjusted in real time.* Rather it is conceived as a scheme for exchange of information in "virtual time" (pseudo-time) by means of which a Walrasian broker or auctioneer struggles to determine prices that will bring supply and demand into balance.[29]

Robust Stability Theorem. *Under assumptions A1, A2, A3, A4, every equilibrium $(\bar{p}, \bar{m}, \bar{x})$ is both semidifferentiably shift-stable and strongly tâtonnement-stable. Moreover, in this setting the demand mappings X_i and the excess demand mapping Z are themselves Lipschitz continuous with one-sided directional derivatives, hence semidifferentiable.*

This will be proved below in Section 3. It should be noted that the combination of the two stability properties says more about tâtonnement stablity than might be apparent from the definition, where (m^0, x^0) is restricted to having $E(m^0, x^0) = (\bar{p}, \bar{m}, \bar{x})$. For any choice of (m^0, x^0) near enough to $(\bar{p}, \bar{m}, \bar{x})$, still in the range of uniqueness of equilibrium but with $E(m^0, x^0) \neq (\bar{p}, \bar{m}, \bar{x})$, the process can be activated anyway, with the only difference that it will converge instead to $E(m^0, x^0)$.

A result related to the theorem's assertions about shift stability was obtained in [18, Theorem 3], but not in the same framework. It will be crucial to our proof.

The stability properties confirmed in the Robust Stability Theorem may help further, beyond the insights of Balasko and others building on his work, to alleviate worries about the fragility of equilibrium. They also open up new horizons for exploration.

The idea that the goods acquired by the agents through trading yield general "holdings" with many conceivable attractions besides immediate "consumption" is crucial for contemplating a broad range of additional possibilities. From this angle, as explained in the introduction, equilibrium does not need be viewed statically. It can interpreted as modeling an observable phenomenon over time in which supply and demand, with respect to maintaining the agents'

[28] A simple tactic is to show by differentiation that $||p(t) - \bar{p}||^2$ is a decreasing function of t that must go to 0.

[29] This is made especially clear by Uzawa in [49]; see also the text of Mas-Colell [38, page 621].

holdings, stay close to being in balance, but the balance continually shifts due to the influence of various factors, both internal and external.[30]

Those factors could have many forms. A good that stands for a consumable commodity, for which the holding is a sort of stockpile, might be subject to a rate of consumption dictated by an agent's needs, or for that matter, deterioration due to environmental circumstances. Money, as a good, could be taken from an agent through exogenously instituted taxation. On the other hand, money or consumable goods could reach some agents through subsidies, and so on.[31] Still another possibility is production: An agent might be continually converting some goods into others, and that would lead to altered holdings; some goods could even be capital goods desired for this purpose. Whether that can convincingly be articulated as an economic model is beyond what we can take up here, and we only suggest for now that many avenues might be explored. The localized (or truncated) mapping $E : (m^0, x^0) \mapsto (p, m, x)$ described in the Robust Stability Theorem can help to clarify what would happen in such circumstances.

Evolutionary Dynamics Theorem. *Suppose $(p(t), m(t), x(t))$ is an equilibrium evolving over some interval of time t in response to modifications to $m(t)$ and $x(t)$ coming in at continuous rates $m_+(t)$ and $x_+(t)$.[32] Then $(p(t), m(t), x(t))$ depends locally Lipschitz continuously on t and has right derivatives $(\dot{p}^+(t), \dot{m}^+(t), \dot{x}^+(t))$ that satisfy the one-sided differential equation*

$$(\dot{p}^+(t), \dot{m}^+(t), \dot{x}^+(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \qquad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}). \tag{18}$$

For a locally Lipschitz continuous function $y(t)$ (which necessarily has a derivative $\dot{y}(t)$ almost everywhere), one-sided differentiability refers to the existence of the right or left limits

$$\dot{y}^+(t) = \lim_{h \to 0^+} \frac{1}{h}[y(t+h) - y(h)], \qquad \dot{y}^-(t) = \lim_{h \to 0^-} \frac{1}{h}[y(t+h) - y(h)]. \tag{19}$$

Differentiability corresponds to having $\dot{y}^+(t) = \dot{y}^-(t)$.

The details of the argument behind this result will appear in Section 3. One-sided, instead of two-sided, differentiability of the trajectory is unavoidable in this result, because, in general, some goods components in the $x(t)$ trajectory could start at 0, or drop to zero at a later time, only to eventually rise up and perhaps again drop down.

This result offers the interesting prospect that, within the confines of the model, both the prices and holdings in an equilibrium will *evolve in time according to a fixed rule, dictated only by the utility functions of the agents*, in response to internally/externally driven inputs. For instance, what might be expected if agents had their money holdings "controlled" by a government? Of course, the limitations of the idea are indeed many, and most important among them is the absence in this formulation of any modeling of uncertainty over the future. But our efforts on this can at least furnish a start for such a new direction of research.

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[30] This vision can be found in the 1941 paper of Samuelson [46] in the era before equilibrium had achieved an adequate mathematical formulation.

[31] This extended view of holdings and their potential persistence also underlies our work with financial market modeling in [28].

[32] The components of these input rate vectors may be positive, negative, or zero, thereby allowing for both additions and subtractions.

The Evolutionary Dynamics Theorem does not directly address the existence of a trajectory $(p(t), m(t), x(t))$ as described, the reason being the one-sidedness in the differential equation. Dealing with that will require mathematical innovations. But the complications in (18) with one-sidedness stem from letting the quantities of attractive goods sometimes be 0. *Around an equilibrium $(\bar{p}, \bar{m}, \bar{x})$ with everything positive, the differential equation (18) loses its one-sided aspects and takes the ordinary form*

$$(\dot{p}(t), \dot{m}(t), \dot{x}(t)) = DE(m(t), x(t); m_+(t), x_+(t)), \quad (p(0), m(0), x(0)) = (\bar{p}, \bar{m}, \bar{x}),$$

in which E is continuously differentiable instead of just semidifferentiable. The standard theory of differential equations is applicable then, and the existence of a trajectory of evolution over a time interval $[0, \varepsilon)$, at least, is fully assured.

This applies in particular to economies in which all goods are indispensable to all agents and thus to economies with utilities in Cobb-Douglas form or Scarf form, if every good is at least attractive to every agent. Of course the Robust Stability Theorem applies to those economies even without that provision of universal attractiveness. The misbehavior of tâtonnement, the demonstration of which was the motivation of Scarf for introducing his utility functions in [48], doesn't conflict with this, because that phenomenon depended on initiating the tâtonnement process from holdings far-enough away from any equilibrium holdings. Here we are focused on being near-enough.

Finally, it may be noted that a differential equation for prices and holdings with some superficial resemblance to (18) has appeared in work on a different topic by Bottazzi [10]. That work is not about evolution of an equilibrium subject to perturbations but rather about a sort of substitute for tâtonnement in which prices and holdings in disequilibrium follow a trajectory towards a state of equilibrium. Another distinction with respect to (18) is the need for parameters controlling the agents' actions to be imposed by an outside entity; the main theorem essentially just confirms that a successful control strategy is sure to exist in broad circumstances, although constructing it could be a more challenging matter. In contrast, (18) requires no outside manager to guide its operation.

3 The arguments behind the theorems

Proof of the Ample Existence Theorem. We rely for this on specializing the existence result of Dontchev and Rockafellar [18, Theorem 1]. The key to that result, besides the utility conditions in A1, is the following replacement for the usual assumption that all agents start with positive quantities of every good. The initial holdings (m^0, x^0) give *ample survivability* if the agents i have choices $(\hat{m}_i, \hat{x}_i)$ with $u_i(\hat{m}_i, \hat{x}_i) > -\infty$ such that

$$\text{(a)}\ \hat{x}_i \leq x_i^0 \ \text{but}\ \hat{m}_i < m_i^0, \ \text{and (b)}\ \textstyle\sum_{i=1}^r \hat{x}_i < \sum_{i=1}^r x_i^0.$$

The interpretation is that the agents could, if they wished, survive without any trading at all and do so with individual surpluses of money and collective surpluses in every other good.

To justify the claim made in the Ample Existence Theorem, we need to verify that

$$\text{all choices of } (m^0, x^0) \text{ with } (m_i^0, x_i^0) \in O_i \text{ give ample survivability.} \tag{20}$$

The argument is elementary and merely depends on observing the extent to which various components in (m^0, x^0) can be lowered slightly to get new holdings $(\hat{m}, \hat{x})$ such that $(\hat{m}_i, \hat{x}_i)$ still lies in O_i. This is evidently possible when the money holdings of all agents are positive and each other good is possessed in positive quantity by at least one agent. The definition of O_i in (1) ensures this through the indispensability in A2. The assertions in the theorem about every equilibrium $(\bar{p}, \bar{m}, \bar{x})$ come from the observations made in (5) and (6).

Proof of the Robust Stability Theorem. The assertions about shift-stability will be an application of the stability result of Dontchev and Rockafellar [18, Theorem 3].

An immediate consequence of (20), in combination with the fact noted earlier in (6) about solutions to the agents' optimization problems, is that

$$\text{in any equilibrium } (\bar{p}, \bar{m}, \bar{x}), \text{ the holdings } (\bar{m}, \bar{x}) \text{ as } (m^0, x^0) \text{ give ample survivability.} \tag{21}$$

Through this, the parametric stability result of Dontchev and Rockafellar [18, Theorem 3] can be applied to an equilibrium with respect to its own holdings. The result then asserts the shift stability of the equilibrium. (The cited result is applicable to any initial holdings (m^0, x^0) in some small-enough neighborhood of $(\bar{m}, \bar{x})$ as long as (m^0, x^0) gives ample survivability. Shift stability of an equilibrium, not defined or considered in [18], needs $(m^0, x^0) = (\bar{m}, \bar{x})$ itself to give ample survivability, and the guarantee of that is what is new here.)[33] The existence of one-sided derivatives of the localized equilibrium mapping is provided by [18, Theorem 3] as well.

That work relies, in particular, on conditions that characterize optimality in the agent's maximization problems. Those conditions will again have to come into play, so we record them next before going on with the remainder of the proof.

Because u_i is concave[34] and differentiable on the convex set O_i where any solution must lie, a condition both necessary and sufficient for (m_i, x_i) to be optimal for a given $p > 0$ can be given in terms of the gradient of u_i at (m_i, x_i) and a Lagrange multiplier λ_i for the budget constraint:

$$(m_i, x_i) \geq (0,0), \quad \lambda_i(1,p) - \nabla u_i(m_i, x_i) \geq (0,0), \quad (m_i, x_i) \cdot [\lambda_i(1,p) - \nabla u_i(m_i, x_i)] = 0, \tag{22}$$

where

$$m_i = m_i^0 + p \cdot (x_i^0 - x_i). \tag{23}$$

The so-called complementary slackness conditions (22), expressed in a manner typical in optimization, say that for each good the corresponding components of the nonnegative vectors (m_i, x_i) and $\lambda_i(1,p) - \nabla u_i(m_i, x_i)$ cannot both be positive; at least one or the other must be 0.

[33] The format in [18] is that of survival sets U_i not necessarily of orthant type, but the cited result depends on having orthant-like structure locally around the equilibrium under investigation. That is true automatically here for the same reasons that have been laid out in deriving (20).

[34] Plain quasi-concavity of the utility function u_i would not suffice for this.

Specializations can be gleaned from the categorization of goods in our model by attractiveness and indispensability. Let the goods other than money be indexed by $j = 1, \dots, n$, so that

$$x_i = (\dots, x_{ij}, \dots) \text{ with } x_{ij} \geq 0, \qquad p = (\dots, p_j, \dots) \text{ with } p_j > 0.$$

Since indispensable goods, including money, occur only in positive amounts in O_i, we can reduce (22) through assumptions A2 and A3 to

$$\begin{array}{ll} \lambda_i = (\partial u_i/\partial m_i)(m_i, x_i) & \text{(hence } \lambda_i > 0\text{)}, \\ \lambda_i p_j = (\partial u_i/\partial x_{ij})(m_i, x_i) & \text{for indispensable goods } j \text{ of agent } i, \\ \lambda_i p_j \geq (\partial u_i/\partial x_{ij})(m_i, x_i) & \text{for attractive but not indispensable goods } j, \\ & \text{with equality holding when } x_{ij} > 0, \\ x_{ij} = 0 \text{ for goods } j \text{ that are not attractive for agent } i. & \end{array} \tag{24}$$

The conditions for solutions to these problems for $i = 1, \dots, r$ to constitute an equilibrium (p, m, x) associated with (m^0, x^0) are the combination of (23) and (24) with

$$\textstyle\sum_{i=1}^r x_{ij} = \sum_{i=1}^r x_{ij}^0 \text{ for } j = 1, \dots, n. \tag{25}$$

The remainder of the proof, which is concerned with tâtonnement stability, must delve deeper into the variational analysis through which the results in [18] that we have been applying were themselves derived. Some background in [17], concerning solution mappings associated with variational inequality models for expressing optimality conditions and equilibrium, will be essential.[35] To make things easier for readers not familiar with that subject, we start with a brief overview.

Variational inequalities. *The variational inequality associated with a nonempty, closed, convex set $C \subset I\!R^N$ and a mapping $f : C \to I\!R^N$ with parameter $p \in I\!R^n$ takes the form finding $w \in C$ such that*

$$-f(p, w) \in N_C(w), \tag{26}$$

where $N_C(w)$ is the normal cone to C at w, defined by

$$v \in N_C(w) \iff w \in C \text{ and } v{\cdot}(w' - w) \leq 0 \text{ for all } w' \in C. \tag{27}$$

The normal cone $N_C(w)$ at any $w \in C$ is closed and convex. It always contains $v = 0$, and that is its only element when w is an interior point of C, which is true of course for every w in the special case when $C = I\!R^N$. The variational inequality reduces then to the vector equation $f(p, w) = 0$, and this is the sense in which variational inequality models expand on equation models. Vectors $v \neq 0$ necessarily exist in $N_C(w)$ when w is a boundary point of C. They can be of any length and are the outward normals to the (closed) supporting half-spaces to C at w.

The *solution mapping* associated with (26), which may be set-valued (i.e., a relation, or a correspondence in terminology common to economics literature), is

$$S : p \mapsto \{\, w \mid -f(p, w) \in N_C(w) \,\}. \tag{28}$$

[35]Variational analysis as laid out in [44] has also been the key to our other papers on economic equilibrium, namely [24], [25], [26] and [28].

Results in [17] generalize the classical implicit function theorem for equations by providing criteria under which, *in localization* around a pair $(\bar{p}, \bar{w})$ with $\bar{w} \in S(\bar{p})$, the mapping S is single-valued and Lipschitz continuous, moreover with one-sided derivatives having a specific formula. The best case, which will be in play here, centers on C being polyhedral, i.e., expressible as the intersection of a finite collection of closed half-spaces. A useful object then is the *critical cone* to C at a point $w \in C$ with respect to a normal $v \in N_C(w)$, which is the polyhedral cone

$$K(w, v) = \{\, w' \in T_C(w) \,|\, v{\cdot}w' = 0 \,\},$$

where $T_C(w)$ is the *tangent cone* to C at w, equal to the polar of $N_C(w)$. All these cones are important in the study of optimality conditions, and to a large extent the passage from equations to variational inequalities is motivated by modeling circumstances that involve first-order optimality conditions associated with inequality constraints, such as the nonnegativity of goods in our economic setting.

The form of generalized implicit function theorem for (26) that was basic in Dontchev and Rockafellar [18], and will be basic here again, refers to the smallest linear subspace $K^+(w, v)$ containing the critical cone $K(w, v)$ as well as the largest linear subspace, $K^-(w, v)$ contained within $K(w, v)$. Under the assumption that the function f in (26) is continuously differentiable, it focuses on a particular solution $w \in S(p)$ and invokes for the normal vector $v = -f(p, w)$ the criterion that

$$w' \in K^+(w, -f(p, w)),\;\; \nabla_w f(p, w)w' \perp K^-(p, -f(p, w)),\;\; w'{\cdot}\nabla_w f(p, w)w' \leq 0 \;\;\Longrightarrow\;\; w' = 0,$$

where $\nabla_w f(p, w)$ denotes the $N \times N$ Jacobian of f with respect to the w argument. The conclusion then is that the solution mapping S does have a single-valued Lipschitz continuous localization around (p, w) for which the one-sided derivatives relative to vectors p' exist and are given by

$$\begin{array}{rl} DS(p; p') = & \text{the unique solution } w' \text{ to the auxiliary variational inequality} \\ & -[\nabla_p f(p, w)p' + \nabla_w f(p, w)w'] \in N_{K(p,w)}(w'). \end{array}$$

This is from [17, Theorem 2E.8]. Note that in the equation case, with $C = I\!R^N$ and $f(p, w) = 0$, the critical cone and its associated subspaces are all just $I\!R^N$ itself. The criterion to be invoked reverts then to having $\nabla_w f(p, w)w' = 0$ imply $w' = 0$, or in other words, the full rank condition on the Jacobian matrix $\nabla_w f(p, w)$, as in the classical implicit function theorem.

For utilizing this general perturbation theory here, the target is the excess demand mapping Z in (9) and specifically the monotonicity-type property we claim for it in (14). That property will be deduced from a formula for one-sided derivatives of Z. Clearly from (9), the key ingredient in that has to be formulas for one-sided derivatives of the agents' demand mappings X_i in (7). From now on the initial holdings (m^0, x^0) will be fixed, so in working with these mappings we can pass to simpler notation:

$$X_i^0(p) = X_i(p; m_i^0, x_i^0), \qquad Z^0(p) = \textstyle\sum_{i=1}^r [X_i^0(p) - x_i^0] = Z(p; m^0, x^0). \tag{29}$$

It has already been noted that (through ample survivability) $X_i^0(p)$ is a uniquely determined goods vector in O_i for every price vector $p > 0$, and indeed that it is the unique solution to the conditions in (24) with m_i given by (23) (and λ_i given by the first line in (24)). We are involved, in other words, with solving these conditions for x_i as a function of p. If it were not for the third line in (24), we could view this from the classical perspective of solving a system of equations and try to apply the implicit function theorem. The inequality complication would drop away, of course, if we could be sure that the demand vector x_i would be > 0 in all its components, but allowing goods that are attractive but not indispensable to have zero demand for some combinations of prices is an important goal of our efforts.

It will help to reconfigure our task as the analysis of the enlarged mapping

$$S_i^0 : p \mapsto \{ (m_i, x_i, \lambda_i) \text{ satisfying (23)–(24) } \}. \tag{30}$$

From that analysis, the properties we require of X_i^0, as a component mapping, will be easy to extract. By interpreting S_i^0 as the solution mapping associated with a "variational inequality" problem, we will have available the above extension of the implicit function theorem, which can handle the inequality condition in (24).

In the case to which we want to apply this, the solution mapping will be S_i^0, already known to be single-valued. This case identifies (23)–(24) with the variational inequality[36]

$$-f_i(p, w_i) \in N_{C_i}(w_i) \text{ for } \begin{cases} w_i = (m_i, x_i, \lambda_i) \in C_i = I\!R_+^{n+1} \times I\!R, \\ f_i(p, w_i) = -(\nabla u_i(m_i, x_i) - \lambda_i(1, p), m_i - m_i^0 - p\cdot(x_i^0 - x_i)). \end{cases} \tag{31}$$

We will be analyzing this relative to an arbitrary $p > 0$ and $(m_i, x_i, \lambda_i) = w_i = S_i^0(p)$. Then $(m_i, x_i) \in O_i$, and since the analysis is local, the fact that ∇u_i is undefined at points of $I\!R_+^{n+1}$ outside O_i will not matter. The analysis will utilize the Jacobian expressions

$$\begin{aligned} &\nabla_p f_i(p, w_i)p' = [\, \lambda_i(0, p'), p'\cdot(x_i - x_i^0)\,], \\ &\nabla_{w_i} f_i(p, w_i)w_i' = [-\nabla^2 u_i(m_i, \bar{x}_i)(m_i', x_i') + \lambda_i'(1, p), -m_i' - p\cdot x_i'\,], \end{aligned} \tag{32}$$

where $\nabla^2 u_i$ is the matrix of second partial derivatives of u_i. It will involve us not only with the normal cone $N_{C_i}(w_i)$, but also its polar, the tangent cone $T_{C_i}(w_i)$, and the "critical cone"

$$K_i(p, w_i) = \{\, w_i' = (m_i', x_i', \lambda_i') \in T_{C_i}(w_i) \mid f_i(p, w_i)\cdot w_i' = 0\, \}. \tag{33}$$

Because C_i is a polyhedral convex set (actually a cone itself), the critical cone $K_i(p, w_i)$ is polyhedral convex as well. The theorem about solution mappings to variational inequalities over polyhedral sets that we are going to apply requires us also to look at

$$\begin{aligned} &K_i^+(p, w_i) = K_i(p, w_i) - K_i(p, w_i) = \text{smallest subspace} \supset K_i(p, w_i), \\ &K_i^-(p, w_i) = K_i(p, w_i) \cap K_i(p, w_i) = \text{largest subspace} \subset K_i(p, w_i). \end{aligned} \tag{34}$$

Perturbation Result to be Applied (as specialized from [17, Theorem 2E.8])**.** *Under the criterion that*

$$w_i' \in K_i^+(p, w_i), \quad \nabla_{w_i} f_i(p, w_i)w_i' \perp K_i^-(p, w_i), \quad w_i'\cdot\nabla_{w_i} f_i(p, w_i)w_i' \leq 0 \implies w_i' = 0, \tag{35}$$

[36]Again, this formulation could not be reached without utility being concave instead of just quasi-concave.

the solution mapping S_i^0 for the variational inequality (31) is Lipschitz continuous in a neighborhood of p and semidifferentiable there with one-sided directional derivatives given by

$$DS_i^0(p;p') = \begin{array}{l}\textit{the unique solution } w_i' \textit{ to the auxiliary variational inequality}\\ -[\nabla_p f_i(p,w_i)p' + \nabla_{w_i} f_i(p,w_i)w_i'] \in N_{K_i(p,w_i)}(w_i').\end{array} \tag{36}$$

The next step is to work out the details of this in our context of (31). The cone $K_i(p,w_i)$ and subspaces $K_i^+(p,w_i)$ and $K_i^-(p,w_i)$ come out, in expression with respect to the goods j, as

$$\begin{array}{l} K_i(p,w_i) = I\!R \times \Pi_{j=1}^n K_{ij}(p_j,m_i,x_{ij},\lambda_i) \times I\!R, \text{ where}\\ \qquad K_{ij}(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!R & \text{if } x_{ij} > 0,\\ I\!R_+ & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij}(m_i,x_i) = \lambda_i p_j,\\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) < \lambda_i p_j,\end{cases}\end{array} \tag{37}$$

$$\begin{array}{l} K_i^+(p,w_i) = I\!R \times \Pi_{j=1}^n K_{ij}^+(p_j,m_i,x_{ij},\lambda_i) \times I\!R, \text{ where}\\ \qquad K_{ij}^+(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!R & \text{if } x_{ij} > 0,\\ I\!R & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) = \lambda_i p_j,\\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i/\partial x_{ij})(m_i,x_i) < \lambda_i p_j,\end{cases}\end{array} \tag{38}$$

$$\begin{array}{l} K_i^-(p,m_i,x_i,\lambda_i) = I\!R \times \Pi_{j=1}^n K_{ij}^-(p_j,m_i,x_{ij},\lambda_i) \times I\!R, \text{ where}\\ \qquad K_{ij}^-(p_j,m_i,x_{ij},\lambda_i) = \begin{cases} I\!R & \text{if } x_{ij} > 0,\\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i)/\partial x_{ij})(m_i,x_i) = \lambda_i p_j,\\ \{0\} & \text{if } x_{ij} = 0 \text{ and } (\partial u_i)/\partial x_{ij}(m_i,x_i)) < \lambda_i p_j.\end{cases}\end{array} \tag{39}$$

In (37), (38) and (39) the same three categories of indices j are involved, and it will be convenient to speak of them as categories 1, 2 and 3. Unattractive goods j are clearly always in category 3, which gives $\{0\}$ in every case. Let J^+ refer to all the indices j in categories 1 and 2, and let J^- refer to those only in category 1. In these terms we proceed toward verifying (35), which can written as

$$\begin{array}{l} \text{only } (m_i',x_i',\lambda_i') = (0,0,0) \text{ satisfies the conditions}\\ \qquad (m_i',x_i',\lambda_i') = w_i' \in K_i^+(p,w_i),\\ \qquad [-\nabla^2 u_i(m_i,x_i)(m_i',x_i') + \lambda_i'(1,p), -m_i' - p{\cdot}x_i'] \perp K_i^-(p,w_i),\\ \qquad (m_i',x_i',\lambda_i'){\cdot}[-\nabla^2 u_i(m_i,x_i)(m_i',x_i') + \lambda_i'(1,p), -m_i' - p{\cdot}x_i'] \le 0,\end{array} \tag{40}$$

The first of the three conditions in (40) narrows our attention to cases of (m_i',x_i',λ_i') having $x_{ij}' = 0$ for all $j \notin J^+$, while the second further narrows it to $\lambda_i' = 0$ and $m_i' + \bar{p}{\cdot}x_i' = 0$, along with having the components of the vector $\nabla^2 u_i(\bar{m}_i,\bar{x}_i)(m_i',x_i')$ be 0 except possibly for some of them that belong to indices $j \notin J^-$. The quadratic expression in the third condition reduces then to $-(m_i',x_i'){\cdot}\nabla^2 u_i(\bar{m}_i,\bar{x}_i)(m_i',x_i')$. Because of the negative definiteness coming from our assumption A4, this expression cannot be ≤ 0 unless $m_i' = 0$ and $x_{ij}' = 0$ for all attractive goods j. But $x_{ij}' = 0$ already for unattractive goods, so we conclude that (40) does hold, and with it the properties of S_i^0 listed in the Perturbation Result above.

It follows then that the demand mapping X_i^0 is likewise locally Lipschitz continuous and semidifferentiable. More specifically, we have from (36) through (32) that

$$\begin{array}{l} DS_i^0(p;p') \text{ is the unique solution } (m_i',x_i',\lambda_i') \text{ to the variational inequality}\\ \qquad -[\lambda_i(0,p') + \lambda_i'(1,p) - \nabla^2 u_i(m_i,x_i)(m_i',x_i'), -p'{\cdot}(x_i - x_i^0) - m_i' - p{\cdot}x_i']\\ \qquad\qquad \in N_{K_i(p,w_i)}(w_i') \text{ with } w_i = (m_i,x_i,\lambda_i),\ w_i' = (m_i',x_i',\lambda_i'),\end{array} \tag{41}$$

where the details of the cone $K_i(p, w_i)$ are in (37). The one-sided directional derivatives of X_i^0 are given then by

$$DX_i^0(p; p') = x_i' \text{ for the } (m_i', x_i', \lambda_i') \text{ in (41).} \tag{42}$$

It is evident now that the excess demand mapping Z^0 in (29) is Lipschitz continuous locally as well, and semidifferentiable with its one-sided derivatives given by

$$DZ^0(p; p') = \sum\nolimits_{i=1}^r x_i' \text{ where } x_i' = DX_i^0(p; p'). \tag{43}$$

This brings us to the stage where we have confirmed all of the claims of the Robust Stability Theorem except for the "monotonicity" property (14). Directional derivatives will help with that, as follows. The inequality in (14) can be equivalently be rewritten (with a change of variables that alters the meaning of p') as

$$-\mu||p'||^2 \;\geq\; ([p+p'] - p){\cdot}(Z^0(p+p') - Z^0(p)) = \int_0^1 p'{\cdot}DZ^0(p + tp'; p')dt,$$

inasmuch as the (Lipschitz continuous) function $z(t) = Z^0(p + tp')$ is differentiable for almost every t with $\dot{z}^+(t) = DZ^0(p + tp'; p')$. Our task in these terms is reduced to demonstrating the existence of $\mu > 0$ for which

$$-\mu||p'||^2 \;\geq\; p'{\cdot}DZ^0(p; p') \;=\; p'{\cdot}\sum\nolimits_{i=1}^r DX_i^0(p; p') \text{ when } p \in N,\ p + p' \in N,\ p' \neq 0, \tag{44}$$

provided that the ball N around the equilibrium price vector $\bar{p}$ and the ball N_0 around the equilibrium holdings $(\bar{m}, \bar{p})$ in (14) are chosen small enough. Here by (42) we have

$$p' \sum\nolimits_{i=1}^r DX_i^0(p; p') = \sum p'{\cdot}x_i' \text{ with } x_i' \text{ from } (m_i', x_i', \lambda_i') \text{ solving (41)} \tag{45}$$

and can make that the platform for our analysis.

It is important now to notice a sort of uniformity in the local behavior of the sets in (37), (38) and (39), namely that

$$\begin{gathered} K_i^-(\bar{p}, \bar{w}_i) \subset K_i^-(p, w_i) \subset K_i(p, w_i) \subset K_i^+(p, w_i) \subset K_i^+(\bar{p}, \bar{w}_i) \\ \text{for } (p, w_i) \text{ near enough to } (\bar{p}, \bar{w}_i). \end{gathered} \tag{46}$$

This is evident from the formulas for these sets and the continuity of the partial derivatives of u_i. A follow-up to this observation, taking advantage of the fact that making p be close to $\bar{p}$ also makes $w_i = S^0(p)$ be close to $\bar{w}_i = S_i^0(\bar{p})$ through the continuity of S_i^0, is that

$$\begin{aligned} &\text{if } (m_i', x_i', \lambda_i') \text{ solves in (41) with } p \text{ near enough to } \bar{p}, \text{ then } (m_i', x_i', \lambda_i') \in K_i^+(\bar{p}, \bar{w}_i), \\ &-[\lambda_i(0, p') + \bar{\lambda}_i'(1, p) - \nabla^2 u_i(m_i, x_i)(m_i', x_i'), -p'{\cdot}(x_i - x_i^0) - m_i' - p{\cdot}x_i'\,] \perp K_i^-(\bar{p}, \bar{w}_i), \\ &(m_i', x_i', \lambda_i'){\cdot}\Big(-[\lambda_i(0, p') + \lambda_i'(1, p) - \nabla^2 u_i(m_i, x_i)(m_i', x_i'), -p'{\cdot}(x_i - x_i^0) - m_i' - p{\cdot}x_i'\,]\Big) = 0. \end{aligned} \tag{47}$$

In view of (38) and (39), we must then have

$$\begin{gathered} p'{\cdot}(x_i - x_i^0) + m_i' + p{\cdot}x_i' = 0 \text{ for all agents } i, \text{ and} \\ x_{ij}' = 0 \text{ when a good } j \text{ is unattractive to agent } i. \end{gathered} \tag{48}$$

The equation in the third condition of (47) reduces in this case to

$$-\lambda_i{\cdot}x_i'{\cdot}p' + \lambda_i'{\cdot}(m_i' + x_i'{\cdot}p) - (m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') = 0.$$

Since $\lambda_i > 0$, we can rewrite this, using the first line of (48), as

$$p'{\cdot}x_i' = \lambda_i^{-1}\Big[(m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') + \lambda_i' p'{\cdot}[x_i - x_i^0]\Big], \tag{49}$$

where moreover $(m_i', x_i'){\cdot}\nabla^2 u_i(m_i, x_i)(m_i', x_i') < 0$ unless $(m_i', x_i') = (0,0)$ through the second line of (48) and the negative definiteness of the submatrix of $\nabla^2 u_i(m_i, x_i)$ with respect to the attractive goods for agent i in our assumption A4. Hence

$$p'{\cdot}\sum\nolimits_{i=1}^r x_i' \;=\; \sum\nolimits_{i=1}^r (m_i', x_i'){\cdot}\Big[\frac{1}{\lambda_i}\nabla^2 u_i(m_i, x_i)\Big](m_i', x_i') + \sum\nolimits_{i=1}^r \frac{\lambda_i'}{\lambda_i} p'{\cdot}[x_i - x_i^0], \tag{50}$$
where the first sum on the right is < 0 unless $(m_i', x_i') = (0,0)$ for all i.

The crux of the matter emerges as making sure that the negativity of the quadratic sum in (50) cannot be overpowered by the second sum. The quadratic sum has the form

$$(m', x'){\cdot}A(m, x, \lambda)(m', x') \;\text{ for }\; m' = (\ldots, m_i', \ldots),\; x' = (\ldots, x_i', \ldots),$$

where $A(m, x, \lambda)$ is a negative definite matrix depending continuously on (m, x, λ), which in turn is comprised of elements $(m_i, x_i, \lambda_i) = S_i^0(p) = S_i(p; m_i^0, x_i^0)$ that depend continuously on p and also on (m^0, x^0).[37] Its eigenvalues can therefore be bounded locally away from 0:

$$\begin{gathered}\text{there exist } \varepsilon > 0 \text{ and closed balls } N \text{ at } \bar{p} \text{ and } N_0 \text{ at } (\bar{m}, \bar{x}) \text{ such that}\\ \sum\nolimits_{i=1}^r (m_i', x_i'){\cdot}\Big[\frac{1}{\lambda_i}\nabla^2 u_i(m_i, x_i)\Big](m_i', x_i') \;\le\; -\varepsilon||(m', x')||^2 \;\text{ when }\; p \in N,\; (m^0, x^0) \in N_0.\end{gathered} \tag{51}$$

An upper estimate of the size of the second sum in (50) must next come into play. For this we return to the conditions in (47). With $(\ldots, \lambda_i', \ldots)$ denoted by λ', let

$$W(p; m^0, x^0) = \{\, (m', x', \lambda', p') \text{ satisfying the first two conditions in (47) for all } i \,\}.$$

Because $K_i^+(\bar{p}, \bar{w}_i)$ and $K_i^-(\bar{p}, \bar{w}_i)$ are linear subspaces, $W(p; m^0, x^0)$ is a linear subspace as well. We claim

$$\begin{gathered}\text{there exist } \rho > 0 \text{ and balls } N' \text{ at } \bar{p} \text{ and } N_0' \text{ at } (\bar{m}, \bar{x}) \text{ such that}\\ ||(p', \lambda')|| \;\le\; \rho||(m', x')||\\ \text{for all } (p', m', x', \lambda') \in W(p; m^0, x^0) \text{ when } p \in N',\; (m^0, x^0) \in N_0'.\end{gathered} \tag{52}$$

If this were not true, there would be sequences of elements

$$\begin{gathered}(p'^k, m'^k, x'^k, \lambda'^k) \in W(p^k; m^{0k}, x^{0k}) \text{ with } p^k \in N',\; (m^{0k}, x^{0k}) \in N_0',\\ \text{such that } ||(p'^k, \lambda'^k)|| = 1 \text{ for all } k \text{ and } |(m'^k, x'^k)|| \to 0.\end{gathered}$$

[37]The continuity of $S_i(p; m_i^0, x_i^0)$ with respect to (m_i^0, x_i^0) is seen from the optimality conditions (23)–(24) for the optimization problem of agent i, which involve (m_i^0, x_i^0) only through (23). A limit of solutions to these conditions coming from a convergent sequence of such initial holdings must be another solution.

Passing to convergent subsequences, we would arrive in the limit at elements

$$(p'^*, m'^*, x'^*, \lambda'^*) \in W(p^*; m^{0*}, x^{0*}) \text{ with } p^* \in N',\ (m^{0*}, x^{0*}) \in N_0',$$
$$\text{such that } (p'^*, \lambda'^*) \neq (0,0) \text{ but } (m'^*, x'^*) = (0,0).$$

This is an impossible situation for the following reason. It entails

$$-[\lambda_i^*(0, p'^*) + \bar{\lambda}_i'^*(1, p^*), -p'{\cdot}(x_i^* - x_i^{0*})\,] \perp K_i^-(\bar{p}, \bar{m}_i, \bar{x}_i, \bar{\lambda}_i),$$

which first implies $\bar{\lambda}_i'^* = 0$ and then that $p_j'^* = 0$ for all goods j having $x_{ij}^* > 0$. But then $p_j'^* = 0$ for all goods j, yielding a contradiction because our assumptions make it impossible for any j to have $x_{ij}^* = 0$ for every agent i. Thus, (52) is confirmed.

Putting (52) now to use, and noting that $||(p', \lambda')|| \leq \rho||(m', x')||$ implies that $||p'|| \leq \rho||(m', x')||$ and $|\lambda_i')| \leq \rho||(m', x')||$ for all i, as well as $||x_i - x_i^0|| \leq ||x - x^0||$, we get the upper bound

$$\sum_{i=1}^r \frac{\lambda_i'}{\lambda_i} p'{\cdot}[x_i - x_i^0] \;\leq\; \rho\Big[\sum_{i=1}^r \frac{1}{\lambda_i}\Big]||x - x^0||\,||(m', x')||^2,$$

which holds when p is close enough to $\bar{p}$ and (m^0, x^0) is close enough to $(\bar{m}, \bar{x})$. Since $\lambda \to \bar{\lambda}$ and $x \to \bar{x}$ as $p \to \bar{p}$ because $(\bar{m}_i, \bar{x}_i, \bar{\lambda}_i) = S_i(\bar{p}; m^0, x^0)$ for all i, there is also a local upper bound

$$\Big[\sum_{i=1}^r \frac{1}{\lambda_i}\Big]||x - x^0|| \leq \nu||\bar{x} - x^0||.$$

Putting all this together with (51), we obtain from (50) and (52) that

$$p'{\cdot}\sum_{i=1}^r x_i' \;\leq\; -\Big(\varepsilon - \nu\rho||\bar{x} - x^0||\Big)||(m', x')||^2 \;\leq\; -\rho^{-2}\Big(\varepsilon - \nu\rho||\bar{x} - x^0||\Big)||p'||^2$$

when p is close enough to $\bar{p}$. The coefficient $\mu = \rho^{-2}(\varepsilon - \nu\rho||\bar{x} - x^0||)$ is sure to be > 0 when x^0 is close enough to $\bar{x}$. We have already determined that having $(m', x') = (0,0)$ is incompatible with $p' \neq 0$, so the desired conclusion, supporting the existence of a neighborhood N as in (44), has been reached.

Proof of the Evolutionary Dynamics Theorem. We are dealing with a time-dependent equilibrium, $(p(t), m(t), x(t)) = E(m(t), x(t))$, affected somehow by additions to $m(t)$ and $x(t)$ that enter at rates $m_+(t)$ and $x_+(t)$. The evolution can be understood through consideration of time increments $h > 0$ that cause $m(t)$ and $x(t)$ to shift approximately to $m(t) + hm_+(t)$ and $x(t) + hx_+(t)$. These could be out of equilibrium and no longer be matched by $p(t)$, but on the basis of the Robust Stability Theorem an adjusted equilibrium

$$(p(t+h), m(t+h), x(t+h)) = E(m(t) + hm_+(t), x(t) + hx_+(t)),$$

identifiable by tâtonnement, will exist uniquely nearby. Then also

$$(p(t+h), m(t+h), x(t+h)) = E(m(t+h), x(t+h)),$$

along with $(p(t), m(t), x(t)) = E(m(t), x(t))$, so the corresponding rate of change in the components of the equilibrium is

$$\begin{aligned}&\tfrac{1}{h}[(p(t+h), m(t+h), x(t+h)) - (p(t), m(t), x(t))]\\ &\qquad = \tfrac{1}{h}[E(m(t) + hm_+(t), x(t) + hx_+(t)) - E(m(t), x(t))].\end{aligned}$$

In taking the limit of this as $h \to 0^+$ we obtain, on the right, the one-sided directional derivative $DE(m(t), x(t); m_+(t), x_+(t))$.

This establishes in particular that the trajectory does have right derivatives. A parallel argument equally well serves to establish that the trajectory has left derivatives. Even though the theorem does not deal with left derivatives, the existence of both, along with the local boundedness of the expression on the right side of the differential equation makes these both one-sided derivatives be locally bounded, and the local Lipschitz continuity in t then follows for the trajectory.

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