Variational Analysis of Preference Relations and Their Utility Representations

R. Tyrrell Rockafellar[1]

Abstract

Preference relations on nonnegative orthants are important in economics for understanding how agents might decide between different vectors of goods. Under commonly accepted assumptions, they can be represented by continuous quasi-concave utility functions that are also quasi-smooth. Here, conditions intrinsic to a preference relation are identified that correspond to the possibility of a representation by a quasi-concave utility function that is continuously differentiable, first-order or second-order. However, even without such conditions, marginal preferences, which operate on an infinitesimal level, turn out to be available always for guidance in optimizing an agent's holdings. Tangent and normal cones to the graph of a preference relation are explored more broadly for additional insights into smoothness from the perspective of variational geometry. The combination of intrinsic second-order smoothness with strong convexity of preference sets furnishes, relative to any compact subset of the positive orthant, a minimally concave $\mathcal{C}^2$ utility function which is necessarily unique up to affine rescaling. Cardinal instead of merely ordinal utility can essentially be relied on then in an agent's decision-making.

Keywords: *preference relations, utility functions, differentiable utility, concave utility, minimal concavity, marginal preferences, variational analysis*

Version of 26 October 2023

[1]University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195-4350;
E-mail: *rtr@uw.edu*, URL: sites.math.washington.edu/∼rtr/mypage.html

1 Introduction

In modeling the behavior of agents such as consumers in mathematical economics, it is essential to know when one vector of goods might be preferred over another. The space of goods vectors of interest to an agent is a subset G of the nonnegative orthant $I\!R^g_+$, where g is the number of goods under consideration; the jth component of $x \in G$ gives the amount of good j. It will be assumed here that G includes the positive orthant $I\!R^g_{++}$,[2] so that

$$\operatorname{int} G = I\!R^g_{++}, \qquad \operatorname{cl} G = I\!R^g_+. \tag{1.1}$$

Preferences in G are articulated as a relation $x \preceq y$, according to which y is at least as desirable as x. The reverse is $x \succeq y$, and if both hold there is said to be indifference between x and y, symbolized by $x \sim y$. The strict relation $x \prec y$ refers to having $x \preceq y$ but not $x \sim y$, and similarly $x \succ y$. The preferences are said to be representable by a utility function u when

$$x \preceq y \Longleftrightarrow u(x) \leq u(y), \qquad x \prec y \Longleftrightarrow u(x) < u(y), \qquad x \sim y \Longleftrightarrow u(x) = u(y), \tag{1.2}$$

where the second and third equivalences follow from the first.

The theory of preference relations was put together by economists with some use of convex analysis but not its extensions into modern variational analysis, as in [15]. Here we draw on such mathematical advances to see what more can be gleaned from commonly employed axioms, specifically the following:

(A1) *for all $x, y \in G$, either $x \preceq y$ or $x \succeq y$ (or both).*

(A2) *if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.*

(A3) *the set $\{ (x, y) \in G \times G \mid x \preceq y \}$ is closed in $I\!R^g_+ \times I\!R^g_+$.*

(A4) *if $x \in G$ and $y \geq x$, $y \neq x$, then $y \in G$ and $y \succ x$.*

(A5) *for every $x \in G$, the set $P(x) = \{ y \in G \mid y \succeq x \}$ is convex.*

(A6) *for $x \in \operatorname{int} G$, the set $P(x)$ has no more than one supporting hyperplane at x.*

The microeconomics textbook of Mas-Colell, Whinston and Green [14] provides valuable resources for this subject. Already from (A1) and (A2), it is known for instance that the preference relation can indeed be represented by a utility function u, which (A3) makes continuous. Under (A4), $u(x)$ must increase with any increase in any goods component of x. Economists often get by with weaker versions of monotonicity, in which $u(x)$ just never decreases when goods components increase, and never reaches a maximum, but (A4) will lead us down a more interesting path. The convexity axiom (A5) makes u be *quasi-concave* in having upper level sets that are convex, while the hyperplane axiom (A6) makes it be *quasi-smooth* by eliminating kinks on the surfaces of those sets.

Because $x \sim y$ corresponds to $u(x) = u(y)$, it's clear that "$\sim$" is an equivalence relation for which the equivalence classes are the level sets $u =$ constant. These are identifiable through (A4) as the boundaries of the convex sets $P(x)$ in (A5), inasmuch as

$$P(x) = P(x') \ \text{ when } \ x \sim x' \tag{1.3}$$

in consequence of (A2). Any other utility function u' for the same preferences has to have these same level sets and therefore can differ from u only by a rescaling function, namely

$$u_*(x) = \theta(u(x)) \ \text{ for some } \theta \text{ that is continuous and always increasing.} \tag{1.4}$$

[2] To avoid the ambiguity of ">" in a vector inequality, we indicate that $x \in I\!R^g_{++}$ by $x \gg 0$ and saying x is *positive*.

A question then is whether rescaling, along with plausible supplements to the assumptions (A1)–(A6) on the preference relation that economists could generally be comfortable with, might enable representation by a utility function having superior properties than those already prescribed. This is the core motivation behind our investigations here.

An example is whether a preference relation can be represented by a utility function that is concave, instead of just quasi-concave. Concavity would be far more advantageous for numerical work in optimization as well as in equilibrium models of variational inequality type, as developed in [9, 10, 11, 12]. For econometric purposes in which an agent's preferences are probed through discrete experiments, modeling with concave utility does alway suffice, as demonstrated by Brown and Shannon [3] (2008). Studies of when concavity can be counted on in theory have been undertaken in the past by de Finetti [7] (1949), Fenchel [8] (1956), Aumann [1] (1975),[3] Debreu [5] (1976), and most definitively Kannai [13] (1977).[4] However, loose ends have remained in identifying elementary features of a preference relation that ensure at least regional representation by a "natural" utility function that is unique up to the choice of units of measurement.

Another example is whether a preference relation can be represented by a utility function that is also differentiable. Continuous differentiability of first-order or second-order is often assumed for convenience, but can that meaningfully be tied to added conditions akin to (A1)–(A6)? Efforts so far have centered on methods of differential geometry that instead require complicated partial differential equations to be satisfied; see for instance Debreu [6] and the history reviewed there. This relates back to the concavity question, because that can benefit from already having $\mathcal{C}^2$ utility at hand.

On a more fundamental level there is the question of whether a preference relation can provide *marginal* preference relations that apply *infinitesimally* to how a goods vector might change.

It will be established here, perhaps surprisingly, that marginal preferences (existing in a precise sense) are guaranteed by (A1)–(A6) even when there is no everywhere differentiable utility representation. On the other hand, simple necessary and sufficient conditions on the preference relation, not invoking equations of differential geometry, will be presented that characterize when a $\mathcal{C}^1$ or a $\mathcal{C}^2$ utility representation does surely exist. In the $\mathcal{C}^2$ case under the assumption that the preference sets $P(x)$ are *strongly* convex (a bit beyond strictly convex in also having reliably curved boundaries),[5] it will be shown that a *concave* utility representation is available over any compact subset of $\mathbb{R}^g_{++}$. Moreover it can be taken to be *minimally* concave, which makes it be *unique up to affine rescaling*, i.e., up to just a change of units of measurement, like switching in temperature between Celcius and Fahrenheit.

The concept of a minimally concave utility representation started with de Finetti [7] and was later taken up by Debreu [5] and finally Kannai [13]. It has several equivalent descriptions, one of them being a concave utility representation from which all other concave utility representations can be derived by rescaling as in (1.4) with a function θ that is *concave*.[6] Previous work established the existence of a minimally concave utility representation in various complicated circumstances, difficult to confirm in practice. That previous work furthermore got minimally concave $\mathcal{C}^2$ utility only by making an *ad hoc* assumption that a $\mathcal{C}^2$ utility representation was available. There is less import then

[3]The author is indebted to a referee for bring this paper to his attention. On the surface it concerns game theory with a continuum of players, but significant contributions to our topic here are buried inside.

[4]More recently, Connell and Rasmusen [4] (2017) have carried out some extensions to domains like Riemannian manifolds that lie beyond our topic here.

[5]Strict convexity unfortunately called strong convexity by Debreu in [6], but the distinction is in convex analysis is longstanding.

[6]Uniqueness up to affine rescaling follows from this description, because if u_1 and u_2 are both minimally concave, there are continuous concave functions θ_1 and θ_2 such that $u_2 = \theta_1 \circ u_1$ and $u_1 = \theta_2 \circ u_2$. But then θ_1 and θ_2 are inverse to each other, hence affine.

to the existence than for our result, where such a representation is developed from elementary axioms on the preference relation.

Aumann in [1] implicitly obtained a result similar to ours on the existence of a concave utility representation, but lacking the claim of minimal concavity with its accompanying property of uniqueness.[7] Where we bring in strong convexity, he appeals to bounds on Gaussian curvature of hypersurfaces. His goods space G is all of $I\!R^g_+$, and he starts by assuming the existence of a utility function that can be extended to be $\mathcal{C}^2$ on a neighborhood of $I\!R^g_+$. That excludes the common case in economics where G is just $I\!R^g_{++}$ and there is no utility representation that extends differentiably, or perhaps even finitely, to the boundary points of $I\!R^g_+$.

More will be explained now about the approach we take in this paper. Assumptions on the geometry of the convex sets $P(x)$ escape existential controversy, since that geometry is intrinsic to the preferences and not some artificial mathematical add-on. The quasi-smoothness condition in (A6) belongs to that category. Economists are content even with stronger versions in which the supporting hyperplane at x depends differentiably on x. But there is another aspect of preferences that relates deeply to "smoothness" as well. Due to (A4), the positive multiples rx of any goods vector $x \in I\!R^g_{++}$ have $rx \preceq r'x$ if and only if $r \leq r'$, and as r increases, eventually $rx \succ z$ for any given $z \in G$. The same holds for the positive multiples sy of any $y \in I\!R^g_{++}$, so there is a one-to-one correspondence between values r and s described by a set $\Theta_{x,y}$ which is the graph of an increasing function $\theta_{x,y}$ from $(0, \infty)$ onto $(0, \infty)$ that by (A3) is continuous:

$$\operatorname{gph} \theta_{x,y} = \Theta_{x,y} = \{\, (r, s) \in I\!R^2_{++} \mid rx \sim sy \,\}. \tag{1.5}$$

By taking possible differentiability properties of the functions $\theta_{x,y}$ into account, we will be able to pin down exactly the conditions on preferences that support representation by smooth utility functions.

Besides the basic scaling comparisons in (1.5), which are new in being made a centerpiece of theory, there are comparisons to look at between r and s scales along line segments $\{\, x + r\xi \mid -\varepsilon < r < \varepsilon\}$ and $\{\, y + s\eta \mid -\delta < s < \delta\}$ through points $x \sim y$. Their role in providing useful information about utility can already be seen in the paper of Kannai [13] and before.

In still more fundamental territory for variational analysis of preferences, unexplored until now, properties akin to differentiability will be investigated in terms of graphical derivatives and coderivatives[8] of the convex-set-valued mapping

$$P : x \in G \mapsto P(x) \subset G. \tag{1.6}$$

Such generalized derivatives correspond to tangent cones and normal cones to

$$R = \{\, (x, y) \in G \times G \mid x \preceq y\} = \operatorname{gph} P, \tag{1.7}$$

the *graph* of P, which is a closed subset of $I\!R^g_+ \times I\!R^g_+$ by (A3). As seen for instance from a utility representation of the preferences in which, under our axioms, (1.2) holds with u continuous and increasing with any increase in any of its arguments,

$$R = \operatorname{cl}[\operatorname{int} R], \quad \text{where} \quad \operatorname{int} R = \{\, (x, y) \in \operatorname{int} G \times \operatorname{int} G \mid x \prec y\}, \tag{1.8}$$

[7] The result is "implicit" because it's embedded in more complicated assertions about infinite collections of preference relations that need to satisfy a number of conditions uniformly in a sense. It appears only as Lemma 15.1 in the lengthy proof of something else in his paper [1].

[8] Developed in [15, Chapter 8].

and furthermore

$$[\operatorname{int} G \times \operatorname{int} G] \cap \operatorname{bdry} R = \{\, (x,y) \in \operatorname{int} G \times \operatorname{int} G \mid x \sim y\}. \tag{1.9}$$

Thus, within $I\!R^g_{++} \times I\!R^g_{++}$, the boundary of R is geometrically a *hypersurface* separating the open sets where $x \prec y$ or $x \succ y$, which are geometric reflections of each other under $(x,y) \longleftrightarrow (y,x)$. The properties of that hypersurface have obvious significance for understanding the information carried by the preference relation. What if the boundary in (1.9) is smooth in the first degree, or in the second degree? If the preferences originate in the mind of a consumer instead of, say, the robotic management of a production process, wouldn't such smoothness be a natural expectation?

The key to understanding the boundary of R and its possible smoothness is taking a close look at the pairs (x,y) in (1.9) and the sets

$$R^\tau_{x,y} = \{\, (\xi,\eta) \in I\!R^g \times I\!R^g \mid x + \tau\xi \;\preceq\; y + \tau\eta \,\} = \frac{1}{\tau}\Big[R - (x,y)\Big] \;\text{ for }\; \tau > 0. \tag{1.10}$$

These sets focus on the local aspects of preferences around x in relation to those around y, starting from having $x \sim y$, and the question is what happens to them as $\tau \searrow 0$. By definition, the *general* tangent cone to R at (x,y) is the outer limit in the sense of set convergence,[9]

$$T_R(x,y) = \limsup_{\tau \searrow 0} R^\tau_{x,y}, \tag{1.11}$$

whereas the *regular* tangent cone is the inner limit in which also (x,y) is approached by pairs (x',y'),

$$\widehat{T}_R(x,y) = \liminf_{\substack{(x',y') \to (x,y) \\ \tau \searrow 0}} R^\tau_{x',y'}. \tag{1.12}$$

Both cones are alway closed, and the regular one is always convex.[10] The equality of the two is called *Clarke regularity*; it entails (1.11) holding as the limit as $\tau \searrow 0$, not just the outer limit, which itself is called *derivability*. To what extent might R be Clarke regular or at least derivable at (x,y), and what would that say about the preferences?

Derivability at (x,x) is especially attractive, because $R^\tau_{x,x}$ satisfies the axioms (A1)–(A6) for a preference relation among vectors ξ and η in the set $G^\tau_{x,x} = \frac{1}{\tau}[G - x]$:

$$\xi \;\preceq^\tau_x\; \eta \quad\Longleftrightarrow\quad x + \tau\xi \;\preceq\; x + \tau\eta. \tag{1.13}$$

It's easy to see moreover that (A1)–(A5), at least, are sure to be preserved if $R^\tau_{x,x}$ approaches a limit as $\tau \searrow 0$, during which $G^\tau_{x,x}$ grows to include all of $I\!R^g \times I\!R^g$. Thus, $T_R(x,x)$ is in that case the graph of another preference relation.

Definition 1.1 (marginal preferences)**.** *The given preference relation will be said to exhibit marginal preferences at a point $x \in G$ if the tangent cone $T_R(x,x)$ to R at (x,x) is derivable, so that $T_R(x,x)$ is itself the graph of a preference relation in $I\!R^g \times I\!R^g$:*

$$\xi \;\preceq_x\; \eta \quad\Longleftrightarrow\quad (\xi,\eta) \in T_R(x,x). \tag{1.14}$$

In Section 2 it will be demonstrated that marginal preferences are exhibited at all points $x \in I\!R^g_{++}$ and are linear in nature, being tied to the hyperplanes in (A6). These marginal preferences are moreover consistent, in that they vary continuously from one point to another.

[9]for background on Kuratowski-Pompeiu set convergences, see [15, Chapter 4].

[10]The variational geometry of tangent and normal cones is explained in [15, Chapter 6].

The meaning of this can easily be visualized by taking the view that, within $\mathbb{R}^g_{++}$, the preference relation corresponds to a "continuous nest" of upward-opening convex sets with kinkfree boundaries. Focus on a point x there and imagine what might happen in zooming in on it and seeing those sets locally in finer and finer detail. The result establishes that, in the limit, what will emerge is again a "continuous nest," but it will be of half-spaces whose hyperplane boundaries are parallel to the hyperplane supporting $P(x)$ at x in (A6).

Also in Section 2, *smoothness* of the given preference relation will be defined in terms of an assumption on the scaling functions $\theta_{x,y}$ in (1.5) and shown to characterize when that relation is representable by utility functions that are continuously differentiable on $\mathbb{R}^g_{++}$. That will then allow us to tie marginal preferences to marginal utility as expressed through partial derivatives of a utility function. But in fact, the marginal preferences only depend on ratios among those partial derivatives, and those ratios emerge from the preference geometry itself.

For (x,y) in (1.9) with $x \neq y$, $R^\tau_{x,y}$ no longer satisfies the axioms of a preference relation, because the transitivity in (A2) fails. Nonetheless, it conveys significant information about scaling and trade-offs, and the property of derivability remains of interest.

Definition 1.2 (marginal co-preferences)**.** *The given preference relation will be said to exhibit marginal co-preferences at a pair of points $x \in G$, $y \in G$, with $x \sim y$, $x \neq y$, if the tangent cone $T_R(x,y)$ to R at (x,y) is derivable. The notation then will be*

$$\xi \preceq_{x,y} \eta \iff (\xi,\eta) \in T_R(x,y) \tag{1.15}$$

and naturally extended to the variants

$$\begin{aligned} &\xi \sim_{x,y} \eta \text{ when both } \xi \preceq_{x,y} \eta \text{ and } \eta \preceq_{y,x} \xi, \\ &\xi \prec_{x,y} \eta \text{ when } \xi \preceq_{x,y} \eta \text{ but not } \eta \preceq_{y,x} \xi, \end{aligned} \tag{1.16}$$

although the transivity of an actual preference relation is lacking.

The co-preference relation $\xi \prec_{x,y} \eta$, for instance, has the interpretation that, in an "infinitesimal sense," in starting from goods vectors x and y that are deemed equivalent and shifting x to $x + \tau\xi$ while shifting y to $y + \tau\eta$, the second will be preferable to the first.

Unlike marginal preferences, marginal co-preferences are not automatically available on the basis of just (A1)-A6). In Section 3, we identify the troublespot and show that it disappears for preference relations that meet our prescription for smoothness. The connection between co-preferences and normal cones to the boundary of R is developed next. The main result is that smoothness of the preference relation is equivalent to first-order smoothness of the boundary of R plus the absence of a kind of "singularity." Such singularity corresponds to the preference mapping P failing at some point x to enjoy a fundamental set-valued version of localized Lipschitz continuity.

Section 4 extends the results on first-order smoothness of preferences to results on second-order smoothness. With $\mathcal{C}^2$ utility then available as a springboard, the existence of a minimally concave utility representation is brought out in the case of preferences that are strongly convex, a natural sharpening of the condition of strict convexity already well appreciated in economics.

More, of course, remains for future work. The minimally concave utility representations are obtained relative to arbitrarily large compact subsets of $\mathbb{R}^g_{++}$ rather than the entire goods space G. That shortcoming may be due in part to our emphasis always on goods vectors in the positive orthant, with nothing explored about goods vectors on the boundary. The utility functions extend uniquely to the boundary, but more attention paid to what happens to them there could lead to better insights. A simpler question is whether, when actually $G = \mathbb{R}^g_{++}$, the assumptions in our result might yield a minimally concave utility represention on all of G.

2 Marginal preferences and basic utility

The utility rescaling in (1.4), while potentially a route to improvement, could also take a nice function u and make it worse. Poor properties of the rescaling function θ, such as kinkiness, could lead to utility properties that are arbitrary and have no source in the preferences themselves.

To provide a platform for coming developments, it will help to single out first a mode of utility representation that avoids such pitfalls. We do this as follows by fixing any positive goods vector as a yardstick to compare with other goods vectors. This isn't a new idea, but our systematic use of it will break new ground.

Theorem 2.1 (basic utility functions). *For any vector $e \gg 0$ and any $x \in G$, there is a unique value $t \geq 0$ such that $te \sim x$. Let that value be denoted by $u_e(x)$. Then u_e is a utility function for the given preferences having the special property that*

$$u_e(te) = t \;\text{ for all }\; t > 0 \tag{2.1}$$

and the descriptions

$$u_e(x) = \min\{\, t \,|\, te \succeq x\},\;\; \text{and for } x \in \mathbb{R}^g_{++} \text{ also }\; u_e(x) = \max\{\, t \,|\, te \preceq x\}. \tag{2.2}$$

The utility function $u_{e'}$ for an alternative vector $e' \gg 0$ relates to u_e by

$$u_{e'}(x) = \theta_{e,e'}(u_e(x)), \;\; \text{with } \theta_{e,e'} \text{ as defined in (1.5).} \tag{2.3}$$

More broadly, any utility function u representing the given preferences has $u(te)$ continually increasing in t and therefore relates to u_e through

$$u(x) = \theta(u_e(x)) \;\text{ and }\; u_e(x) = \theta^{-1}(u(x)), \;\; \text{where }\; \theta(t) = u(te). \tag{2.4}$$

In particular, $u = u_e$ if and only if $u(te) = t$ for all $t > 0$.

Proof. The preferences in G induce preferences on the set $\{\, te \,|\, t > 0\}$ that, under (A1), (A2), (A3) and (A4), must correspond to the real-number ordering of the values t. It's not possible to have $te \sim t'e$ with $t' \neq t$ or, through (A2), to have $te \sim x$ for more than one value of t. For any $x \in G$ there exists high enough t such that $te \geq x$, and then $te \succeq x$ by (A4). Accordingly there must be, by (A3), a lowest t with $te \succeq x$, in which case $te \sim x$. In taking that t as $u_e(x)$ we meet the prescription for the function u_e and confirm that it satisfies the minimization formula in (2.2).

When $x \in \mathbb{R}^g_{++}$, there exists t such that $te \leq x$ and therefore $te \preceq x$ by (A4). Then likewise there must be a highest t with $te \preceq x$, and for that t necessarily $te \sim x$ by (A2). That verifies the maximization formula in (2.2).

For e' in comparison to e, $u_{e'}(te)$ gives the t' such that $t'e' \sim te$, and that by definition in (1.5) is the value $\theta_{e,e'}(t)$. Since $t = u_e(x)$ means $te \sim x$, while $t' = u_{e'}(x)$ means $t'e' \sim x$, the conversion in (2.3) is correct.

The assertions in the theorem about other utility functions u follow simply from the general fact that all utility represenions are derivable from each other by rescaling in the manner of (1.4). $\square$

The case of u_e with $e = (1, 1, \ldots, 1)$, at least, has long been familiar in microeconomics. It has been put to use, for instance, in demonstrating the existence of a continuous utility representation by way of arguments like those in the proof of Theorem 2.1, as in the textbook [14, pp. 47-47]. What's different here will be our emphasis on general e and the spectrum of relationships in (2.3) and (2.4).

We will call u_e the *basic utility* function associated with e. Its properties reflect, and fully embody, those of the given preference mapping P and its graph R. Nothing is added or lost in working with u_e, although that could not be said of passing to a representation by any arbitrary utility function u. Of course, the choice of e is arbitrary, but that can be viewed in the bigger picture of considering alternative choices e'. In particular, *all aspects of the rescaling functions $\theta_{e,e'}$ in (2.3) are firmly grounded in P and R, and any condition that might be placed on them to secure advantageous behavior is a condition on preferences themselves, not an extraneous mathematical construct.*

Temporarily fixing a particular basic utility function u_e can help in working with the preference sets $P(x)$, which really only depend on the equivalence classes under "$\sim$" as in (1.4). Clearly

$$P(x) = U_e(t) \text{ for } t = u_e(x), \text{ where } U_e(t) = \{\, y \,|\, u_e(y) \geq t\} = \{\, y \,|\, y \succeq te\}. \tag{2.5}$$

With this in hand, we can proceed to say more about axiom (A6) and what it entails.

The language in (A6) concerning supporting hyperplanes to $P(x)$ at x refers to sets of the form $H = \{\, y \,|\, v{\cdot}[y-x] = 0\}$, with $v \neq 0$, such that $v{\cdot}[y-x] \leq 0$ for all $y \in P(x)$. Such vectors v, necessarily negative by (A4), comprise along with the zero vector the normal cone $N_{P(x)}(x)$ to $P(x)$ at its boundary point x in the sense of convex analysis, and they always exist. The uniqueness condition imposed by (A6) means that this normal cone consists of a single ray. In our context of preferences and utility with an "upward orientation," it will be best to work with this by introducing

$$n(x) = \text{unique vector } \gg 0 \text{ such that } N_{P(x)}(x) = \{\, -\lambda n(x) \,|\, \lambda \geq 0\} \text{ and } |n(x)| = 1, \tag{2.6}$$

where $|\cdot|$ denotes the canonical Euclidean norm in $I\!R^g$. Then the normal cones to the sets $U_e(t)$ in (2.5) are given by

$$N_{U_e(t)}(x) = \{\, -\lambda n(x) \,|\, \lambda \geq 0\} \text{ at its boundary points } x \in I\!R^g_{++}. \tag{2.7}$$

Theorem 2.2 (existence and consistency of marginal preferences). *The marginal preference relation $\xi \preceq_x \eta$ in (1.14) is available at every $x \in I\!R^g_{++}$ in the sense that the outer limit in (1.11) is always manifested as a full limit and the axioms (A1)–(A6) are again satisfied with respect to ξ and η in place of x and y. The marginal preferences have the linear utility representation*

$$\xi \preceq_x \eta \iff n(x){\cdot}\xi \leq n(x){\cdot}\eta \tag{2.8}$$

and are consistent in the sense that $n(x)$ depends continuously on x.

Proof. The consistency claim at the end can be taken care of first. It is based on the fact that convergence of a sequence of convex sets brings with it graphical convergence of the normal cone mappings associated with those sets as the subgradient mappings associated with their indicator functions [15, 12.35]. We apply this to the sets $U_e(t)$, which depend continuously on t because $U_e(t')$ increases to become $U_e(t)$ as $t' \searrow t$, whereas $U_e(t')$ shrinks to $U_e(t)$ as $t' \nearrow t$. In terms of sequences indexed by ν, the rule assures in particular that, as $t^\nu \to t$ and $x^\nu \in U_e(t^\nu)$ approaches $x \in U_e(t)$, all limits of sequences of vectors v^ν selected from the normal cones $N_{U_e(t^\nu)}(x^\nu)$ must be in $N_{U_e(t)}(x)$. Here the normal cones, at points x^ν and x in $I\!R^g_{++}$, are the rays generated by the vectors $-n(x^\nu)$ and $-n(x)$. The limit property therefore just comes down to $n(x^\nu) \to n(x)$.

Without loss of generality in the rest, we can focus on the case of $x = e$, inasmuch as e can be freely chosen anywhere from $I\!R^g_{++}$. For any $\tau > 0$, the preference relation in (1.13) then corresponds to $u_e(e+\tau\xi) \leq u_e(e+\tau\eta)$ and thus has a utility representation by the function $\xi \to u_e(e+\tau\xi)$, or as will suit us better, the difference quotient function

$$\tilde{u}^\tau_e(\xi) = [u_e(e+\tau\xi) - u_e(e)]/\tau, \text{ with } \tilde{u}^\tau_e(se) = s \text{ by (2.1)}, \tag{2.9}$$

its domain being $\frac{1}{\tau}[G-e]$ with interior consisting of the vectors $\xi \gg (-1/\tau)e$. The corresponding upper level sets

$$\tilde{U}_e^\tau(s) = \{\,\eta \,|\, \tilde{u}_e^\tau(\eta) \geq s\} \;\text{ for }\; s \in (-\tau^{-1}, \infty), \;\text{ with } se \text{ on the boundary, ‘} \tag{2.10}$$

capture the preference relation $\sim_e^\tau$, and the question is what happens to them as $\tau \searrow 0$. We have

$$\begin{aligned} &\eta \in \tilde{U}_e^\tau(s) \iff \tau^{-1}[u_e(e+\tau\eta)-1] \geq s \iff e+\tau\eta \in U_e(1+\tau s) \\ &\iff \eta \in \tau^{-1}[U_e(1+\tau s)-(1+\tau s)e]+se = \tau^{-1}[P((1+\tau s)e)-(1+\tau s)e]+se. \end{aligned} \tag{2.11}$$

Thus, as $\tau \searrow 0$ with s fixed, the set $\tilde{U}_e^\tau(s)$ will approach a limit if and only if that holds for the set $\tau^{-1}[P((1+\tau s)e)-(1+\tau s)e]$. Observe that, for any $\varepsilon > 0$,

$$\varepsilon^{-1}[P((1+\tau s)e)-(1+\tau s)e] \subset \{\,\eta \,|\, n((1+\tau s)e){\cdot}\eta \geq 0\}, \tag{2.12}$$

with the set on the right being the unique supporting half-space to the set on the left at the origin, which is a common boundary point of both sets. We know from earlier that $P((1+\tau s)e)$, as $U_e(1+\tau s)$, depends continuously on τ, so as $\tau \searrow 0$ in (2.12) with ε fixed, both sides in (2.12) converge with

$$\begin{aligned} \lim_{\tau \searrow 0} \varepsilon^{-1}[P((1+\tau s)e)-(1+\tau s)e] &= \varepsilon^{-1}[P(e)-e] \\ &\subset \{\,\eta \,|\, n(e){\cdot}\eta \geq 0\} = \lim_{\tau \searrow 0} \{\,\eta \,|\, n((1+\tau s)e){\cdot}\eta \geq 0\}. \end{aligned} \tag{2.13}$$

Any decrease in ε causes the convex sets to which it is attached in (2.13) to grow larger, because they are all anchored at the origin. Since τ eventually falls below any ε as $\tau \searrow 0$, it follows that

$$\begin{aligned} \varepsilon^{-1}[P(e)-e] &\subset \liminf_{\tau \searrow 0} \tau^{-1}[P((1+\tau s)e)-(1+\tau s)e] \\ &\subset \limsup_{\tau \searrow 0} \tau^{-1}[P((1+\tau s)e)-(1+\tau s)e] \subset \{\,\eta \,|\, n(e){\cdot}\eta \geq 0\}. \end{aligned} \tag{2.14}$$

But the limit of $\varepsilon^{-1}[P(e)-e]$ as $\varepsilon \searrow 0$ is the tangent cone to $P(e)$ at e, equal to the half-space $\{\,\eta \,|\, n(e){\cdot}\eta \geq 0\}$. Therefore, the inner and outer limits in (2.14) must coincide with the half-space on the right. Applying this in (2.11), we see that, for all s,

$$\begin{aligned} \lim_{\tau \searrow 0} \tilde{U}_e^\tau(s) &= \{\, se+\eta \,|\, n(e){\cdot}\eta \geq 0\} = \{\,\xi \,|\, n(e){\cdot}(\xi - se) \geq 0\} \\ &= \{\,\xi \,|\; n(e){\cdot}\xi \geq [n(e){\cdot}e]s\} = \{\,\xi \,|\; [n(e){\cdot}e]^{-1}n(e){\cdot}\xi \geq s\}. \end{aligned} \tag{2.15}$$

Thus, the marginal preferences exist and correspond to the nest of closed half-spaces that are the upper level sets of the linear function $\xi \mapsto [n(e){\cdot}e]^{-1}n(e){\cdot}\xi$. That function then gives a utility representation, but so too does the simpler function $\xi \mapsto n(e){\cdot}\xi$, as claimed in (2.8). □

Theorem 2.2 captures a sort of universal geometric differentiability of the preference relation, but we can also inquire about differentiability of the associated basic utility functions. The following concept will have a big role in this.

Definition 2.3 (smooth preferences, first-order)**.** *The preference relation will be said to be first-order smooth if, along with the quasi-smoothness already stipulated in (A6), it has the property that the basic scaling functions $\theta_{x,y}$ in (1.5) are not just continuous, but have derivatives $\theta'_{x,y}(r)$ that are continuous with respect to r and y.*

It has to be noted that some reseachers in economics, such as Debreu [5] and Aumann [1] speak of preferences being smooth only when they satisfy stronger conditions which imply that the associated demand mappings are single-valued and continuously differentiable. To our thinking, it would be good then to speak instead of *demand-smooth* preferences.

Theorem 2.4 (first-order smoothness of basic utility). *If the preference relation is first-order smooth, the basic utility function u_e for any choice of $e \gg 0$ will be continuously differentiable on $\mathbb{R}^g_{++}$, and conversely. Gradients are given by*

$$\nabla u_e(x) = \mu_e(x) n(x), \quad \text{where } \mu_e(x) = \frac{\theta'_{x,e}(1)}{n(x)\cdot x} > 0, \quad \text{hence } \nabla u_e(x) \neq 0. \tag{2.16}$$

Proof. More can be gleaned from the set limits determined in the proof of Theorem 2.2. The hypograph of the difference quotient function $\tilde{u}^\tau_e$ in (2.9) can be described in terms of its upper level sets $\tilde{U}^\tau_e(s)$ in (2.10) as

$$\operatorname{hypo} \tilde{u}^\tau_e = \bigcup_{s > -\frac{1}{\tau}} (\tilde{U}^\tau_e(s), s), \quad \text{where } U^\tau_e(s') \supset U^\tau_e(s) \text{ when } s' > s. \tag{2.17}$$

We saw in (2.15) that $U^\tau_e(s)$ converges for each s as $\tau \searrow 0$ to $\{\, \xi \mid n_e(e)\cdot\xi \geq s\}$, where

$$n_e(e) = n(e)/[n(e)\cdot e].$$

But then we see in (2.17) that $\operatorname{hypo} \tilde{u}^\tau_e$ converges to the hypograph of the linear function $\xi \mapsto n_e(e)\cdot\xi$. This hypoconvergence[11] covers the property in terms of sequences indexed by ν that

$$n_e(e)\cdot\xi = \lim_{\nu\to\infty} \tilde{u}^{\tau^\nu}_e(\xi^\nu) = \lim_{\nu\to\infty} [u_e(e + \tau^\nu \xi^\nu) - u_e(e)]/\tau^\nu \text{ when } \tau^\nu \searrow 0,\ \xi^\nu \to \xi,$$

which means that u_e is differentiable at e with $\nabla u_e(e) = n_e(e)$.

To verify that u_e is differentiable also at other points of $\mathbb{R}^g_{++}$, not just at e, we can denote that point by e' and apply the formula in (2.3) in the reverse form of rescaling $u_{e'}$ by $\theta_{e',e}$ to get u_e. Because $\theta_{e',e}$ is continuously differentiable by assumption, that yields

$$\nabla u_e(x) = \theta'_{e',e}(u_{e'}(x)) \nabla u_{e'}(x) \text{ if } u_{e'} \text{ is differentiable at } x.$$

The argument already given for the differentiablity of u_e at e holds equally well, though, for $u_{e'}$ at e', where $u_{e'}(e') = 1$ by (2.1). It tells us that $\nabla u_{e'}(e') = n_{e'}(e')$. Thus, u_e is differentiable at e' and

$$\nabla u_e(e') = \theta'_{e',e}(1) n_{e'}(e') = [\theta'_{e',e}(1)/n(e')\cdot e'] n(e'). \tag{2.18}$$

This furnishes the gradient formula in (2.16) by taking $e' = x$.

The assumed continuous dependence of $\theta'_{x,e}(1)$ on x guarantees by this formula and the continuous dependence of $n(x)$ on x in Theorem 2.2 that $\nabla u_e(x)$ in (2.16) depends continuously on x. Necessarily $\theta'_{x,e}(1) > 0$, because the increasing functions $\theta_{x,e}$ and $\theta_{e,x}$ are inverse to each other, and their derivatives at 1, when they both exist, must therefore be reciprocals of each other. Of course, $n_e(x)\cdot x > 0$ as well, inasmuch as both vectors have all positive components, so the claim in (2.16) that $\mu_e(x) > 0$ is valid.

Conversely, if all the basic utility functions associated with the preferences are continuously differentiable, then the functions $\theta_{e,e'}$ in (2.3) will be continuously differentiable with their derivatives depending continuously also on e and e'. But e and e' are just stand-ins for the general x and y in $\mathbb{R}^g_{++}$ in (1.5). The property in Definition 2.3 will thus be at hand, with the preference relation therefore being deemed first-order smooth. $\square$

[11] The results in variational analysis ordinarily focus on epiconvergence, cf. [15, Chapter 7], but their hypoconvergence counterparts are obvious.

Corollary 2.5 (smoothness of general utility)**.** *A preference relation is smooth if and only if it can be represented by a utility function u that is continuously differentiable on $I\!R^g_{++}$ with its gradient $\nabla u(x)$ never 0, in fact always $\gg 0$.*

Proof. If the preference relation is smooth, its representations by basic utility functions u_e fit the prescription according to the theorem. On the other hand, if there is a representation by a continuously differentiable u with nonzero gradients, we know that, for any $e \gg 0$, the function $t \to u(te)$ will be continuously differentiable with positive derivatives. By taking the inverse of that function to be θ and passing to $\theta \circ u$, we get an alternative smooth utility function having $(\theta \circ u)(te) = t$. But then $\theta \circ u = u_e$ by (2.4), so the conclusion is that every basic utility function is smooth, and by the theorem, the preference relation is then smooth. □

Note that a smooth preference relation can also be represented by a $\mathcal{C}^1$ utility function having gradients sometimes zero: take $u(x) = \theta(u_e(x))$ for a $\mathcal{C}^1$ increasing function θ such that $\theta'(t)$ is sometimes zero. Thus, being representable by smooth utility might not quite imply smoothness of preferences as in Definition 2.3.

Corollary 2.6 (nonsmooth representations of smooth preferences)**.** *For a preference relation that is smooth, a utility function that is continuously differentiable along any single ray $\{\, te \,|\, t > 0\}$ must be continuously differentiable on all of $I\!R^g_{++}$.*

Only a utility function u having the form $u(x) = \theta(u_e(x))$ for a rescaling function θ that lacks continuously differentiability somewhere in $(0, \infty)$ can fail to have this property. In that case there is at least one equivalence set of vectors x under "$\sim$" where the gradient $\nabla u(x)$ is missing at every x.

Proof. This is immediate from Theorem 2.4 by way of (2.4). □

It might be wondered whether the extra property in Definition 2.3 needs explicitly to be assumed in order to arrive at the smoothness in Theorem 2.4 and Corollaries 2.5 and 2.6. Is it maybe automatic already from (A1)–(A6)? Here is a counterexample.

Example 2.7 (a preference relation with kinky rescaling)**.** *Let the goods space G consist of all the vectors $x = (x_1, x_2) \geq (0, 0)$ in $I\!R^2$. For $t \geq 1$, let $L(t)$ be the line in $I\!R^2$ through (t, t) with slope $-m(t)$, where m is an increasing, continuously differentiable function with $m(1) = 1$, $m'(1) > 1$, such that $m(t) \to \infty$ as $t \to \infty$ but slowly enough that the intercept of $L(t)$ with the x_1-axis, which starts at 1, likewises increases and goes to ∞ as $t \to \infty$. For $t \in (0, 1)$, just let $L(t)$ be the line through (t, t) with slope -1. For each $t \in (0, \infty)$ and $x \in L(t)$, define $u(x) = t$.*

Then u is a utility function for a preference relation "$\preceq$" that satisfies (A1)–(A6). In fact, $u = u_e$ for $e = (1, 1)$. But for goods vectors $x \gg 0$ in the same equivalence set at e, the scaling function $\theta_{x,e}$ has different right and left derivatives at the origin and thus fails to be differentiable there.

Detail. The construction makes it obvious that (A1)–(A6) are fulfilled, and because $(t, t) \in L(t)$, it yields $u(te) = t$ for $e = (1, 1)$. Then $u = u_e$ by (2.4). Now fix any positive vector y in $L(1)$ that differs from e, i.e.,

$$\text{fix } y_1 > 0 \text{ and } y_2 > 0 \text{ with } y_1 + y_2 = 2 \text{ but } y_1 \neq 1,\ y_1 \neq 1. \tag{2.19}$$

The scaling function $\theta_{y,e}$, which by definition in (1.5) assigns to each $s > 0$ the t such that $sy \sim te$, is the function $s \mapsto u_e(sy) = u(sy)$. It has $t = \theta_{y,e}(s)$ if and only if $sy \in L(t)$, and this also describes the inverse relationship, $s = \theta_{e,y}(t)$, with $s = 1$ corresponding to $t = 1$. For values of s and t below 1, the relations reduce to $s = t$ because the lines in question are parallel, so values above 1 are the key. Then the line $L(t)$, in passing through (t, t) and having slope $-m(t)$, contains (x_1, x_2) if and only if $(x_2 - t) = -m(t)(x_1 - t)$, or equivalently $m(t)x_1 + x_2 = t(1 + m(t))$. Specializing to $x = sy$, we see that $t = \theta_{y,e}(s)$ correponds to $s[m(t)y_1 + y_2] = t(1 + m(t))$ and allows us easily to express s as a function of t:

$$\theta_{e,y}(t) = \frac{t(1 + m(t))}{m(t)y_1 + y_2} \ \text{ when } \ t \in [1, \infty). \tag{2.21}$$

This has derivative

$$\theta'_{e,y}(t) = \frac{(1 + m(t) + tm'(t))(m(t)y_1 + y_2) - t(1 + m(t))m'(t)y_1}{(m(t)y_1 + y_2)^2}$$

which at $t = 1$ stands for a right derivative rather than a two-sided derivative and, because $m(1) = 1$ and $m'(1) > 1$, calculates out through (2.19) as

$$\frac{(1 + 1 + m'(1))(y_1 + y_2) - (1 + 1)m'(1)y_1}{(y_1 + y_2)^2} = 1 + m'(1)\frac{1 - y_1}{2} \neq 1.$$

In contrast, the left derivative does equal 1, inasmuch as $\theta_{e,y}(t) = t$ for $t \leq 1$. Thus $\theta_{e,y}$ has a kink at 1, and the same must be true of its inverse $\theta_{y,e}$. □

3 Marginal co-preferences and graphical smoothness

Theorem 2.2 guarantees the availability of marginal preferences in Definition 1.1 just on the basis of (A1)–(A6), but there is no such universal availability of the marginal co-preferences in Definition 2.2, and it's not hard to see why. Marginal preferences can be visualized as emerging when we zoom in on the picture of preferences around a point x, but in the case of co-preferences we have to keep an eye on two different points, x and y. The view will settle down eventually around either one, but co-preferences require a sort of coordination in the view, and there could be a mismatch then in the rate of zooming.

The way to understand that is through a comparison of scales which resembles that in (1.5) but involves instead the unit vectors $n(x)$ and $n(y)$ at points x and y in $\mathbb{R}^g_{++}$ such that $x \sim y$ but $x \neq y$. Because the components of $n(x)$ are all positive, and likewise $n(y)$,

$$\begin{array}{l} \{\, (r, s) \,|\, x + rn(x) \sim y + sn(y)\} \ \text{ is the graph of a continuous increasing function} \\ \gamma_{x,y} \ \text{ from an interval around } r = 0 \text{ to an interval around } s = 0, \text{ with } \ \gamma_{x,y}(0) = 0. \end{array} \tag{3.1}$$

The key is the "marginal" behavior of $\gamma_{x,y}$ at 0.

It will also be necessary, however, to take into account special *local* representations of utility, as follows:

$$v_x(\xi) = \ \text{ the unique } t \text{ such that } \ x + \xi \sim x + tn(x) \tag{3.2}$$

for ξ in a small-enough neighborhood of 0. Then locally

$$x + \xi \preceq x + \xi' \quad \Longleftrightarrow \quad v_x(\xi) \leq v_x(\xi'). \tag{3.3}$$

Similarly for $v_y(\eta)$ at y.

Theorem 3.1 (availability of marginal co-preferences)**.** *For $x \gg 0$ and $y \gg 0$ with $x \sim y$ but $x \neq y$, marginal co-preferences will be exhibited when the function $\gamma_{x,y}$ in (3.1) is differentiable at 0. They will then have the representation*

$$\xi \ \preceq_{x,y} \ \eta \quad \Longleftrightarrow \quad n(x){\cdot}\xi \ \leq \ \gamma'_{x,y}(0)n(y){\cdot}\eta. \tag{3.4}$$

Proof. Looking at $x+\tau\xi$ and $y+\tau\eta$ for small $\tau>0$, we see that $x+\tau\xi \sim x+v_x(\tau\xi)n(x)$ through (3.2) and $y+\tau\eta \sim y+v_y(\tau\eta)n(y)$. On the other hand $x+v_x(\tau\xi)n(x) \sim y+\gamma_{x,y}(v_x(\tau\xi))n(y)$ by (3.1). Then

$$x+\tau\xi \preceq y+\tau\eta \iff y+\gamma_{x,y}(v_x(\tau\xi))n(y) \preceq y+v_y(\tau\eta)n(y),$$

so the set $R^\tau_{x,y}$ in (1.10) that we want to actually to converge to the tangent cone $T_R(x,y)$, which its outer limit in (1.11), has the description that

$$(\xi,\eta)\in R^\tau_{x,y} \iff \gamma_{x,y}(v_x(\tau\xi)) \le v_y(\tau\eta) \iff \tau^{-1}\gamma_{x,y}(\tau[\tau^{-1}(v_x(\tau\xi)]\,) \le \tau^{-1}v_y(\tau\eta). \tag{3.5}$$

Applying what we know about marginal preferences in Theorem 2.2 to (3.2), and likewise with $v_y(\eta)$ replacing $v_x(\xi)$, we see that

$$\tau^{-1}v_x(\tau\xi) \to n(x){\cdot}\xi \ \text{ and } \ \tau^{-1}v_y(\tau\eta) \to n(y){\cdot}\eta \ \text{ as } \ \tau\searrow 0. \tag{3.6}$$

At the same time, the differentiability assumed for $\gamma_{x,y}$ at 0, where its value is 0, tells us that, for any expression $\rho(\tau)$ that converges to a value $\rho(0)$ as $\tau\searrow 0$, we will have $\tau^{-1}\gamma_{x,y}(\tau\rho(\tau)) \to \gamma'_{x,y}(0)\rho(0)$. Through (3.6), therefore, the function inequality on the right of (3.5) transforms to the one on the right of (3.4). In that process the sets defined by those inequalities themselves converge. That means that the tangent cone $T_R(x,y)$ is indeed derivable and is decribed by (3.4), as claimed. □

Theorem 3.1 reveals how the foundation for marginal co-preferences might be disrupted if the difference quotients behind the differentiability of $\gamma_{x,y}$ at 0 failed to converge. In fact an example of just such behavior can be gleaned from Example 2.7 as a case where there are left and right derivatives with a gap between them.

However, Theorem 3.1 also reveals a potential source of discomfort even with the solid result that it provides. What if the derivative $\gamma'_{x,y}(0)$ in (3.4) equals 0? Then we would have $\xi \preceq_{x,y} \eta$ corresponding just to $n(x){\cdot}\xi \le 0$, without any involvement from η at all! Could that really happen? Yes, by the example coming next.

Example 3.2 (singular co-preferences). *In $G=\mathbb{R}^2_{++}$, let $C=\{\,x=(x_1,x_2)\in G \mid x_1x_2 \ge 1\}$ and $e=(1,1)$. For $t>0$, let $U(t)=G\cap(\rho(t)e+t^{-1}[C-e])$ for an increasing, continously differentiable function ρ from $(0,\infty)$ onto $(0,\infty)$ having $\rho(1)=1$ and $\rho'(1)=0$. Then $\{U(t)\}_{t>0}$ is a nest of closed convex sets serving as the preference sets for a preference relation that satisfies (A1)–(A6). A utility function for that relation is obtained by taking $u(x)$ for $x\in G$ to be the unique t such that x lies on the boundary of $U(t)$; this corresponds to $e+t[x-\rho(t)e]$ lying on the boundary of C, so that*

$$u(x)=t \iff (tx_1+[1-t\rho(t)])(tx_2+[1-t\rho(t)])=1. \tag{3.7}$$

Specialize the function $\gamma_{x,y}$ of (3.1) to $x\sim e$, $y=e$. Then $\gamma'_{x,y}(0)=0$.

Detail. Observe first that, as t increases, $t^{-1}[C-e]$ continually shrinks and goes in the limit to $\mathbb{R}^2_+$. At the same time, $\rho(t)e$ continually progresses "towards the northeast." The combination of those properties underlies the claim that the sets $U(t)$ form a continuous nest of the kind required for a preference relation; u is then a continuous function that increases in both arguments, and so forth.

Because e lies on the boundary of $C=U(1)$, and $x\sim e$, we have x also on that boundary with $u(x)=u(e)=1$, where $n(x)=(x_2,x_1)/|(x_1,x_2)|$ and $n(e)=(1,1)/|(1,1)|$. The function $\gamma_{x,e}$ has

$$\operatorname{gph}\gamma_{x,e}=\{\,(r,s)\mid u(e+sn(e))=u(x+rn(x))\}, \tag{3.8}$$

and our claim about its derivative at 0 means that the ratio of s to r approaches 0 as $(r,s)\to(0,0)$ in this graph.

To confirm this, we can work with t alongside of r and s and view the equation in (3.8) as the double equation $u(e+sn(e)) = t$ and $u(x+rn(x)) = t$ in which for convenience in taking $a = 1/|(1,1)| = 1/\sqrt{2}$ and $b = 1/|(x_1,x_2)| = 1/\sqrt{x_1^2+x_2^2}$, we have

$$e + sn(e) = (1+as)e, \qquad x + rn(x) = (x_1 + rbx_2, x_2 + rbx_1). \tag{3.9}$$

From the basic construction, it's evident then that

$$u(e+sn(e)) = t \iff n((1+as)e) = t \iff 1 + as = \rho(t). \tag{3.10}$$

On the other hand from (3.7) and (3.9),

$$u(x) = t \iff (t[x_1 + rbx_2] + [1 - t\rho(t)])(t[x_2 + rbx_1] + [1 - t\rho(t)]) = 1.$$

This could explicitly be solved for r as a function of t, but there is no need for that, because we can get a hold on the derivative of that function by differentiating on both sides of the equation:

$$\begin{aligned} 0 = {} & ([x_1 + rbx_2] + tbx_2\tfrac{dr}{dt} - \rho(t) - t\rho'(t))(t[x_2 + rbx_1] + [1 - t\rho(t)]) \\ & + ([x_2 + rbx_1] + tbx_1\tfrac{dr}{dt} - \rho(t) - t\rho'(t))(t[x_1 + rbx_2] + [1 - t\rho(t)]). \end{aligned} \tag{3.11}$$

Since $t = 1$ corresponds to $(r,s) = (0,0)$, we can find out what happens to s/r as $(r,s) \to (0,0)$ by evaluating the ratio of ds/dt to dr/dt at $t = 0$. By setting $r = 0$ and $t = 1$ in (3.11), we get

$$0 = (x_1 + bx_2\frac{dr}{dt} - 1)x_2 + (x_2 + bx_1\frac{dr}{dt} - 1)x_1 = 2x_1x_2 + b(x_1^2 + x_2^2)\frac{dr}{dt} - x_1 - x_2,$$

where $x_1x_2 = 1$ by choice and $x_1^2 + x_2^2 = b^{-2}$ by definition, and therefore

$$\frac{dr}{dt} = b(x_1 + x_2 - 2) \neq 0 \text{ when } t = 0,$$

because $x_1 + x_2 > 2$ when $x_1x_2 = 1$ but $x_1 \neq x_2$. From (3.10), however, we have

$$\frac{ds}{dt} = a^{-1}\rho'(t) = 0 \text{ when } t = 0.$$

Thus the ratio of $\frac{ds}{dt}$ to $\frac{dr}{dt}$ is indeed 0 when $t = 0$. □

What accounts for this strange possibility? It turns out to signal the absence of a desirable Lipschitz-type property of the preference mapping P. To get to an understanding of that, we need to turn to an investigation of the normal cones to the set $R = \operatorname{gph} P$ at its boundary points in (1.9), along with the tangent cones already introduced in (1.11) and (1.12).

For our purposes, instead of going back to the definitions themselves, we can skip ahead to general tangent-normal relationships [15, Chapter 6], starting with the fact that, in terms of the tangent cone $T_R(x,y)$, the *regular* normal cone to R at (x,y) is given by

$$\widehat{N}_R(x,y) = \{\, (p,-q) \in I\!R^g \times I\!R^g \mid (p,-q)\cdot(\xi,\eta) \leq 0, \forall(\xi,\eta) \in T_R(x,y)\}. \tag{3.12}$$

The *general* normal cone is then

$$N_R(x,y) = \limsup_{(x',y')\to(x,y)} \widehat{N}_R(x',y') \supset \widehat{N}_R(x,y). \tag{3.13}$$

The regular tangent cone relates to the general normal cone by

$$\widehat{T}_R(x,y) = \{\, (\xi,\eta) \in I\!R^g \times I\!R^g \,|\, (p,-q)\cdot(\xi,\eta) \le 0,\ \forall (p,-q) \in N_R(x,y)\}. \tag{3.14}$$

Because of (A4), the tangent cones in our context satisfy

$$T_R(x,y) \supset \widehat{T}_R(x,y) \supset I\!R^g_- \times I\!R^g_+, \tag{3.15}$$

and that guarantees through (3.12) that

$$p \ge 0 \text{ and } q \ge 0 \text{ for all } (p,-q) \in N_R(x,y). \tag{3.16}$$

It might be thought that because the monotonicity in (A4) is strict — any increase in any single goods component increases utility — the pairs $(p,-q) \neq (0,0)$ in $N_R(x,y)$ would have $p \gg 0$ and $q \gg 0$, but Example 3.2 shows there can be exceptions. In Theorem 3.1, the co-preference relation $\xi \preceq_{x,y} \eta$ refers to (ξ,η) belonging to $T_R(x,y)$, and the characterization of co-preferences in (3.4) says that

$$(\xi,\eta) \in T_R(x,y) \iff (\xi,\eta)\cdot(p,-q) \le 0 \text{ for } p = n(x) \text{ and } q = \gamma'_{x,y}(0)n(y). \tag{3.17}$$

Then $(p,-q) \in \widehat{N}_R(x,y)$ by (3.12), hence also $(p,-q) \in N_R(x,y)$ by (3.13). But in Example 3.2 we have an instance of this in which $p \gg 0$, yet $q = 0$.

This phenomenon fits a fundamental pattern in the theory of set-valued mappings concerned with graphical derivatives and co-derivatives [15, Chapter 8]. The *graphical derivative* of the preference mapping P at x for an element y of $P(x)$ is by definition the mapping $DP(x\,|\,y) : I\!R^g \rightrightarrows I\!R^g$ such that

$$\eta \in DP(x\,|\,y)(\xi) \iff (\xi,\eta) \in T_R(x,y), \tag{3.18}$$

whereas the *graphical coderivative* $D^*P(x\,|\,y) : I\!R^g \rightrightarrows I\!R^g$ has

$$p \in D^*P(x\,|\,y)(q) \iff (p,-q) \in N_R(x,y). \tag{3.19}$$

The condition that

$$p \in D^*P(x\,|\,y)(0) \implies p = 0, \tag{3.20}$$

as a way of forbidding combinations $(p,-q) \in N_R(x,y)$ with $p \ge 0$ but $q = 0$, is well understood in the general theory and recognized for its major implications [15, 9.40].

Here it will connect up with a property of the set-valued mapping P called sub-Lipschitz continuity in variational analysis [15, 9E]. For this we draw on the notation that

$$I\!B = \text{closed unit ball in } I\!R^g. \tag{3.21}$$

Definition 3.3 (sub-Lipschitz continuity of preferences). *Preferences will be called sub-Lipschitz continuous if for every closed, bounded subset $B \subset I\!R^g_{++}$ there exists $\kappa > 0$ and such that*

$$P(x') \cap B \;\subset\; P(x) + \kappa|x'-x| I\!B \text{ for all } x,x' \in B. \tag{3.22}$$

This property would be *Lipschitz continuity* without the truncation of $P(x')$ to $P(x') \cap B$. That truncation is important for general reasons in dealing with mappings having unbounded values like P, but because P is convex-valued, the sub-Lipschitz continuity in Definition 3.3 is equivalent actually to the Lipschitz continuity of the mappings $P_B : x \in B \mapsto P(X) \cap B$ for convex B by [15, 9.33].

Instead of the Euclidean unit ball in (3.21), the unit cube could be employed with $|x'-x|$ replaced then in (3.22) by the norm $||x'-x||_\infty$. That would be more natural for goods vectors, but further specialization is also in the offing. We know that, for any $e \gg 0$, the sets $P(x)$ for $x \sim te$ are the same, namely equal to $U_e(t) = P(te)$. Instead of the unit cube, we can just as well appeal to $[-e, e]$, and instead of general B we can focus on intervals $[t_0e, t_1e]$ with $0 < t_0 < t_1$. We can also take advantage of having $U_e(t') \supset U_e(t)$ when $t' < t$. That way, the sub-Lipschitz continuity in Definition 3.3 can be construed equivalently as requiring for each interval $[t_0, t_1] \subset (0, \infty)$ the existence of $\kappa > 0$ such that

$$U_e(t') \cap [t_0e, t_1e] \;\subset\; U_e(t) - \kappa|t'-t|e \;\text{ when }\; t_0 \le t' < t \le t_1. \tag{3.23}$$

Another ingredient of the theorem we are leading up to is smoothness of R, which likewise is tied to normal cones. We saw in (3.17) a situation in which the tangent cone $T_R(x, y)$ is a half-space, and this can be pursued as a property of interest in itself.

Definition 3.4 (graphical smoothness of preferences)**.** *Preferences will be called smooth graphically if, at each boundary point (x, y) of $R = \operatorname{gph} P$ within $I\!R^g_{++} \times I\!R^g_{++}$, the tangent cone $T_R(x, y)$ is a half-space, and the unit normal to that half-space depends continuously on (x, y).*

We combine this now into a definitive statement about smoothness from more than one angle.

Theorem 3.5 (smoothness versus graphical smoothness)**.** *Preferences are first-order smooth if and only if they are sub-Lipschitz continuous and graphically smooth. Then, for any $\mathcal{C}^1$ utility function u with nonzero gradients and any boundary point (x, y) of R in $I\!R^g_{++} \times I\!R^g_{++}$,*

$$N_R(x, y) \;\textit{ is the ray generated by }\; (\nabla u(x), -\nabla u(y)), \tag{3.24}$$

and co-preferences are guided by

$$\xi \;\preceq_{x,y}\; \eta \quad\Longleftrightarrow\quad \nabla u(x){\cdot}\xi \;\le\; \nabla u(y){\cdot}\eta. \tag{3.25}$$

Proof. Starting from the assumption that the preference relation is first-order smooth, we have at our disposal in Corollary 2.5 a $\mathcal{C}^1$ utility function u with nonzero gradients. For such a function, the set $R^\tau(x, y)$ in (1.10) consists of the pairs (ξ, η) such that $u(x + \tau\xi) \le u(y + \tau\eta)$ and thus has the description

$$R^\tau_{x,y} = \{\, (\xi, \eta) \,|\, \varphi^\tau(\xi, \eta)\}, \;\text{ with }\; \varphi^\tau(\xi, \eta) = \frac{1}{\tau}[u(x + \tau\xi) - u(x)] - \frac{1}{\tau}[u(y + \tau\eta) - u(y)]. \tag{3.26}$$

This takes advantage of $u(x)$ equaling $u(y)$ because $x \sim y$. The differentiability of u makes φ^τ converge uniformly on bounded sets to the nonzero linear function $\varphi(\xi, \eta) = \nabla u(x){\cdot}\xi - \nabla u(y){\cdot}\eta$. The sets defined by the inequalities in (3.26) correspondingly converge to the closed half-space for that linear function. This tells us that the tangent cone $T_R(x, y)$ is derivable and equals that half-space, having $(\nabla u(x), -\nabla u(y))$ as outward normal. Then from (3.12), the regular normal cone $\widehat{N}_R(x, y)$ is the ray generated by $(\nabla u(x), -\nabla u(y))$, and it depends continuously on x and y.

That fits the prescription for graphical smoothness and precludes having $(p, -q) \in N_R(x, y)$ with $p \ne 0$ but $q = 0$. Then $D^*P(x\,|\,y)(0) = \{0\}$, which according to [15, 9.40] is equivalent to the mapping P having what is called the Aubin property at x for y. That being true for all pairs $(x, y) \in I\!R^g_{++} \times I\!R^g_{++}$, we have P locally sub-Lipschitz continous on the basis of [15, 9.38, 9.31], so the criterion for sub-Lipschitz continuity of preferences in Definition 3.3 is met.

Working backwards now from such sub-Lipschitz continuity combined with graphical smoothness, the sub-Lipschitz continuity forbids through [15, 9.31, 9.38, 9.40] the existence of $(p, -q) \in N_R(x, y)$

with $p \neq 0$ but $q = 0$, while the graphical smoothness in particular makes $N_R(x,y)$ be a ray generated by a pair $(p,-q)$ such that the co-preference relation $\xi \preceq_{x,y} \eta$ corresponds to $p{\cdot}\xi \leq q{\cdot}\eta$. Likewise the co-preference relation $\eta \preceq_{y,x} \xi$ has a representation $q'{\cdot}\eta \leq p'{\cdot}\xi$, but it's obvious from the definitions that $\xi \sim_{x,y} \eta$ is equivalent to $\eta \sim_{y,x} \xi$, with the indiference relations corresponding to the equations $p{\cdot}\xi = q{\cdot}\eta$ and $q'{\cdot}\eta = p'{\cdot}\xi$, hence $p' = p$ and $q' = q$. In other words, having $N_R(x,y)$ be the ray generated by $(p,-q)$ corresponds to having $N_R(y,x)$ be the ray generated by $(q,-p)$. That symmetrically precludes having $q \neq 0$ but $p = 0$.

We can normalize by making $N_R(x,y)$ be the ray generated by a pair denoted by $(p(x,y), -q(x,y))$ with $|p(x,y)| = 1$. The assumed graphical smoothness makes $p(x,y)$ and $q(x,y)$ depend continuously on the boundary pair (x,y) in $I\!R^g_{++} \times I\!R^g_{++}$ with both nonzero vectors being ≥ 0, at least, by (3.16).

Now fix any $e \gg 0$ and view the minimization formula for u_e in (2.2) in inf-projection mode as

$$u_e(x) = \min_{t>0} \varphi_e(x,t), \;\text{ where }\; \varphi_e(x,t) = t + \delta_R(A_e(x,t)) \;\text{ for }\; A_e : (x,t) \mapsto (x, te). \tag{3.27}$$

Subgradient calculus can be applied to this in order to determine subgradients of u_e and confirm that u_e really is differentiable. First, we apply such calculus to φ_e, using the sum rule in [15, 8.8(c)] and the chain rule in [15, 10.6] to see that its general subdradients and horizon subgradients satisfy

$$\begin{aligned} \partial\varphi_e(x,t) &\subset \{\, (0,1) + A_e^*(p,-q) \,|\, (p,-q) \in \partial\delta_R(A_e(x,t))\}, \\ \partial^\infty\varphi_e(x,t) &\subset \{\, (0,0) + A_e^*(p,-q) \,|\, (p,-q) \in \partial^\infty\delta_R(A_e(x,t))\}, \end{aligned} \tag{3.28}$$

where A_e^* is the adjoint of the linear transformation A_e and takes $(p,-q)$ to $(p, -q{\cdot}e)$. Here both $\partial\delta_R(x,y)$ and $\partial^\infty\delta_R(x,y)$ are just the normal cone $N_R(x,y)$, which we know to consist of all the multiples $\lambda(p(x,y), -q(x,y))$, $\lambda \geq 0$. The inclusions (3.28) reduce therefore to

$$\begin{aligned} \partial\varphi_e(x,t) &\subset \{\, (\lambda p(x,y), 1 - \lambda q(x,y){\cdot}e) \,|\, \lambda \geq 0\}, \\ \partial^\infty\varphi_e(x,t) &\subset \{\, (\lambda p(x,y), -\lambda q(x,y){\cdot}e) \,|\, \lambda \geq 0\}. \end{aligned} \tag{3.29}$$

Next we appeal to the rule associated with (3.27) that

$$\begin{aligned} \partial u_e(x) &\subset \{\, p \,|\, \exists\, t \in \text{argmin} \;\text{ with }\; (p,0) \in \partial\varphi_e(x,t)\}, \\ \partial^\infty u_e(x) &\subset \{\, p \,|\, \exists\, t \in \text{argmin} \;\text{ with }\; (p,0) \in \partial^\infty\varphi_e(x,t)\}, \end{aligned} \tag{3.30}$$

[15, 10.13]. Of course, the minimum in (3.27) is achieved uniquely by $t = u_e(x)$.

Putting these things together, we ask first whether $\partial^\infty u_e(x)$ might contain a vector $p \neq 0$. By the horizon subgradient inclusions in (3.29) and (3.30), that would entail the existence of $\lambda > 0$ such that $(p,0) = \lambda(p(x,y), q(x,y){\cdot}e)$, where $y = te$ for $t = u_e$). That's impossible because $q(x,te){\cdot}e > 0$, due to having $e \gg 0$ and $q(x,te) \geq 0$, $q(x,te) \neq 0$. Thus, $\partial^\infty u_e(x)$ is just $\{0\}$. But that characterizes u_e as being Lipschitz continuous on a neighborhood of 0 [15, 9.13]. Then $\partial u_e(x)$ is a nonempty, bounded set [15, 9.13].

What might that set contain? The general subgradient inclusions in (3.29) and (3.30) narrow the possibilities down to vectors $\lambda p(x,te)$ with $\lambda > 0$ and such that $1 - \lambda q(x,te){\cdot}e = 0$, which comes down to $\lambda = 1/q(x,te)$. Thus, $\partial u_e(x)$ is the singleton consisting of the vector obtained by dividing $p(x, u_e(x)e)$ by $q(x, u_e(x)e){\cdot}e > 0$, with both of those depending continuously on x. Singleton subgradient sets for Lipschitz continuous functions indicate strict differentiability with the vector in question being the gradient [15, 9.18]. In summary, u_e is differentiable at x with the gradient depending continuously on x. This has been established for any $e \gg 0$ and $x \in I\!R^g_{++}$, so we can conclude from Theorem 2.4 that the preface relation is first-order smooth. $\square$

4 Second-order smoothness and concavity

So far, our results have exploited two aspects of smoothness of the preference relation in partnership. The first is the supporting hyperplane axiom (A6) and the second is the scaling property in Definition 2.3. The first got translated into the availability of the unit vectors $n(x)$ in (2.6), which in fact automatically then depend continuously on x, as seen in Theorem 2.2. The second started from the continuity of the basic scaling functions $\theta_{x,y}$ in (1.5) which comes from (A3) and then upped that to the continuous differentiability of those functions, with the derivatives depending continuously also on (x, y). We look now at a partnership of second-order enhancements of those two properties.

Definition 4.1 (smooth preferences, second-order)**.** *The preference relation will be said to be second-order smooth if the following pair of conditions holds:*

(a) *the unit normals $n(x)$ to the supporting hyperplanes in (A6) depend in a continuously differentiable way on $x \in \mathbb{R}^g_{++}$.*

(b) *the basic scaling functions $\theta_{x,y}$ for positive x and y are have derivatives $\theta'_{x,y}(r)$ that are not just continuous, but continuously differentiable with respect to r and y.*

Condition (a) already has a significant history in economics in connection with "demand-smooth" preferences, as for instance in Debreu's paper [6]. Condition (b), however, is brought to the fore only here, in contrast to the equations of differential geometry imposed in [5].

Theorem 4.2 (second-order smooth utility)**.** *The preference relation is second-order smooth if and only if it can be represented by a utility function u that is twice-continuously differentiable on $\mathbb{R}^g_{++}$ with its gradient $\nabla u(x)$ never 0. In particular, the basic utility functions u_e associated with a second-order smooth preference relation serve to represent it that way.*

Proof. The formula in (2.16) for the gradients of a basic utility function u_e reveals immediately that those gradients, known to be nonzero always, will be continuously differentiable with respect to x in the presence of the properties in Definition 4.1. In the other direction, if the preferences can be represented by a utility function u with the properties described, then those properties carry over to every basic utility function u_e through the relationships in (2.4) of Theorem 2.1. The mutual rescalings in (2.3) of Theorem 2.1 confirm then, since any positive vectors x and y can be taken as e and e', that part (b) of Definition 4.1 is fulfilled. Because $n(x) = \nabla u(x)/|\nabla u(x)|$, part (a) of Definition 4.1 is fulfilled as well. □

Utility functions have very often been assumed to be $\mathcal{C}^2$ for convenience in microeconomics, but without the axiomatic support provided now by Theorem 4.2 through the properties in Definition 4.1. Those properties solidify exactly what it needed in terms of "substitution" effects among goods that economists have anyway understood must somehow underlie second-order smoothness.[12] Here we are able to furnish not only the characterization in Theorem 4.2 but also, next, its counterpart in terms of the graph of the preference relation.

Definition 4.3 (graphical double smoothness of preferences)**.** *Preferences will be called doubly smooth graphically if, at each boundary point (x, y) of $R = \operatorname{gph} P$ in $\mathbb{R}^g_{++} \times \mathbb{R}^g_{++}$, the tangent cone $T_R(x, y)$ is a half-space, the unit normal to which depends continuously differentiably on (x, y).*

This condition means geometrically that the boundary is a second-order smooth hypersurface, an embedded $\mathcal{C}^2$ manifold of dimension $2g - 1$ in $\mathbb{R}^g_{++} \times \mathbb{R}^g_{++}$.

[12]On page 49 of the textbook [14], for example, it says that: "Intuitively, what is required is that the indifference sets be smooth surfaces that fit together nicely so that the rates at which commodities substitute for each other depend differentiably on the consumption levels."

Theorem 4.4 (second-order smoothness versus graphical double smoothness). *Preferences are second-order smooth if and only if they are sub-Lipschitz continuous and graphically doubly smooth.*

Proof. This is quickly derived as an extension of Theorem 3.5 enabled by Theorem 4.2. The only question to resolve is how the behavior of normal cones in Definition 4.3 relates equivalently to the differentiability of utility gradients $\nabla u(x)$. But the answer to that is obvious from (3.14). □

What we call graphical double smoothness was earlier investigated by Debreu in [5]. In a framework like ours where preferences satisfy (A1)–(A6), he argued it was equivalent to the existence of a utility representation that is $\mathcal{C}^2$. However, that's not consistent with the combination of Theorems 4.2 and 4.4, because the sub-Lipschitz continuity that is essential (in going back to Theorem 3.5) is left out.

The challenge we take up next is determining when the preference relation can be represented by a utility function that is actually concave. There has been broadly based work on that in the past, as explained in Section 1, but here we are concentrating on preferences that satisfy the axioms (A1)–(A6) and furthermore are second-order smooth in the sense of Definition 4.1. That makes available, through Theorem 4.2, representations by $\mathcal{C}^2$ utility functions u. Then Hessian matrices $\nabla^2 u(x)$ are at hand as supplements to the gradient vectors $\nabla u(x)$. Concavity of u is characterized by $\nabla^2 u(x)$ being negative-semidefinite for $x \in \mathbb{R}^g_{++}$.

The key idea is producing a concave utility function u as $\theta \circ u_e$ from a quasi-concave basic utility function u_e that is $\mathcal{C}^2$ by virtue of Theorem 4.2. For this to work, the rescaling function θ must itself be concave and $\mathcal{C}^2$, inasmuch as $\theta(u_e(te)) = \theta(t)$. That greatly narrows down the search.

Guidance will come from a look at properties on the interface between convex geometry and classical differential geometry and how they fit into modern varational analysis.

Property (a) in Definition 4.1 implies that the boundaries of the convex sets $P(x)$ are not only $\mathcal{C}^1$-smooth hypersurfaces in $\mathbb{R}^g_{++}$, as was ascertained in Theorem 2.2 to follow from (A6), but $\mathcal{C}^2$ smooth. Curvature aspects of those convex hypersurfaces are captured at their points x by the second partial derivatives of a $\mathcal{C}^2$ utility function u in restriction to the supporting hyperplane, i.e., by the partial quadratic forms

$$\xi \in H(x) \mapsto \frac{\xi{\cdot}\nabla^2 u(x)\xi}{|\nabla u(x)|} \;\text{ where }\; H(x) = \{\, \xi \mid \nabla u(x){\cdot}\xi = 0 \} = \{\, \xi \mid \xi \perp n(x) \}. \tag{4.1}$$

The equivalent descriptions of the hyperplane $H(x)$ here come from the gradient formula in Theorem 2.4 and the fact that rescaling a utility function merely changes its gradient in length, not direction. The reason for dividing by $|\nabla u(x)|$ in (4.1) is that structure independent of the particular utility representation is secured in that way:

$$\text{if } u_* = \theta \circ u \text{ with } \theta \in \mathcal{C}^2, \text{ then } \frac{\xi{\cdot}\nabla^2 u_*(x)\xi}{|\nabla u_*(x)|} = \frac{\xi{\cdot}\nabla^2 u(x)\xi}{|\nabla u(x)|} \;\text{ for all }\; \xi \in H(x), \tag{4.2}$$

as seen from calculating the first and second derivatives with respect to τ of $u_*(x+\tau\xi) = \theta(u(x+\tau\xi))$ and evaluating them at $\tau = 0$; those steps lead to

$$\begin{aligned} &\nabla u_*(x){\cdot}\xi = \theta'(u(x))\nabla u(x){\cdot}\xi, \;\text{ so }\; |\nabla u_*(x)| = \theta'(u(x))|\nabla u(x)|, \\ &\text{and }\; \xi{\cdot}\nabla^2 u_*(x)\xi = \theta'(u(x))[\xi{\cdot}\nabla^2 u(x)\xi] + \theta''(u(x))[\nabla u(x){\cdot}\xi]^2, \end{aligned} \tag{4.3}$$

from which (4.2) is apparent because the final term drops off when $\xi \in H(x)$. *The partial quadratic forms in (4.1) are thus independent of the particular utility representation and are intrisic companions of the second-order smooth preference relation itself.*[13]

[13] Another way of reaching the same conclusion through variational analysis alone is to look at the second-order epi-derivative of the indicator of the preference set $P(x)$ at x [15, Chapter 13]. That comes out as the function that agrees on H with the given quadratic expression, but is ∞ outside of H.

We could proceed by placing assumptions directly on those partial quadratic forms, but will instead first recall enhancements of convexity that can be contemplated without immediately appealing to smoothness. The simplest is *strict* convexity of a closed set C; it adds to plain convexity the requirment that the line segment joining two different boundary points must, except for those points, lie in int C. This can also be articulated in a local sense, with strict convexity around a boundary point $\bar{x}$ of C meaning that the line-segment property holds in a neighborhood of $\bar{x}$. Such local strict convexity has a dual description in terms of the normal cones to C in the sense of convex analysis, given by

$$N_C(x) = \{\, v \mid v{\cdot}(x' - x) \le 0,\, \forall x' \in C \} \;\text{ for }\; x \in C, \tag{4.4}$$

for which in general $(v' - v){\cdot}(x' - x) \ge 0$ when $v \in N_C(x)$ and $v' \in N_C(x')$. It corresponds namely to having $(v' - v){\cdot}(x' - x) > 0$ when v and v' are nonzero, unless $x' = x$.

Beyond strict convexity, there is *strong* convexity of C around a boundary point $\bar{x}$, which is the property we'll really need. Its tighter normal cone description is that

$$\begin{array}{l} \exists\, \sigma > 0 \;\text{ and a neighborhood } V \text{ of } \bar{x} \text{ such that }\; (v' - v){\cdot}(x' - x) \ge \sigma |x' - x|^2 \\ \text{for }\; x, x' \in V \;\text{ when }\; v \in N_C(x') \;\text{ and }\; v' \in N_C(x') \;\text{ with }\; |v| = 1 \;\text{ and }\; |v'| = 1. \end{array} \tag{4.5}$$

Other, primal, descriptions of strong convexity can be furnished that bring in locally supporting parabolic surfaces, for instance, instead of just hyperplanes, to ensure reliable "curvature," but for our purposes the following example is central.

Example 4.5 (strong convexity from a constraint representation)**.** *Let $C = \{\, x \mid f(x) \le 0\}$ for a $\mathcal{C}^2$ function f, and let $\bar{x} \in C$ have $f(\bar{x}) = 0$ and $\nabla f(\bar{x}) \ne 0$. Then C is strongly convex around $\bar{x}$ if and only if*

$$\xi \ne 0,\; \nabla f(\bar{x}){\cdot}\xi = 0 \quad\Longrightarrow\quad \xi{\cdot}\nabla^2 f(\bar{x})\xi > 0. \tag{4.6}$$

Detail. This is a local matter, and we can simplify through a change of coordinates to posing it with C the epigraph of a function g on $I\!R^n$ that has $g(0) = 0$ and $\nabla g(0) = 0$. That way $f(x, \alpha) = g(x) - \alpha$ on $I\!R^{n+1}$ with $f(0, 0) = 0$ and $\nabla f(0, 0) = (0, -1)$. The condition in (4.6) comes out then as

$$\xi{\cdot}\nabla^2 g(0)\xi > 0 \;\text{ for }\; \xi \in I\!R^n \;\text{ with }\; |\xi| = 1 \tag{4.7}$$

and corresponds to g being strongly convex at the origin of $I\!R^n$. The claim is that this is equivalent to the strong convexity of the epigraph of g around the origin of $I\!R^{n+1}$ in the sense of (4.5).

The normal cone to the epigraph at a point $(x, g(x))$ has a unique unit element, $c(x)(\nabla g(x), -1)$ for $c(x) = 1/\sqrt{|\nabla g(x)|^2 + 1}$. The condition in (4.5) thus requires the inequality

$$\sigma |(x', g(x')) - (x, g(x))|^2 \le [c(x')(\nabla g(x'), -1) - c(x)(\nabla g(x), -1)]{\cdot}[(x', g(x')) - (x, g(x))] \tag{4.8}$$

to hold for x and x' in a neighborhood of 0. Elaborating the right side of (4.8) as

$$\begin{array}{l} c(x)[\nabla g(x'), -1) - (\nabla g(x), -1)]{\cdot}[(x', g(x')) - (x, g(x))] \\ \qquad\qquad +[c(x') - c(x)](\nabla g(x'), -1){\cdot}[(x', g(x')) - (x, g(x))] \end{array}$$

and expressing x' as $x + \tau\xi$ for $\tau > 0$ and ξ having $|\xi| = 1$, we are able to convert (4.8) into

$$\begin{array}{l} \sigma |(\tau\xi, g(x + \tau\xi) - g(x))|^2 \le c(x)[(\nabla g(x + \tau\xi) - \nabla g(x), 0){\cdot}(\tau\xi, g(x + \tau\xi) - g(x)) \\ \qquad\qquad\qquad +[c(x + \tau\xi) - c(x)](\nabla g(x + \tau\xi), -1){\cdot}(\tau\xi, g(x + \tau\xi) - g(x)), \end{array}$$

where

$$g(x + \tau\xi) - g(x) = \tau\nabla g(x){\cdot}\xi + o(\tau), \qquad \nabla g(x + \tau\xi) - \nabla g(x) = \tau\nabla^2 g(x)\xi + o(\tau).$$

On dividing by τ^2, that yields equivalently

$$\sigma\Big|\Big(\xi,\nabla g(x){\cdot}\xi+\tfrac{o(\tau)}{\tau}\Big)\Big|^2\le c(x)\Big[\xi{\cdot}\nabla^2 g(x)\xi+\tfrac{o(\tau)}{\tau}\Big]$$
$$+[c(x+\tau\xi)-c(x)](\nabla g(x+\tau\xi),-1){\cdot}\Big(\xi,\nabla g(x){\cdot}\xi+\tfrac{o(\tau}{\tau}\Big).$$

This brings the question of whether the strong convexity property in (4.5) is satisfied in our epigraphical setting down to whether the ratio

$$\frac{c(x)\Big[\xi{\cdot}\nabla^2 g(x)\xi+\tfrac{o(\tau)}{\tau}\Big]+[c(x+\tau\xi)-c(x)](\nabla g(x+\tau\xi),-1){\cdot}\Big(\xi,\nabla g(x){\cdot}\xi+\tfrac{o(\tau}{\tau}\Big)}{\Big|\Big(\xi,\nabla g(x){\cdot}\xi+\tfrac{o(\tau)}{\tau}\Big)\Big|^2},$$

is bounded below by some $\sigma>0$ for all unit vectors ξ when τ is small and x is near enough to the origin. In view of the boundedness of the term in the denominator and the fact that $c(x+\tau\xi)-c(x)\to 0$ as $\tau\searrow 0$ and $x\to 0$, the positive definiteness in (4.7) is seen to be necessary and sufficient, exactly as claimed. □

Strict convexity of preference relations is understood in economics as the enhanced version of the convexity axiom (A5) in which

$$y_0\succeq x,\ y_1\succeq x,\ y_0\neq y_1,\ \lambda\in(0,1)\quad\Longrightarrow\quad(1-\lambda)y_0+\lambda y_1\succ x. \tag{4.9}$$

This is a strict convexity property of the sets $P(x)$. We need to replace such strict convexity by strong convexity.

Definition 4.6 (strongly convex preferences)**.** *The preference relation will be called strongly convex if the sets $P(x)$ are strongly convex everywhere within $I\!R^g_{++}$.*

In terms of the description taken here as the definition of strong convexity in (4.5), we can draw on the fact that the unique unit vector in the normal cone to $P(x)$ at x is $-n(x)$ to express the property as the local existence, around any $\bar{x}\in I\!R^g_{++}$, of $\sigma>0$ such that

$$x\sim\bar{x},\ x'\sim\bar{x}\quad\Longrightarrow\quad[n(x')-n(x)]{\cdot}[x-x]\le-\sigma|x'-x|^2. \tag{4.10}$$

However, an alternate characterization is available from Example 4.5 in the case of a second-order smooth preference relation, for which the sets $P(x)$ can be represented as $\{\,x\,|\,u(x)\ge t\}$ for a $\mathcal{C}^2$ utility function with nonzero gradients. Then *strong convexity of preferences corresponds to the partial quadratic forms in (4.1) being always negative-definite*, and that is how it will come into play below. It is also how strong convexity has entered utility developments in the past, as seen for instance in Debreu [6].

In the coming statement of our result about the availability of concave utility representations, *minimally concave* utility will be important. We referred to this in Section 1 as indicating a concave u such that every other concave utility function u_* can be obtained from u as $u_*=\theta\circ u$ for a concave rescaling function θ. That property was introduced by Debreu [5] but called by him *least concavity.* Kannai [13] showed through his developments that it was equivalent to the different property formulated earlier by de Finetti [7], which we needn't get into here; "minimally" rather than "least" seems to be the better term.

Kannai in [13, Section 5] pointed out that his method of constructing a concave utility function, if successful, led to a minimally concave utility when carried out "in the most natural way." Here we follow much the same method but, at first anyway, don't attempt to concavify over the entire goods space G. We settle instead for a compact convex subset (arbitrarily large) within $I\!R^g_{++}$, where we are able to demonstrate that strong convexity of preferences gives a sufficient boost to secure the existence.

Theorem 4.7 (existence of minimally concave utility representations)**.** *For a second-order smooth, strongly convex preference relation, and any compact convex subset B of $I\!R^g_{++}$, there exists a representation by a $\mathcal{C}^2$ utility function with nonzero gradients on $I\!R^g_{++}$ that is concave relative to B and minimally so, being unique then up to affine rescaling.*

Proof. We have at our disposal from Theorem 4.2 utility representations that are $\mathcal{C}^2$ and can fix a basic one, u_e, targeting as our goal a rescaled $\mathcal{C}^2$ utility $u(x) = \theta(u_e(x))$, again with nonzero gradients, that over some region, at least, will be concave. Because $u_e(te) = t$, we will end up with $u(te) = \theta(t)$ in this pattern, so it's essential that the function θ be $\mathcal{C}^2$ and concave as well as increasing, moreover with $\theta'(t) > 0$ to ensure that $\nabla u(x)$ won't vanish anywhere.

The concavity of u will correspond to having $\xi{\cdot}\nabla^2 u(x)\xi \leq 0$, with the quadratic form in question being derivable from the one for u_e by the calculation we employed to get (4.3) in confirming (4.2):

$$\xi{\cdot}\nabla^2 u(x)\xi \;=\; \theta'(u_e(x))[\xi{\cdot}\nabla^2 u_e(x)\xi] + \theta''(u_e(x))[\nabla u_e(x){\cdot}\xi]^2. \tag{4.11}$$

We know from our assumption about the preference relation being strongly convex that the restriction of this quadratic form to the hyperplane subspace $H(x)$ in (4.1) is already negative-definite, yielding $\xi{\cdot}\nabla^2 u(x)\xi < 0$ there for nonzero ξ. A general ξ can be articulated as the sum of something in that subspace and a multiple of the normal vector $n(x)$, which aligns with $\nabla u_e(x)$, but the particular multiple won't matter. We can concentrate therefore on arranging that

$$(\xi + \nabla u_e(x)){\cdot}\nabla^2 u(x)(\xi + \nabla u_e(x)) \leq 0 \;\text{ for all }\; \xi \in H(x). \tag{4.12}$$

Plugging this specialization into the formula in (4.11) gives us, since $\nabla u(x){\cdot}\xi = 0$ when $\xi \in H(x)$,

$$\begin{aligned}(\xi + \nabla u_e(x)){\cdot}\nabla^2 u(x)(\xi + \nabla u_e(x)) &= \theta'(u_e(x))[(\xi + \nabla u_e(x)){\cdot}\nabla^2 u_e(x)(\xi + \nabla u_e(x))] \\ &\qquad + \theta''(u_e(x))[\nabla u_e(x){\cdot}(\xi + \nabla u_e(x))]^2 \\ = \theta'(u_e(x))[\xi{\cdot}\nabla^2 u_e(x)\xi] &+ 2\theta'(u_e(x))[\xi{\cdot}\nabla^2 u_e(x)\nabla u_e(x)] \\ &\qquad + \theta'(u_e(x))[\nabla u_e(x){\cdot}\nabla^2 u_e(x)\nabla u_e(x)] + \theta''(u_e(x))|\nabla u_e(x)|^2.\end{aligned}$$

Since the derivatives of θ are positive, and $\nabla u_e(x) \neq 0$, the key to (4.12) is having

$$\begin{gathered}-\frac{\theta''(u_e(x))}{\theta'(u_e(x))}|\nabla u_e(x)|^2 - [\nabla u_e(x)){\cdot}\nabla^2 u_e(x)\nabla u_e(x)] \geq \sup_{\xi \in H(x)}\Big\{v_e(x){\cdot}\xi - \xi{\cdot}Q_e(x)\xi\Big\}, \\ \text{where }\; v_e(x) = 2\nabla^2 u_e(x)\nabla u_e(x) \;\text{ and }\; Q_e(x) = -\nabla^2 u_e(x).\end{gathered} \tag{4.13}$$

This is where the negative-definiteness of the quadratic form on $H(x)$ crucially comes on stage, because we see in the supremum the operation of calculating the conjugate of a strongly convex quadratic function on $H(x)$ as a vector space of dimension $g-1$ within $I\!R^g$. For $\xi \in H(x)$ the inner product $v_e(x){\cdot}\xi$ depends only on the projection of $v_e(x)$ on $H(x)$, and the supremum determines its value $V_e(x) \geq 0$ at that projection. This value depends continuously on x, so we can rewrite the desired inequality in (4.13) as

$$-\frac{\theta''(u_e(x))}{\theta'(u_e(x))} \geq \frac{V_e(x) + [\nabla u_e(x)){\cdot}\nabla^2 u_e(x)\nabla u_e(x)]}{|\nabla u_e(x)|^2} \tag{4.14}$$

with the right side being a continuous function of x. In our search for θ satisfying this, our attention is focused on having nonpositive θ'' along with positive θ', so we can refine (4.14) to

$$-\frac{\theta''(u_e(x))}{\theta'(u_e(x))} \geq W_e(x) \;\text{ for }\; W_e(x) = \max\left\{0, \frac{V_e(x) + [\nabla u_e(x)){\cdot}\nabla^2 u_e(x)\nabla u_e(x)]}{|\nabla u_e(x)|^2}\right\}. \tag{4.15}$$

Consider now a compact convex subset B of $I\!R^g_{++}$. Enlarge it to have the vector interval form $[t_0 e, t_1 e]$, which is convenient because then t_0 and t_1 are the minimum and maximum values of $u_e(x)$ for $x \in B$. Define

$$\begin{aligned} &\lambda_B(t) = \max\{\, W_e(x) \,|\, x \in B,\ u_e(x) = t\} \ \text{ for } \ t \in [t_0, t_1], \\ &\lambda_B(t) = \lambda_B(t_0) \ \text{ for } \ t \in (0, t_0), \quad \lambda_B(t) = \lambda_B(t_1) \ \text{ for } \ t \in (t_1, \infty). \end{aligned} \tag{4.16}$$

Then λ_B is a continuous nonnegative function on $(0,\infty)$ in terms of which the targeted condition (4.15), with respect to $x \in B$, becomes

$$-\frac{\theta''(t)}{\theta'(t)} \geq \lambda_B(t) \ \text{ for } \ t \in [t_0, t_1], \ \text{ where } \ \frac{\theta''(t)}{\theta'(t)} = \frac{d}{dt} \log \theta'(t). \tag{4.17}$$

By design, this is *necessary and sufficient* for $u = \theta \circ u_e$ to be concave on $B = [t_0 e, t_1, e]$. Now define

$$\begin{aligned} \theta_B = \ & \text{the unique } \mathcal{C}^2 \text{ concave function on } (0,\infty) \text{ having} \\ & -\frac{\theta_B''(t)}{\theta_B'(t)} = \lambda_B(t) \ \text{ for all } \ t, \quad \theta_B(t_0) = 0, \quad \theta_B'(t_0) = 1. \end{aligned} \tag{4.18}$$

This is obtained by integrating $-\lambda_B(t)$ to get $\log \theta_B'(t)$ and then integrating the exponential of that to get $\theta_B(t)$; the two constants of integration are fixed by specifying the values of θ_B and θ_B' at t_0. Concavity is guaranteed because the nonnegativity of $\lambda_B(t)$ dictates nonpositivity of $\theta_B''(t)$ in (4.18). Then $u_B(x) = \theta_B(u_e(x))$ furnishes a $\mathcal{C}^2$ utility representation that is concave relative to B.

We argue next that u_B is minimally concave in this respect. Consider any other $\mathcal{C}^2$ utility function u that is concave on B. It must be given by $\theta(u_e(x))$ for some $\mathcal{C}^2$ concave rescaling function θ, but the claim is that also $u(x) = \theta_*(u_B(x))$ for some $\mathcal{C}^2$ concave rescaling function θ_*, in which case necessarily

$$\theta(t) = \theta_*(\theta_B(t)), \qquad \theta'(t) = \theta_*'(\theta_B(t))\theta_B'(t). \tag{4.19}$$

There is no loss of generality in normalizing θ to match θ_B in having $\theta(t_0) = 0$ and $\theta'(t_0) = 1$. That pins down θ_* to having $\theta_*(0) = 0$ and $\theta_*'(0) = 1$.

To derive the existence of θ_*, we appeal to the fact that θ has to satisfy (4.17), whereas θ_B comes from (4.18), and therefore

$$0 \geq \frac{\theta''(t)}{\theta'(t)} - \frac{\theta_B''(t)}{\theta_B'(t)} = \frac{d}{dt}\log\theta'(t) - \frac{d}{dt}\log\theta_B'(t) = \frac{d}{dt}\log\rho(t) \ \text{ for } \ \rho(t) := \frac{\theta'(t)}{\theta_B'(t)} > 0. \tag{4.20}$$

Then ρ is a nonincreasing $\mathcal{C}^1$ function for which $\theta_*'(\theta_B(t)) = \rho(t)$ by (4.19). Equivalently we have $\theta_*'(s) = \rho(\theta_B^{-1}(s))$, hence necessarily

$$\theta_*''(s) = \rho'(\theta_B^{-1}(s))\,[\theta_B^{-1}]'(s), \ \text{ or } \ \theta''(\theta_B(t)) = \frac{\rho'(t)}{\theta_B'(t)}.$$

Then $\theta_*''(t) \leq 0$, because $\rho'(t) \leq 0$ and $\theta_B'(t) > 0$. Obviously θ_* as a $\mathcal{C}^2$ concave function, increasing and having $\theta_*(0) = 0$ and $\theta_*'(0) = 1$, can be obtained from this by integrating the expression derived for $\theta_*''(s)$. Thus, u_B is minimally concave relative to B. □

Theorem 4.7 may be compared with a result of Aumann that can be distilled from a complicated game-theoretic setting in [1, Lemma 15.1] where other issues are simultaneously involved for an uncountably infinite collection of preference relations indexed by a measure space. He limits consideration to the case where the goods space is all of $I\!R^g_+$ and simply assumes from the start that the preference relation can be represented by a utility function which can be extended to be $\mathcal{C}^2$ on an open set containing $I\!R^g_+$.[14] Instead of appealing to strong convexity as we have laid out, he invokes bounds on the Gaussian curvatures of the boundaries of the sets $P(x)$ (which is ultimately equivalent, although much more complicated to formulate). He then gets, for any bounded subset B of $I\!R^g_+$ (not just of $I\!R^g_{++}$) a $\mathcal{C}^2$ utility representation on $I\!R^g_+$ that is concave on B. Whether or not this is minimally concave is not addressed. The uniquess up to affine rescaling is thereby missed, and with it the conclusion that cardinal, rather than just ordinal, utility is at hand.

Question 4.8 (global concave utility representations)**.** *When can the result in Theorem 4.7 be extended to obtaining a $\mathcal{C}^2$ utility representation that is minimally concave on all of G, not just relative to arbitrarily large compact convex subsets B of $I\!R^g_{++}$?*

Answer. Full information on that is contained in the proof of Theorem 4.7 in terms of the functions V_e and W_e developed in (4.14) and (4.15). Define

$$\lambda(t) = \sup \{ W_e(x) \,|\, x \in I\!R^g_{++},\ u_e(x) = t\} \ \text{ for } \ t \in (0, \infty). \tag{4.22}$$

If this function is finite — and continuous — then all the constructions in the proof of Theorem 4.7 go through perfectly well in a global manner with λ replacing λ_B. □

The catch, of course, is knowing that $\lambda(t) < \infty$, for which the proof resorted to truncating from G to the compact subset B. The continuity might conceivably be tricky as well with the supremum in (4.22) being over an unbounded set. Both could be taken care of by asking the indifference sets in the preference relation all to be bounded, but that's rather special and anyway would need to be combined with conditions on how those sets meet the boundary of $I\!R^g_+$.

The good thing is that every aspect of the function λ in (4.22) is inherent in the preference relation itself, so any assumption about λ would be a condition on the relation and not an artificial construct. The continuity and boundedness of λ could just be adopted under some name like "tameness" as the preference relation property that exactly corresponds to global representability by a $\mathcal{C}^2$ concave utility function which is "natural" in being minimally concave and unique in that up to affine rescaling. Still, a truly satisfying extension of Theorem 4.7 would require bringing such a condition down to a property that is more on the surface of preference behavior and easier to understand.

[14] This is a serious restriction which importantly leaves out, for example, preferences associated with Cobb-Douglas utilities.

References

[1] R. J. Aumann, *Values of markets with a continuum of traders*, Econometrica **43** (1975), 611-646.

[2] D. Brown and F. K. Kubler, *Computational Aspects of General Equilibrium Theory*, Lecture Notes in Economics and Mathematical Systems No. 604, Springer-Verlag, 2008.

[3] D. Brown and C. Shannon, *Uniqueness, stability and comparative statics in rationalizable markets*, in [2](2008), 27-40.

[4] C. Connell and E. B. Rasmusen, *Concavifying the quasiconcave*, Journal of Convex Analysis **24** (2017), 1239–1262.

[5] G. Debreu, *Least concave utility functions*, Journal of Mathematical Economics **3** (1976), 121–129.

[6] G. Debreu, *Smooth preferences*, Econmetrica **40** (1972), 603–616.

[7] B. de Finetti, *Sulle stratificazioni convesse*, Annali di Matematica Pura ed Applicata **30** (1949), 173–183.

[8] W. Fenchel, *Über konvexe Funcktionen mit vorgeschriebenen Niveaumannigfaltigkeiten*, Mathematische Zeitschrift **63** (1956), 496–506.

[9] A. Jofré, R. T. Rockafellar and R. J-B Wets, *Variational inequalities and economic equilibrium*, Math. of Operations Research **32** (2007), 32–50.

[10] A. Jofré, R. T. Rockafellar and R. J-B Wets, *Convex analysis and financial equilibrium*, Mathematical Programming B **148** (2014), 297–154.

[11] A. Jofré, R. T. Rockafellar and R. J-B Wets, *General economic equilibrium with financial markets and retainablity*, Economic Theory **63** (2017), 309–345.

[12] A. Jofré, R. T. Rockafellar and R. J-B Wets, *On the stability and evolution of economic equilibrium*, Journal of Convex Analysis **30** (2023).

[13] Y. Kannai, *Concavifiability and construction of concave utility functions*, Journal of Mathematical Economics **4** (1977), 1–56.

[14] A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic Theory*, Oxford University Press, 1995.

[15] R. T. Rockafellar and R. J-B Wets, *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften No. 317, Springer-Verlag, 1997.