REACHING AN EQUILIBRIUM OF PRICES AND HOLDINGS OF GOODS THROUGH DIRECT BUYING AND SELLING

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Abstract. The Walras approach to equilibrium focuses on the existence of market prices at which the total demands for goods are matched by the total supplies. Trading activities that might identify such prices by bringing agents together as potential buyers and sellers of a good are charateristically absent, however. Anyway, there is no money to pass from one to the other as ordinarily envisioned in buying and selling. Here a different approach to equilibrium — what it should mean and how it may be achieved — is offered as a constructive alternative.

Agents operate in an economic environment where adjustments to holdings have been needed in the past, will be needed again in a changed future, and money is familiar for its role in facilitating that. Marginal utility provides relative values of goods for guidance in making incremental adjustments, and with money incorporated into utility and taken as numéraire, those values give money price thresholds at which an agent will be willing to buy or sell. Agents in pairs can then look at such individualized thresholds to see whether a trade of some amount of a good for some amount of money may be mutually advantageous in leading to higher levels of utility. Iterative bilateral trades in this most basic sense, if they keep bringing all goods and agents into play, are guaranteed in the limit to reach an equilibrium state in which the agents all agree on prices and, under those prices, have no interest in further adjusting their holdings. The results of computer simulations are provided to illustrate how this works.

Key Words. Equilibrium of prices and holdings, exchange of goods, bilateral trades, markets, money, marginal utility, buy-sell thresholds, bid-ask spreads, convergence of prices.

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1 Introduction

For agents with holdings in various goods and preferences given by utility functions, exchanges in quantities of the goods may be undertaken to mutual advantage. Those exchanges might be facilitated by a "market," but how would that operate and what would it achieve?

In the usual view of a market, there are prices at which an agent can buy or sell goods subject to the budget constraint that the total value of the purchases is covered by the total value of the sales. An *equilibrium of prices and holdings* (in the terminology we use here) is at hand when no agent has a prospect of buying and selling that could lead to holdings with a higher level of utility. What might bring this about? Could agents proceed directly with each other, making advantageous exchanges and, in so doing, gradually reach a consensus on prices and thereby an equilibrium?

Answering this question seems essential for understanding the very fundamental concept of a market, but past research on equilibrium has mostly taken a different path. The standard line of inquiry, opened by Walras in the 1870s and placed on firm mathematical footing by Debreu [**?**] in 1959, is whether, for the initial holdings of the agents, there exist prices such that, when each agent individually determines an exchange that would maximize utility under the budget constraint, the resulting aggregate demands for goods would match the overall supplies. Then indeed, after those desired adjustments, the prices and holdings will be in equilibrium, with no agent interested in further exchange. A balance in competition is depicted this way as arising from individual agents acting solely out of self-interest. Prices coordinate them and appear to serve as a means of decentralization. But there are flaws in this portrayal.

Although the existence of Walrasian prices has been established under various mathematical assumptions, explaining how they might actually come into being has been harder. The main rationale has been some version of tâtonnement, in which prices are adjusted up or down by an abstract entity who repeatedly seeks feedback from the agents. This process is usually modeled by a differential equation; see [22, 1, 19, 23, 13, 14], and more recently [12]. In certain cases tâtonnement does identify prices fitting the Walras prescription,[1] yet it suffers from requiring a central coordinator instead of relying on agent interactions directly. And it lacks justification as an information process backed by economic observation.

Another serious shortcoming of the Walrasian market is that the picture of buying and selling is left blank. When people think of a real market, they contemplate circumstances in which someone can buy or sell a quantity of some good *for a quantity of money*, but in the Walrasian tradition no money changes hands. All talk of money is in fact set aside. Prices are only *relative*, with their ratios indicating the values of goods relative to each other. The agents, having determined what they want to acquire or give up, while respecting the budget constraint (which is insensitive to the scaling behind the relative prices), present their decisions to a "clearing house entity." That entity then somehow executes the desired transfers of goods as a simultaneous *grand exchange*. This is not at all a matter of direct buying or selling by agents, and it can hardly be

[1] Contrary to wide-spread pessimism about the ability of the process to succeed, it can generally be counted on to converge to Walrasian prices when initiated from holdings not too far from equilibrium holdings, even in a setting where agents are allowed to hold zero quantities of some of the goods; see [12].

viewed as signaling economic decentralization.

Might support for the Walras approach be found by demonstrating that the grand exchange could be carried out by the agents themselves, trading together directly in some order? An investigation by one of this paper's coauthors (Rockafellar) in [17] showed that this might be possible up to a point, with two agents at a time bartering one good for another at the given prices. But there was a hitch: no guarantee that each trade would result in higher utility for both. With agents permitted to barter *bundles of several* goods in each transaction, a utility-raising trade was determined in [17] always to exist, but whether a sequence of such trades could bring about the specified grand exchange was elusive and left unanswered. In fact, this was only the latest inquiry into a genuine market justification for Walrasian prices. Starr [21] (1972) and Ostroy [16] (1973) already derived conditions under which the grand exchange might be achieved through budget-balanced multigood trades among agents with money in a helping role. Their trades, though, don't meet the criterion underscored in [17] that utility should always go up for both parties.

Doubts about the credibility of the Walras scheme led others to abandon it in favor of alternative models of exchange and price formation. Shapley and Shubik [20] in 1977 had agents make money-bids for quantities of goods which trading posts for those goods would settle at auction-fixed prices. This replaced the centralized clearing house by a noncooperative game with utility pay-offs. It marked the beginning an elaborate theory of *strategic market games*, envisioned as perhaps building a bridge from microeconomics to macroeconomics. For a bigger view of this over the years, see Dubey and Shubik [**?**] (1980), Sorin [**?**] (1996), and Levando [**?**] (2012). A strategic market game achieves a different equilibrium than the one in the Walras model, and that equilibrium might not even provide an allocation of goods that is "efficient" in the sense of Pareto optimality with respect to agent utility. Getting to equilibrium would seem moreover to require a degree of organization and rule enforcement that raises many questions.[2] It's hard not to see a plausiblity gap with respect to what really goes on in economic behavior.

Still other researchers have departed from the Walras model and the prices it offers without passing to game formulations. How might agents, through repeated bilateral exchanges to the mutual benefit of both, eventually achieve Pareto optimality and with it some particular equilibrium of prices and holdings? This is the direction that especially interests us here.

Feldman [4] in 1973 addressed this with each pair of agents persisting in multigood trades until all possibilities of improvement have been exhausted. Only then could another pair take the stage. He showed that cluster points of the sequence of goods allocations would be Pareto optimal as long as one of the goods was "money-like" in a sense. Eckalbar [3] in 1986 improved the picture greatly by focusing on pairs of agents who *exchange a single good at a time for money* — with one being the buyer and the other the seller. The marginal utility of money relative to the marginal utility of the good, for each of the agents, guides whether the deal is worthwhile. This is the pattern we will adopt as well, in our developments below, but our mechanism for reaching an equilibrium will be very different from Eckalbar's, which was a process in continuous time rather than one of iterative discrete exchange. He poses a differential equation in which the

[2] For instance, agents in [20] must back up their bids with money commitments before knowing what the equilibrium prices might turn out to be, and this can present issues of insolvency that have to be handled.

right side is supposed to have certain behaviors (not fully pinned down in terms of the properties of the function itself), and the claim is made that trajectories generated from this will converge to Pareto optimality. However, his proof of this claim falls short.[3] Another approach to reaching an equilibrium through continuous trading governed by a differential equation was proposed by Bottazzi [2] in 1994. There, trading is bilateral in a generalized sense, but it doesn't come down to simple buying/selling of individual goods and indeed, there is no money. A drawback to both of these differential equation schemes, from the perspective of economics, is that they tacitly require some societal entity to select the particular equation and compel the agents to submit to it. Bottazzi also expects agents to wish for "steepest ascent" with respect to utility, but that is highly artificial, because the direction of steepest ascent varies with the choice of units in which the goods are measured. Why should the choice of units affect an agent's behavior?

Bilateral trading in discrete time has been taken up more recently by Flåm in [5], [6], [7].[4] In [5] utility is somehow measured directly in money units and trading is accompanied by monetary side payments. Like Bottazzi, Flåm insists on directions of "steepest ascent" for trades between two agents.[5] Besides the artificiality of those directions, they generally force agents to trade "full bundles" of goods, instead of a single good at a time. Bilateral trading of a different sort, not involving money, was taken up from another angle by Flåm and Gramstad in [6]. A subsequent contribution of Flåm in [7] brings in money as numéraire and proposes a scheme in which pairs of agents trade bundles of goods at money prices, but without being obliged to exchange money itself. Negative prices and negative holdings of money are permitted, and besides having utility functions, pairs of agents are viewed as possessing various *joint* properties to guide their trading with eachother that aren't normally a part of equilibrium models. Despite those innovations, the assumptions and arguments in [7] don't rise to the level of proving that the iterative adjustments will converge to an equilibrium configuration, or for that matter even produce an equilibrium as a cluster point. Another thing to note is that a transaction between two agents in [7] isn't required to be advantageous to both parties. One of the agents could actually see a decrease in utility, and the question then comes up as to why the transaction would be carried out. It seems that an enforcer would be needed in the background.

Most recently in [8, 9], Flåm has explored different money-price schemes in a game framework based once more on a Walras-like clearing house operation. Double auctions are employed in a process of identifying prices that might ultimately signal Pareto optimality, but bilateral trades are no longer seen. This draws in part on ideas of Gintis and Mandel [10, 15], who have agents participating in a game where they choose price vectors as strategies and get utility pay-offs from the resulting redistributions of goods coming from processes in a centeralized clearing house system. More about double auctions in determining a price for a good, and the stability of that process or lack of it, can be learned from current work of Rasooly [**?**].

[3] It takes for granted that the trajectory, being bounded, must converge to a particular point, whereas it conceivably might just approach a "limit cycle."

[4] Coauthor Rockafellar of the current paper is grateful to Flåm for introducing him to this subject during a two-week visit to the University of Bergen in June of 2015.

[5] He relies on the Euclidean norm for this. But that amounts to identifying the space of goods vectors with the dual space of price vectors. This has no economic basis and thus undermines the economic rationale for his trading prescriptions.

In contrast, our approach in this paper is far simpler, yet demonstrates how an equilbrium of prices and holdings is sure to emerge in time from elementary bilateral transactions which agents can undertake solely for reasons of immediate self-interest. Another major difference is that, instead of multigood trades, we emphasize like Eckalbar the *buying and selling of a single good at a time* and insist on *a payment of money always being made*. The amount of payment emerges from utility of the two agents involved. We are able to show that interactions on this utterly fundamental level lead inevitably to prices accepted by all the agents, which identify a particular equilibrium of prices and holdings — and *as a true limit*, not merely some cluster point. Pareto optimality of the agents' ultimate holdings is revealed in this way to be a natural consequence of direct deals that have no need for a clearing house entity.[6] Nothing like that has been established before now.[7]

Another distinctive feature of our contribution is that we accompany the money-based trading between agents by computational experiments which illustrate how it works out. In a numerical model with Cobb-Douglas-type utility functions, we show that an exchange equilibrium can readily be attained through the decentralized process we propose, even with many goods and many agents. Different trading equilibria are seen to be reached by the agents even in starting from the same holdings, because of the random patterns in which the agents can come together in pairs and select a good to buy/sell. This is completely "economic" and underscores that our equilibrium prices and holdings come from decentralized market dynamics without a higher entity dictating a plan rigid about who-does-what-when. The Walras approach, in contrast, lacks representation of mutually beneficial exchange activity betwen agents in pairs. Very differently, the prices it assigns to a configuration of initial holdings rely on having a centralized mechanism for an all-at-once redistribution of goods, as explained earlier. That makes Walrasian prices economically unconvincing, and not just to us, as the literature reveals. The realization that simple buying and selling, as we describe it, doesn't lead to Walrasian prices, unless by accident, highlights the artificiality of those prices all the more.

In the plan for the rest of this paper, we start in Section 2 by explaining the mathematical details of our model of utility and what extra help it offers to the agents. We continue in Section 3 by showing how agents are able to identify mutually beneficial opportunities for acting as buyer or seller of a good, and how the absence of such opportunities characterizes a state of equilibrium. Section 4 develops how iterative buying and selling produces convergence to such equilibrium. This is then brought to life by computer simulation in Section 5.

2 Marginal utility and price thresholds

The goods space in our setting is the nonnegative orthant $I\!R^{n+1}_+$. The goods are indexed by $j = 0, 1, \ldots, n$, with good 0 standing for money. The agents, indexed by $i = 1, \ldots, m$, deal with

[6] Because an equilibrium of prices and holdings can be viewed as the case of a Walras equilibrium in which the initial holdings coincide with the final holdings, its Pareto optimality is assured by the First Welfare Theorem.

[7] These results, written up in a working paper of April 2018, were presented by co-author Jofré in June 2018 at a meeting of economists in Paris.

holdings of goods represented by vectors $x_i = (x_{i0}, x_{i1}, \ldots, x_{in})$, and they have utility functions u_i for comparing such holdings. Money will serve as numéraire, but it will have a role for us beyond that of a customary numéraire good like gold, because the buyer of any other good will have to transfer to the seller a quantity of money in payment.

Although it may be deemed controversial in some quarters to have utility apply to money, this notion goes back to the early days of the topic we are addressing. We have argued moreover, in previous papers devoted to economic equilibrium [11, 12], that there are solid reasons for allowing it, citing strong support from Keynes. Much of the resistance to money is likely due anyway to the Walrasian model of equilibrium as a circumstance with no past and no future. In thinking of it instead as a transient phenomenon in which agents agree on prices, but no one wishes to buy or sell anything, the picture is different. Equilibrium holdings can be affected by outside actions like consumption, production, taxation or subsidy. The resulting disequilibrium that can then be countered by renewed buying and selling. From that angle, agents can well have an understanding of money and its ongoing importance in transactions, and may therefore wish to hold a quantity of it. The 2018 paper of Dubey, Sahi and Shubik [?] provides additional insights into money along with other helpful references.

Utility functions will do more for us than just capturing the preferences that an agent might have for one vector of holdings over another. They will be important also in assessing quantitatively the effects of *gradual* shifts in holdings. That will give us a handle on how trading might be executed when two agents get together. Specifically, suppose agent i has current holdings given by x_i and is contemplating a shift from that to $x_i + \tau\Delta x_i$ for some "step size" $\tau \geq 0$ with respect to a vector Δx_i having components that may be positive, negative, or zero. The choice of τ must take into account the utility value that corresponds to it, namely

$$\theta(\tau) = u_i(x_i + \tau\Delta x_i). \tag{2.1}$$

For our purposes, we want not only to know whether a choice τ_1 is better than a choice τ_0, in the sense that $\theta(\tau_1) > \theta(\tau_0)$, but also to be able to work with derivatives

$$\theta'(\tau) = \nabla u_i(x_i + \tau\Delta x_i){\cdot}\Delta x_i \tag{2.2}$$

as capturing *marginal utility* with respect to an increase or decrease in τ.

How realistic is that, though, from the perspective of the basic theory of preference relations and their utility representations? Standard axioms only produce a utility function that is continuous, quasi-concave and perhaps quasi-smooth in the sense that its convex upper level sets have kink-free boundaries. Differentability is unaddressed, and derivatives might not be quantitatively meaningful anyway because of the potential arbitrariness in nonlinear rescaling of utility values. However, recent advances in [18] are illuminating in this respect. Natural properties *of the preference relation itself* are identified there which characterize whether it can be represented by a utility function that is $\mathcal{C}^1$ or $\mathcal{C}^2$. In the $\mathcal{C}^2$ case, if the preference sets are *strongly* convex (strictly convex with reliable curvature), there is sure to exist, relative any compact convex subset C of the positive orthant, a $\mathcal{C}^2$ utility representation that is concave on C, moreover in a minimal way that makes it *unique up to affine rescaling*, i.e., up to the choice of units in which utility is to be measured. We align ourselves here with those preference-relation-grounded characteristics.

In preparation for stating the assumptions, we assign to each agent i a set X_i of *admissible* holdings x_i, taking it to have the form

$$X_i = \left\{ x_i \in I\!R^{n+1}_+ \,\middle|\, x_{ij} > 0 \text{ for } j \in J_i \right\} \text{ for a collection } J_i \text{ of goods } j. \tag{2.3}$$

The goods in J_i will be called the *essential* goods for agent i, with other goods being *inessential.* Money is assumed to always be essential:

$$0 \in J_i \text{ for all agents } i. \tag{2.4}$$

Utility assumptions. *Agent i has on X_i a concave utility function u_i that is twice continuously differentiable.*[8] *The gradient vectors $\nabla u_i(x_i)$ have positive components and the hessian matrix $\nabla^2 u_i(x_i)$ is negative-definite on the subspace orthogonal to $\nabla u_i(x_i)$. Furthermore,*

$$x_i \in X_i \implies \left\{ x_i' \in X_i \,\middle|\, u(x_i') \geq u(x_i) \right\} \textit{ is a closed subset of } I\!R^{n+1}. \tag{2.5}$$

The condition assumed on gradients guarantees that u_i increases on X_i with respect to any increase in good:

$$u_i(x_i') > u_i(x_i) \text{ when } x_i' \geq x_i,\ x_i' \neq x_i. \tag{2.6}$$

The concavity and differentiability assumption makes the expressions $\theta(\tau) = u_i(x_i + \tau\Delta x_i)$ in (2.1) be concave functions of τ with continuous second derivatives,

$$\theta''(\tau) = \Delta x_i \cdot \nabla^2 u_i(x_i + \tau\Delta x_i)\Delta x_i.$$

The partial negative-definiteness condition on hessians[9] assures that $\theta''(\tau) < 0$ unless Δx_i is a multiple of $\nabla u_i(x_i + \tau\Delta x_i)$, which will not come up because we will only be looking at adjustment vectors Δx_i that have both a positive component and a negative component. Thus, the functions $\theta(\tau)$ we work with will always be *strongly* concave. The closedness in (2.5) ensures further that, for $x_i \in X_i$ and mixed-sign vectors Δx_i, the set

$$\left\{ \tau \geq 0 \,\middle|\, x_i + \tau\Delta x_i \in X_i,\ u_i(x_i + \tau\Delta x_i) \geq u_i(x_i) \right\} \text{ is compact.} \tag{2.7}$$

A key role will be played by the partial derivative $\frac{\partial u_i}{\partial x_{i0}}(x_i)$, which gives the *marginal utility of money* associated with the holdings vector x_i. It is the derivative $\theta'(0)$ in (2.1) in the case of Δx_i having 1 as its initial component, but 0 for all the others. More generally, $\frac{\partial u_i}{\partial x_{ij}}(x_i)$ gives the *marginal utility of good j* at x_i. These marginal utilities are always positive under our assumptions.

[8] The interpretation of this for points of X_i on its boundary is that the first and second derivatives on the interior of X_i extend to them as unique limits.

[9] This corresponds to locally strong convexity of upper level sets of u_i, instead of the more commonly assumed strict convexity; see [18, Section 4].

Price thresholds for goods. *For an agent i and a good $j \neq 0$, the ratio*

$$p_{ij}(x_i) = \frac{\partial u_i}{\partial x_{ij}}(x_i) \Bigg/ \frac{\partial u_i}{\partial x_{i0}}(x_i) > 0 \tag{2.8}$$

will be called the price threshold of agent i for selling or buying good j when the current holdings are given by the goods vector x_i.

This terminology will be justified in the analysis of selling and buying that comes next. Observe that when the the marginal utility $\frac{\partial u_i}{\partial x_{ij}}(x_i)$ is interpreted as being measured in units of utility of agent i per unit of good j, and $\frac{\partial u_i}{\partial x_{ij}}(x_i)$ is interpreted as being measured in units of utility of agent i per unit of money, the prices $p_{ij}(x_i)$ correctly come out in units of money per unit of good j.[10] It's worth noting that *the threshold values $p_{ij}(x)$ are inherent in the underlying preference relation and independent of the particular scaling of the function u_i*; rescaling can change the length of gradient vectors but has no effect on the ratios between the components of those vectors.

We postulate, as is usual in models of exchange aimed at equilibrium analysis, that agents have in X_i initial holdings x_i^0 which they may wish to trade for other holdings they would like better. All goods are assumed to be in positive *total* supply, and those amounts won't be altered by the trading. The trades will be executed in such a way that the utility of an agent's holdings never decreases. Because of (2.7), the holdings of agent i will always be in X_i. Then in particular, through (2.4) and (2.5), agents will always have some money at their disposal.

This feature leads to a major difference between our model and previous models. We not only approach trading iteratively through pairs of agents, but insist on *bringing money into every transaction*. The considerations behind that will be laid out now in terms of agents selling or buying an amount of some good $j \neq 0$ for some amount of the money good $j = 0$.

Agents as sellers. For an agent i and a good $j \neq 0$, what needs to be considered in contemplating the sale of a quantity $\xi_j > 0$ of that good at a price $\pi_j > 0$? This consists of subtracting ξ_j from the amount of good j currently held, and therefore must be subject to the constraint $\xi_j \leq x_{ij}$. On the other hand, it involves adding the money amount $\pi_j \xi_j$ to the current holding of money, namely x_{i0}. Thus, it shifts x_i to $x_i + \xi_j[\pi_j, -1]$, where the notation is that

$$[\pi_j, -1] \;=\; \text{the vector } \Delta x_i \text{ having components } \begin{cases} \Delta x_{i0} = \pi_j, \\ \Delta x_{ij} = -1, \\ \Delta x_{ik} = 0 \text{ for other goods } k. \end{cases} \tag{2.9}$$

In evaluating the utility of such a sale, we are in the framework of (2.1) for this special Δx_i with ξ_j in place of τ and are looking at

$$\theta_+(\xi_j) = u_i(x_i + \xi_j[\pi_j, -1]\,). \tag{2.10}$$

[10]Note that the monotonicity of the partial derivatives of u_i, following from our assumption that u_i increases with respect to upward shifts of components of x_i, doesn't necessarily carry over to monotonicity of the prices $p_{ij}(x_i)$, due to the division in (2.8).

For selling to be attractive, starting from an arbitrarily small amount, the marginal utility $\theta'_+(0)$ should be > 0. But

$$\theta'_+(\xi_j) = \frac{\partial u_i}{\partial x_{i0}}(x_i + \xi_j[\pi_j, -1]\,)\pi_j - \frac{\partial u_i}{\partial x_{ij}}(x_i + \xi_j[\pi_j, -1]\,), \tag{2.11}$$

so that

$$\theta'_+(0) = \frac{\partial u_i}{\partial x_{i0}}(x_i)\Big[\pi_j - p_{ij}(x_i)\Big], \tag{2.12}$$

where the initial factor, the marginal value of money, is positive. Therefore,

$$\begin{array}{l}\text{selling good } j \text{ at price } \pi_j \text{ will be attractive to agent} \\ i \text{ with current holdings } x_i \text{ if and only if } \pi_j > p_{ij}(x_i).\end{array} \tag{2.13}$$

When selling is attractive in this way, how much should agent i consider selling? This is where our assumptions on second derivatives come in. Under those assumptions the function $\theta(\xi_j)$ in (2.6) is concave with $\theta'_+(\xi_j)$ strictly decreasing from its initial value $\theta'_+(0) > 0$ as ξ_j rises from 0. The interval of values of ξ_j such that $x_i + \xi_j[\pi_j, -1] \in X_i$ lies in $[0, x_{ij}]$ (and equals it unless j is an essential good). Hence by (2.7) there will be a unique maximizing quantity

$$\xi_j^+(x_i, \pi_j) = \operatorname*{argmax}_{\xi_j}\Big\{\, u_i(x_i + \xi_j[\pi_j, -1]\,) \,\Big|\, x_i + \xi_j[\pi_j, -1] \in X_i \,\Big\}. \tag{2.14}$$

It will be either x_{ij} or the unique value of ξ_j where $\theta'_+(\xi_j) = 0$, whichever is reached first as ξ_j increases, and only starting then will selling cease to be attractive.

The consequence for agent i of selling that optimal amount would be to replace x_i by $x'_i = x_i + \xi_j[\pi_j, -1]$ for $\xi_j = \xi_j^+(x_i, \pi_j)$ and thereby change the price threshold $p_{ij}(x_i)$ for good j to $p_{ij}(x'_i)$. The case in (2.14) where optimality is attained with $\theta'_+(\xi_j) = 0$ corresponds to having $p_{ij}(x'_i) = \pi_j$, in contrast to $p_{ij}(x_i) < \pi_j$. The case where it is attained by hitting supply constraint for good j while θ'_+ is still increasing, corresponds instead to having $p_{ij}(x'_i) < \pi_j$. To summarize,

$$\begin{array}{c}\text{for the resultant holdings } x'_i \text{ at optimality in (2.14),} \\ p_{ij}(x'_i) = \pi_j \text{ if } x'_{ij} > 0, \text{ but } p_{ij}(x'_i) \le \pi_j \text{ if } x'_{ij} = 0.\end{array} \tag{2.15}$$

Note, however, that the case of $x'_{ij} = 0$ can't come up if j is an essential good for agent i, namely $j \in J_i$. This is guaranteed by (2.7).

Agents as buyers. For an agent i and a good $j \neq 0$, what needs to be considered in contemplating the purchase of a quantity $\xi_j > 0$ of that good at a price $\pi_j > 0$? This consists of adding ξ_j to the amount of good j currently held while subtracting the money amount $\pi_j\xi_j$ from the current holding of money, namely x_{i0}, leaving some remainder to 0. It shifts x_i to $x_i - \xi_j[\pi_j, -1]$, where the notation is again that of (2.9). We are in a situation similar to that of a seller, but are looking this time at

$$\theta_-(\xi_j) = u_i(x_i - \xi_j[\pi_j, -1]\,) \tag{2.16}$$

for which

$$\theta'_-(\xi_j) = -\frac{\partial u_i}{\partial x_{i0}}(x_i - \xi_j[\pi_j, -1]\,)\pi_j + \frac{\partial u_i}{\partial x_{ij}}(x_i - \xi_j[\pi_j, -1]\,), \tag{2.17}$$

For buying to be attractive, starting from an arbitrarily small amount ξ_j, the marginal utility $\theta'_-(0)$ should be > 0. This time, however,

$$\theta'_-(0) = \frac{\partial u_i}{\partial x_{i0}}(x_i)\Big[p_{ij}(x_i) - \pi_j\Big]. \tag{2.18}$$

Therefore,

$$\begin{array}{l}\text{buying good } j \text{ at price } \pi_j \text{ will be attractive to agent}\\ i \text{ with current holdings } x_i \text{ if and only if } \pi_j < p_{ij}(x_i).\end{array} \tag{2.19}$$

When buying is attractive in this way, how much should agent i consider buying? Again the function $\theta_-(\xi_j)$ is concave with $\theta'_-(\xi_j)$ strictly decreasing in ξ_j and $\theta''_-(\xi_j) < 0$, and the amount of money available for the purchase is $x_{i0} > 0$. That amount can't end up being totally committed, though, because we know from (2.7) that the interval of ξ_j values such that $x_i - \xi_j[\pi_j, -1] \in X_i$ and has utility at least that of x_i is a compact interval within $[0, x_{i0})$. Thus, buying will continue to be attractive until reaching the level of

$$\xi_j^-(x_i, \pi_j) = \operatorname*{argmax}_{\xi_j}\Big\{\, u_i(x_i - \xi_j[\pi_j, -1]\,) \,\Big|\, x_i - \xi_j[\pi_j, -1] \in X_i \,\Big\}, \tag{2.20}$$

which will be the unique value of ξ_j yielding $\theta'_-(\xi_j) = 0$.

The characterization of the optimality in (2.20) is more elementary than it was for (2.14), where we had to cope with the possibility that the entire supply of good j might be sold. Here, if full purchase went through,, the new holdings vector would be $x'_i = x_i - \xi_j[\pi_j, -1]$ for $\xi_j = \xi_j^-(x_i, \pi_j)$, which comes unambiguously from achieving $\theta'_-(\xi_j) = 0$. The new price threshold $p_{ij}(x'_i)$ for good j that replaces $p_{ij}(x_i)$ would emerge only out of that, hence

$$\text{for the resultant holdings } x'_i \text{ at optimality in (2.20), } p_{ij}(x'_i) = \pi_j. \tag{2.21}$$

Utility function status. Like the thresholds $p_{ij}(x)$ in the conditions (2.13) and (2.19) that open the way to selling and buying, the quantities indicated in (2.14) and (2.20) for the amounts desired to be sold or bought *depend only on the preference relation, not the particular utility function u_i with its scaling.* That's because the optimization behind those quantities refers to proceeding along a line to a point of highest utility, and that point doesn't change with a change in scaling. Agents could be tasked with locating that point "geometrically" with respect to the preference sets in the relation, but it's nicer to think of them working with a utility function u_i, as described, which entails no loss of generality.

Connections with other work. Mandel and Gintis [15] propose in their game model that agents have "private" vectors of prices which can be deployed strategically, but they offer no basis in marginal utility, such as we provide here. In the literature on strategic market games mentioned in our introduction, agents are expected to come up with bids for buying or selling up to some amount of a good, but the origin of such bids is left open. Such price-quantity pairs can now be seen as coming, for instance, from our analysis of potential buyers and sellers in association with their price thresholds.

3 Bilateral trading in one good at a time

With the characteristics and interests of sellers and buyers in hand, we can proceed with ideas about how a seller and a buyer might come together through information revealed by one or the other, or both, and proceed with a money-based transaction in a good $j \neq 0$.

Price premiums and bid-ask spreads. It has been seen that an agent i with threshold price $p_{ij}(x_i)$ for good j only wants to sell at a higher price than that, and only wants to buy at a lower price. How should those prices be set? Our approach is that agent i has at any given stage a "premium" $\delta_{ij} > 0$ in mind to make a transaction in good j worth undertaking. The selling and buying prices are set by the agent to maintain that premium:

$$p^+_{ij}(x_i) = p_{ij}(x_i) + \delta_{ij}, \qquad p^-_{ij}(x_i) = p_{ij}(x_i) - \delta_{ij}. \tag{3.1}$$

The particular premiums are only temporary. Eventually they will need to be reduced more and more if trading is going to be able to identify an equilibrium to utmost precision.

Agent i can reveal being open to selling a good j at price $p^+_{ij}(x_i)$ or being open to buying it at price $p^-_{ij}(x_i)$, or both, or neither — in the context of holding off until later. Just how this may be articulated over time will be the subject of discussion later, in the Section 4.

Trading action based on proposed prices. *A bilateral trade is available in good j between agent i_1 as seller and agent i_2 as buyer at a price π_j if, with respect to (3.1),*

$$x_{i_1 j} > 0, \qquad p^+_{i_1 j}(x_{i_1}) \leq p^-_{i_2 j}(x_{i_2}), \qquad \pi_k \in [p^+_{i_1 j}(x_{i_1}), p^-_{i_2 j}(x_{i_2})] \tag{3.2}$$

In acting on this availability (if they choose to do so), the amount of good j transfered from the seller to the buyer will be

$$\xi_j = \min\{\xi^+_j(x_{i_1}, \pi_j),\, \xi^-_j(x_{i_2}, \pi_j)\} \tag{3.3}$$

as determined from (2.14) and (2.20). In return for this amount of good j, the buyer will pay the seller the money amount $\pi_j \xi_j$.

It would not really be necessary to trade all the way up to the profitability-limiting level in (3.3), but this rule is the simplest. It guarantees that the agents will have activated at least one of the optimality rules in (2.14) or (2.20), and that will be useful to us later.

Along with these specifics about how agents can mutually improve the utility values of their holdings by trading individual goods, it will be important to understand the situation in which the limits of such trading have been reached. It turns out that this is the case where an equilibrium of prices and holdings is at hand.

Theorem 1 (seller-buyer characterization of an equilibrium of prices and holdings)**.** *Suppose the agents have holdings $\bar{x}_i$ such that no bilateral trade is available in any good at any premium levels $\delta_{ij} > 0$, or in other words:*

$$\begin{array}{l} \textit{for every good } j \neq 0 \textit{ and potential seller } i_1 \textit{ and buyer } i_2 \\ \textit{enabled by a supply } \bar{x}_{i_1 j} > 0, \textit{ one has } p_{i_1 j}(\bar{x}_{i_1}) \geq p_{i_2 j}(\bar{x}_{i_2}). \end{array} \tag{3.4}$$

Then the price vector

$$\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n) \text{ with } \bar{p}_j = \max_{i=1,\ldots,m} p_{ij}(\bar{x}_i) > 0 \tag{3.5}$$

for the nonmonetary goods (the money price of money being 1) furnishes an equilibrium with those holdings:

$$\begin{array}{l} \bar{x}_i \text{ maximizes } u_i(x_i) \text{ over } x_i \in X_i \text{ subject to the} \\ \text{budget constraint } x_{i0} + \sum_{j=1}^n \bar{p}_j x_{ij} = \bar{x}_{i0} + \sum_{j=1}^n \bar{p}_j \bar{x}_{ij}. \end{array} \tag{3.6}$$

Conversely, every equilibrium of prices and holdings can be described in this manner.

Proof. If the threshold inequality at the end of (3.4) were strict, and $\bar{x}_{i_2 j} > 0$ as well, one could consider reversing the roles of buyer and seller and thereby opening up a trade. Thus (3.4) really says for each good $j \neq 0$ that

$$\begin{array}{l} p_{ij} \text{ has the same value for all agents } i \text{ having } \bar{x}_{ij} > 0, \\ \text{whereas } p_{ij} \text{ is } \leq \text{ that value for agents } i \text{ with } \bar{x}_{ij} = 0. \end{array} \tag{3.7}$$

Storing this observation temporarily, let's next analyze the optimality in (3.6) from the perspective of convex analysis. In terms of the linear function

$$l_i(x_i) = x_{i0} + \sum_{j=1}^n \bar{p}_j x_{ij} = x{\cdot}(1, p) \tag{3.8}$$

(3.6) says $\bar{x}_i$ maximizes $u_i(x_i)$ over $\mathbb{R}^n_+$ subject to $l_i(x_i) \leq l_i(\bar{x}_i)$, but that's the same as saying $\bar{x}_i$ minimizes $l_i(x_i)$ over the set $C_i = \{ x_i \geq 0 \,|\, u_i(x_i) \geq u_i(\bar{x}_i) \}$. That set is convex, so this holds if and only if the normal cone $N_{C_i}(\bar{x}_i)$ to C_i at $\bar{x}_i$ contains the vector $-\nabla l_i(\bar{x}_i) = -(1, p)$. The normal cone is the sum of the ray generated by $\nabla u_i(\bar{x}_i)$ and the normal cone to $\mathbb{R}^n_+$ at $\bar{x}_i$, consisting of all $v \leq 0$ such that $v{\cdot}\bar{x}_i = 0$. Having $-(1, p)$ belong to it corresponds therefore to the existence of $\lambda_i > 0$ such that

$$\nabla u_i(\bar{x}_i) - \lambda_i(1, p) \geq 0 \text{ with equality in the } j\text{th component if } \bar{x}_{ij} > 0, \tag{3.9}$$

where in particular $\bar{x}_{i0} > 0$ as enforced by (2.4)–(2.5). Component by component, this condition that is equivalent to (3.6) means

$$\frac{\partial u_i}{\partial x_{ij}}(\bar{x}_i) \begin{cases} = \lambda_i \text{ for } j = 0, \\ = \lambda_i \bar{p}_j \text{ for } j \neq 0 \text{ with } \bar{x}_{ij} > 0, \\ \leq \lambda_i \bar{p}_j \text{ for } j \neq 0 \text{ with } \bar{x}_{ij} = 0. \end{cases} \tag{3.10}$$

Another way of stating it, in terms of the definition of price thresholds, and understanding that the first line of (3.12) identifies λ_i as the money partial deriative $\frac{\partial u_i}{\partial x_{i0}}(\bar{x}_i)$, is that

$$p_{ij}(\bar{x}_i) \begin{cases} = \bar{p}_j \text{ for } j \neq 0 \text{ with } \bar{x}_{ij} > 0, \\ \leq \bar{p}_j \text{ for } j \neq 0 \text{ with } \bar{x}_{ij} = 0. \end{cases} \tag{3.11}$$

But this version of (3.6) obviously is the combination of (3.7) and (3.5), and we are done. □

The new and valuable conclusion here is that *bilateral trades involving bundles of several goods at a time aren't required in support of equilibrium*, not to speak of multilateral trades in which several agents are simultaneously engaged. Feldman already in 1973 in [4] showed that pairwise Pareto optimality, in which no agent pair can jointly improve utility by a mutual exchange of goods, implies full Pareto optimality if some good is always attractive to every agent — like money. But this was with multigood trades.

Trading as we depict it conserves goods; after each transaction the total of each good $j = 0, 1, \ldots, n$ held by the agents agrees with the amount that was present initially. Thus, the equilibria potentially reachable from total inital supplies s_j correspond to the holdings vectors $\bar{x}_i$ and prices $\bar{p}_j$ satisfying (3.11) along with

$$\sum_{i=1}^{m} \bar{x}_{ij} = s_j \text{ for } j = 0, 1, \ldots, n. \tag{3.12}$$

These conditions describe the "equilibrium manifold" associated with the given utility functions u_i and supply vector $s = (s_0, s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ with respect to the $m(n+1)+n$ unknowns $\bar{x}_{ij}$ and $\bar{p}_j$. When all goods are essential, the inequality case drops out of (3.11), leaving mn equations in the unknowns to be combined with the $n+1$ equations in (3.12), so that the number of equations exceeds the number of unknowns by $m-1$. If degeneracy doesn't intervene in the equations, they ought to produce a kind of $(m-1)$-dimensional "surface" as the manifold in question. *This nonuniqueness is what we expect and embrace.* We aren't in the Walras framework and are instead exploring an approach to equilibrium based squarely on buying and selling as ordinarily envisioned, instead of relying on an abstract societal entity to coordinate the agents' wishes and engineer a simultaneous grand exchange of goods among them.

4 Evolving toward equilibrium

The central question we now wish to answer, at least to some level of satisfaction, is whether, under additional specifics and assumptions if necessary, the kind of direct trading that has been described will lead in the limit to equilibrium holdings as characterized in Theorem 1.

Getting back first to pinning down how a seller and buyer may come together and come up with a price π_j for a good $j \neq 0$, we can forsee several possibilities. After a seller i_1 has proposed a price $p^+_{i_1 j}$, a buyer i_2 may respond by simply accepting that price as π_j, as long as $p^+_{i_1 j} \leq p^-_{i_2 j}$. In that case the buyer acts without revealing $p^-_{i_2 j}$. Likewise, a seller i_1 could respond to a price $p^-_{i_2,j}$ proposed by a buyer i_2 and accept that as π_j, as long as $p^+_{i_1 j} \leq p^-_{i_2,}$, and thereby act without revealing $p^-_{i_1 j}$. More complicatedly, if both i_1 and i_2 have revealed prices acceptable to them, and those prices are different, $p^+_{i_1 j} < p^-_{i_2 j}$, they could take π_j to be the average, say. Many details could obviously be added also to explain how agents find each other. There might be "information boards" or even trading posts with double auctions in various goods — whatever could work in the end to effect elementary exchanges.

The important thing, regardless, is that each time a bilateral trade occurs in some good the utility values of the holdings of both agents in the trade are improved (while those of the other agents are unaffected). We envision a scheme in which prices based on premiums are proposed by buyers and sellers to an extent that, in successive iterations, all agents and goods are repeatedly brought into the picture. Because utility values of the holdings of the agents go up each time for seller and buyer, and none ever go down, it's impossible that the configuration of holdings can "recycle." After each iteration it will be different, and in a Pareto sense always better.

Our hope is that eventually in this manner a stage will be reached in which no bilateral trade in any good is available — at the current premium levels δ_{ij}. At that point the premiums can be lowered and the process restarted.[11] But how do we know that the iterations won't stagnate through utility improvements falling short of what they ought to be?

To head off such stagnation, we'll introduce a restriction on the scope of our goods model in (2.3), namely that

$$\text{henceforth } \textit{all} \text{ goods are essential, so that } X_i = \mathbb{R}^n_{++} \text{ for all agents } i. \tag{4.1}$$

Then, as explained at the end of Section 3, we anticipate facing an equilibrium manifold of dimension $m-1$ as the target to be reached somewhere by the trading scheme. Of course, the particular equilibrium that might take shape from the bilateral activity will likely be affected by the random order in which agents come together and choose goods to look at.

The key consequence of assumption (4.1), which is common to much of the literature on equilibrium, will be to provide quantitative bounds on the sizes of the improvements in utility that result from trading. Something weaker than (4.1) might well be possible through further research. An easy step toward relaxation will be explained at the end of this section, in particular.

Theorem 2 (lower bounds on improvements from trading)**.** *Under the utility assumptions strengthened by (4.1) there exists $\mu > 0$, independent of anything other than the utility functions and the agents' initial holdings, for which the following property holds. When a trade in good j takes place between a seller i_1 with premium level $\delta_{i_1 j}$ and a buyer i_2 with premium level $\delta_{i_2 j}$, their utility levels will impove to at least the degree that, when the current holdings x_{i_1} and x_{i_2} are replaced by the after-trade holdings x'_{i_1} and x'_{i_2}, one will have*

$$u_{i_1}(x'_{i_1}) \geq u_{i_1}(x_{i_1}) + \mu\delta^2_{i_1 j}, \qquad u_{i_2}(x'_{i_2}) \geq u_{i_2}(x_{i_2}) + \mu\delta^2_{i_2 j}. \tag{4.2}$$

Proof. Let $s_j > 0$ denote the total supply of good j, which starts as $\sum_{i=1}^m x^0_{ij}$ and is forever maintained under trading. Let $s = (s_0, s_1, \ldots, s_n)$. The holdings x_i of agent i will always then lie in the compact convex set

$$C_i = \left\{ x_i \,\middle|\, u_i(x_i) \geq u_i(x^0_i),\ x_i \leq s \right\} \subset \mathbb{R}^{n+1}_{++}. \tag{4.3}$$

Because u_i is a twice-continuously differentiable function on $\mathbb{R}^{n+1}_{++}$, its first and second partial derivatives are bounded on C_i. In particular, by virtue of lower and upper bounds on the first partial derivatives, these necessarily being positive, there will exist positive lower and upper bounds on the price thresholds $p_{ij}(x_i)$ in (2.4) as x_i ranges over C_i. Since any price π_j that might

[11] Of course this could be triggered good-by-good instead.

enter a transaction for good j has to lie between a seller's threshold and a buyer's threshold, such prices are thus limited to some interval $[\pi_j^-, \pi_j^+] \subset (0,\infty)$.

Consider now the function $\theta_+(\xi_j)$ in (2.10) with notation $[\pi_j, -1]$ as explained in (2.9). This is the utility expression that agent i as a seller would be involved with optimizing over a line segment necessarily lying within C_i. We have already calculated its derivative at 0 in (2.12) as $\frac{\partial u_i}{\partial x_{i0}}(x_i)[\pi_j - p_{ij}(x_i)]$, but here $\pi_j - p_{ij}(x_i) \geq \delta_{ij}$, so can say that

$$\theta_+'(0) \geq \alpha_i \delta_{ij} \text{ where } \alpha_i > 0 \text{ is a lower bound for } \frac{\partial u_i}{\partial x_{i0}}(x_i) \text{ on } C_i. \tag{4.4}$$

The second derivative of $\theta_+(\xi_j)$ can be calculated in turn from (2.10) as

$$\theta_+''(\xi_j) = [\pi_j, -1]{\cdot}\nabla^2 u_i(x_i')[\pi_j, -1] \text{ for } x_i' = x_i + \xi_j[\pi_j, -1]. \tag{4.5}$$

Because π_j must be in the interval $[\pi_j^-, \pi_j^+]$, while x_i and x_i' must lie in C_i, compactness and continuity ensure that the values in (4.5) over all such possible arguments are bounded from below. Thus, there exists $\beta_{ij} > 0$ such that

$$\theta_+''(\xi_j) \geq -\beta_{ij} \text{ for all feasible } \xi_j, \tag{4.6}$$

where the feasibility refers to ξ_j being admissible in any transaction that agent i might enter in selling good j. The combination of (4.4) and (4.6) implies that

$$\theta_+'(\xi_j) \geq \alpha_i\delta_{ij} - \beta_{ij}\xi_j, \qquad \theta_+(\xi_j) \geq \theta_+(0) + \alpha_i\delta_{ij}\xi_j - \tfrac{1}{2}\beta_{ij}\xi_j^2, \tag{4.7}$$

for all feasible ξ_j. Therefore

$$\max_{\substack{\xi_j \geq 0 \\ x_i+\xi_j[\pi_j,-1]\in X_i}} \theta_+(\xi_j) \geq \max_{\substack{\xi_j \geq 0 \\ x_i+\xi_j[\pi_j,-1]\in X_i}} \left\{ \theta_+(0) + \alpha_i\delta_{ij}\xi_j - \tfrac{1}{2}\beta_{ij}\xi_j^2 \right\}, \tag{4.8}$$

as well as

$$\operatorname*{argmax}_{\substack{\xi_j \geq 0 \\ x_i+\xi_j[\pi_j,-1]\in X_i}} \theta_+(\xi_j) \geq \operatorname*{argmax}_{\substack{\xi_j \geq 0 \\ x_i+\xi_j[\pi_j,-1]\in X_i}} \left\{ \theta_+(0) + \alpha_i\delta_{ij}\xi_j - \tfrac{1}{2}\beta_{ij}\xi_j^2 \right\}, \tag{4.9}$$

where, due to (4.3), each maximum is obtained with a value of ξ_j such that $x_i + \xi_j[\pi_j, -1]$ belongs to the interior of the goods orthant, hence simply by setting the derivative equal to 0. On the right of (4.9), that leads to specifically to

$$\xi_j = \frac{\alpha_i}{\beta_{ij}}\delta_{ij}, \text{ yielding max value } \theta_+(0) + \frac{\alpha_i}{2\beta_{ij}}\delta_{ij}^2.$$

The conclusion reached is that the new holdings vector x_i' obtained by agent i as seller will have

$$u_i(x_i') \geq u_i(x_i) + \mu_{ij}\delta_{ij}^2 \text{ for } \mu_{ij} = \frac{\alpha_i}{2\beta_{ij}} > 0.$$

Treating agent i as buyer instead of seller leads to the same end through analysis of the function $\theta_-(\xi_j)$ in (2.12), because all that really changes is that $[\pi_j, -1]$ is replaced by $-[\pi_j, -1]$, and that has no effect on the hessian expression for second derivatives. Taking μ to be the smallest of the factors μ_{ij} over all i and j justifies the double improvement claim in (4.2). □

A definitive answer to the question of convergence to an equilibrium is now at hand for our scheme in which agents interact on the basis of particular price premiums $\delta_{ij} > 0$ as long as trades are possible, but reduce those premiums once trades are blocked, and keep doing that as the premium levels tend to 0.

Theorem 3 (convergence to an equilibrium)**.** *Under the additional assumption in (4.1), the outlined trading scheme for agents at a given level of price premiums $\delta_{ij} > 0$ can proceed for only finitely many trades before no further trades are available. If the trading is restarted then at lower levels of premiums, and the process repeats iteratively with still lower levels, tending to 0, the holdings x_i and price thresholds $p_{ij}(x_i)$ converge to some particular equilibrium of prices and holdings as characterized in Theorem 1, moreover with*

$$\text{all goods } j \neq 0 \text{ having } p_{ij}(\bar{x}_i) = \bar{p}_j \text{ for every agent } i.$$

Proof. The trading scheme, however articulated among the traders, generates for each agent i a sequence of holdings vectors x_i within the compact set C_i in (4.3). For a cluster point $\bar{x}_i$ of that sequence, $p_{ij}(\bar{x}_i)$ will be a cluster point of the corresponding price thresholds $p_{ij}(x_j)$ for each good j. We claim two things. First, $p_{ij}(\bar{x}_j)$ will have the same value for every agent i, so that this value, as $\bar{p}_j$ will provide prices needed along with the holdings $\bar{x}_i$ to constitute an equilibrium as described in Theorem 2. And second, that the cluster points are unique: even though the existence of more than one equilibrium isn't excluded, the sequences that are generated converge to particular limits.

The process under fixed premiums results in improvements as in Theorem 2 after each transaction. There is a lower bound that way to the size of the utility improvements. Because the utility functions u_i are bounded from above over the set C_i, in which holdings always remain, stagnation is impossible.

Once the stage is reached when no further transactions are possible at the current premium levels δ_{ij}, we must have for every potential seller i_1 and buyer i_2 that $p^+_{i_1 j} > p^-_{i_2,j}$ for the acceptable prices in (3.1). This means that $p_{i_1 j}(x_{i_1}) + \delta_{i_1 j} > p_{i_2 j}(x_{i_2}) - \delta_{i_2 j}$, but since all agents have positive holdings in all goods in consequence of (4.7), the roles of seller and buyer can be interchanged. Thus, we must have

$$|p_{i_1 j}(x_{i_1}) - p_{i_2 j}(x_{i_2})| < \delta_{i_1 j} + \delta_{i_2 j} \text{ for all pairs of agents } i_1 \text{ and } i_2. \tag{4.10}$$

Clearly then, as the trading restarts and proceeds over and over with lower premium levels tending to 0, it must be that, for the subsequence of holdings x_i that converges to $\bar{x}_i$ at the beginning of this proof, we end up for each agent i and good j with the same value for $p_{ij}(\bar{x}_i)$.

For the second claim, we recall that levels of utility only increase for the agents as the trading proceeds, never decrease. The vector of those levels thus tends upward to a vector

$(\bar{u}_1, \ldots, \bar{u}_m)$. Any array $[\bar{x}_1, \ldots, \bar{x}_m]$ of equilibrium holdings that is approached as a cluster point of the (bounded) sequence of holdings vectors generated by the trading must have

$$u_i(\bar{x}_i) = \bar{u}_i \text{ for } i = 1, \ldots, m. \tag{4.11}$$

Suppore that, along with $[\bar{x}_1, \ldots, \bar{x}_m]$, there is another array $[\bar{x}'_1, \ldots, \bar{x}'_m]$ that is an equilibrium cluster point, likewise then having

$$u_i(\bar{x}'_i) = \bar{u}_i \text{ for } i = 1, \ldots, m. \tag{4.12}$$

Because total supplies of goods are maintained in the process, we know that

$$\sum\nolimits_{i=1}^m \bar{x}_i = \sum\nolimits_{i=1}^m \bar{x}'_i = \sum\nolimits_{i=1}^m x_i^0. \tag{4.13}$$

Let $x_i^* = \frac{1}{2}[\bar{x}_i + \bar{x}'_i]$, so that again

$$\sum\nolimits_{i=1}^m x_i^* = \sum\nolimits_{i=1}^m x_i^0.$$

Because each u_i is strictly (even strongly) concave along line segment joining $\bar{x}_i$ and $\bar{x}'_i$ inasmuch as the difference is a mixed-sign vector when nonzero by (4.13), and some differences are indeed nonzero, we have

$$u_i(x_i^*) \geq \tfrac{1}{2}u_i(\bar{x}_i) + \tfrac{1}{2}u_i(\bar{x}'_i) = \bar{u}_i \text{ for all } i, \text{ with strict inequality for some } i.$$

In view of (4.11) and (4.12), this says that neither $[\bar{x}_1, \ldots, \bar{x}_m]$ nor $[\bar{x}'_1, \ldots, \bar{x}'_m]$ is Pareto optimal. But Pareto optimality holds for any exchange equilibrium as a special case of it holding for any Walras equlibrium. □

The achievement of an equilibrium as a limit and not just as a cluster point is a unique contribution here. In other work, such as that of Flaam in [5], only a cluster point level of convergence is obtained, if anything is obtained at all.

Remark about premiums. The scheme in Theorem 3 of having the agents progressively lower their price premiums when trading opportunities are lacking is important for the purpose of proving eventual convergence. From a practical perspective in economics, however, it might be more natural to imagine these money-denominated premiums as characteristics of the agents along with their utility functions. With their levels fixed, buying and selling would come to a halt short of a full equilibrium. But this would reflect reality. Money prices anyway are always truncated in practice. The smallest money unit ever contemplated in market transactions could serve for instance uniformly as every premium δ_{ij}.

Concluding comments about the utility assumptions. The convergence in Theorem 3 rests on the lower bounds for improvements that were derived in Theorem 2, and this is where our second-order assumptions on the utility function u_i and its concavity are most important. How plausible are those assumptions from the angle of economics? In assessing this, we have background support from [18, Theorem 4.2] in regarding the representability of a preference relation by a $\mathcal{C}^2$ quasi-concave utility as a property of the basic workings of that the relation, not a wishful construct. It can well be a natural feature of the agents' preferences in our model. Local strong convexity of the preference sets can be interpreted like that as well.

It helps now to keep in mind that the agents in our setting will not make use of the entire goods orthant but only a limited portion indicated by the compact sets C_i in (4.3). According to [18, Theorem 3.5], a strongly convex preference relation for agent i that's open to representation by a $\mathcal{C}^2$ utility function u_i has one which is concave relative to the set C_i, and minimally so, with it then being unique up to just the choice of units in which utility is to be measured.

That puts everything on solid ground. But it might still be wondered whether an agent could know such a function and be able to deploy it — pure existence might not be enough. In our scheme, though, such knowledge actually isn't required! We pointed out at the end of Section 2 that the optimization steps in selling and buying can, in principle, be carried out directly in terms of the preference relation. The utility representation essentially serves only for convenience. And as for the strong concavity yielding the bounds in Theorem 2, it merely serves in the proof of convergence through its existence. An agent doesn't have to do anything to take advantage of it or even be aware of it.

An easy extension. The extra assumption in (4.1) that we made in order to get the improvement guarantee in Theorem 2 as a key ingredient of the convergence proof in Theorem 3 makes every agent to insist on holding always a positive amount of every good. That's an unpleasant requirement, especially in view of the preceding analysis of buying and selling, which didn't need it. It's good to note, therefore, that (4.1) can be replaced by a somewhat weaker assumption which permits agents to hold zero quantities of goods in which they have no interest in at all. For that, the sets X_i in (2.3) can be modified from the beginning to

$$X_i = \left\{ x_i \in I\!R^{n+1}_+ \,\middle|\, x_{ij} > 0 \text{ for } j \in J_i,\ x_{ij} = 0 \text{ for } j \in J_i^0 \right\} \\ \text{for disjoint collections } J_i \text{ and } J_i^0 \text{ of goods } j. \tag{4.13}$$

The utility assumptions can be altered so as not to apply to the goods $j \in J_i^0$, which have no effect on them. Everything continues to work in this mode with only minor and obvious adjustments to the story.

5 Computational experiments

The effectiveness of the bilateral trading dynamic in Sections 3 and 4 is open to being explored by computer simulation. Examples of that will now be presented. After giving a formal statement of the trading procedure as Algorithm 1, we apply it first to situations in which agents have utility functions in the Cobb-Douglas family and the holdings of all goods are necessarily positive, in accordance with assumption (4.1). The conditions behind Theorem 3 are met, and numerical convergence to a market equilibrium is consistently obtained, as expected.

We then experiment with whether the trading procedure can reach an equilibrium even in a example where the interiority assumption (4.1) is not in force, so that agents can hold zero quantities of some goods. In that example the utility functions are not Cobb-Douglas. Convergence

Algorithm 1 Bilateral Trading Scheme

1: **procedure** BTS(Agent parameters $\{\beta_{ij}\}$, $\{x^0_{ij}\}, \{\delta^0_{ij}\}$, Algorithm parameters ε_δ, ε_p, λ)
2: Define $p^0_{ij} = p_{ij}(x^0_i), \forall i, j$.
3: **for** $k = 1, \ldots$ **do**
4: **if** $\max_j \text{stdev}\,(p^k_{ij}) < \varepsilon_p$ **then return Equilibrium found**.
5: **else if** $\max_{ij}\{\delta^k_{ij}\} < \varepsilon_\delta$ **then return** Max of deltas is less than the given tolerance level of ε_δ
6: **else**
7: **for** $i_1 \in \Pi(1, \ldots, m)$ **do**
8: **for** $i_2 \in \Pi(1, \ldots, m)$ **do**
9: **for** $j \in \Pi(1, \ldots, n)$ **do**
10: **if** Trade between agents agents i_1 and i_2 of good j is possible **then**
11: Update holdings $x^{k+1}_{i_1,j}$, $x^{k+1}_{i_2,j}$
12: Update agents' prices $p^{k+1}_{i_1}$, $p^{k+1}_{i_2}$.
13: Go to Step 4
14: **end if**
15: **end for**
16: **end for**
17: **end for**
18: **if** No trade is possible **then**
19: $\delta^{k+1}_{ij} \leftarrow \lambda\delta^k_{ij}$, make $k \leftarrow k + 1$ and go to Step 7
20: **end if**
21: **end if**
22: **end for**
23: **end procedure**

to an equilibrium is sometimes obtained but can also bog down. The lesson is that a refined substitute for the utility growth guarantee in Theorem 2 is definitely needed for convergence outside of assumption (4.1) or its relaxation described at the end of the preceding section.

In the inspection process for determining the existence of bilateral trade opportunities, we rely here on a naturally *random trading order*. This means that we inspect potential buyer/seller partners at random, traveling through all possible combinations and for each one looking randomly through all the goods to possibly trade, but stopping at the first trading opportunity encountered. If the inspections comes to an end with no trading opportunity at all having emerged, the agents all reduce their premium levels δ_{ij}, and at those new levels the inspection resumes. The ultimate convergence of this scheme is guaranteed by Theorem 3. In Steps 7, 8 and 9 of Algorithm 1 the functions Π represent the generated permutations of agents or goods.

When the procedure terminates in Step 4 in iteration k with an equilibrium having been found — approximately — under the stopping criterion, the prices and holdings associated with it are taken to be

$$\bar{p}_j = \frac{1}{m} \sum\nolimits_{i=1}^{m} p_{ij}^k, \qquad \bar{x}_{ij} = x_{ij}^k. \tag{5.1}$$

We developed a Python 3.8.11 code for this algorithm, making it freely available in our repository[12] along with the parameters and documentation. We ran our examples on an Apple Mac Mini 2018 (Intel Core i5 CPU@3GHz, 32 GB memory), operating with macOS Big Sur.

Example 1. *This concerns an economy with $n = 9$ non-money goods and $m = 5$ agents, in which each agent i has a utility function of Cobb-Douglas type:*

$$u_i(x_i) = \prod_{j=0}^{n} x_{ij}^{\beta_{ij}}, \text{ with } 0 < \beta_{ij} < 1, \quad \sum_{j=0}^{n} \beta_{ij} < 1. \tag{5.2}$$

The values assigned to the exponents β_{ij} are shown in Table 1 along with the quantities taken as the agents' initial holdings x_{ij}^0, all of which are positive.

Agent	Parameter	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=1$	β_1	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
	x_1^0	59	76	10	37	54	99	73	30	25	20
$i=2$	β_2	0.05	0.1	0.17	0.02	0.16	0.1	0.16	0.07	0.03	0.04
	x_2^0	14	40	63	57	69	39	34	86	10	56
$i=3$	β_3	0.06	0.05	0.09	0.15	0.07	0.08	0.14	0.02	0.11	0.13
	x_3^0	19	57	43	65	78	40	9	82	71	82
$i=4$	β_4	0.01	0.15	0.01	0.11	0.11	0.16	0.03	0.14	0.09	0.09
	x_4^0	10	65	35	43	63	74	79	38	20	27
$i=5$	β_5	0.03	0.13	0.05	0.16	0.16	0.07	0.08	0.1	0.08	0.04
	x_5^0	37	70	40	94	83	15	34	97	35	34

Table 1: Utility parameters β_{ij} and initial holdings x_{ij}^0 in Example 1

[12] Github repository https://github.com/jderide/DeJoRo-Bilateral

We ran our algorithm 50 times with a random inspection strategy and observed that the trading scheme reached a market equilibrium every time in the sense of the stopping criterion. The precision level for that criterion in Step 4 of the algorithm was taken to to have $\varepsilon_p = 10^{-6}$,, so the equilibrium price vector $\bar{p}$ reported as indicated in (5.1) was such that its distance from every one of the agent price vectors p_i^k was less than 10^{-6}. The median execution time for these runs was 2.12 seconds in 3093 iterations.

Because of the random inspection strategy, each run could, and did, end with a different equilibrium, and significant differences in the resulting prices were observed. Rather that reporting the results of all 50 runs, we selected the four with the highest absolute price differences, namely runs 5, 4, 9, 1, for display of their equilibrium prices in Table 2.

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$\overline{p}^5$	1.0000	0.9278	1.2147	1.0510	0.9491	1.0213	1.2575	0.6828	1.4315	1.0215
$\overline{p}^4$	1.0000	0.9105	1.2176	1.0077	0.9356	0.9999	1.2394	0.6736	1.3860	0.9888
$\overline{p}^9$	1.0000	0.9468	1.2214	1.0493	0.9678	1.0303	1.2552	0.7005	1.4263	1.0077
$\overline{p}^1$	1.0000	0.9192	1.2000	1.0141	0.9387	1.0009	1.2229	0.6841	1.3907	0.9836
$\overline{p}^W$	1.0000	0.9575	1.2218	1.0569	0.9680	1.0594	1.2609	0.7102	1.4501	1.0371

Table 2: Selected outcomes for equilibrium prices in Example 1

The corresponding money-denominated prices p_j^W for a Walras equilibrium with these utility functions and initial holdings are also shown in Table 2 for comparison and possible side interest in underscoring a difference in aims and concept. The Walrasian model *lacks true market legitimacy*, as explained in Section 1 and [17]. There is no reason why a market equilibrium achieved through our bilateral trading scheme would reproduce its configuration of terminal holdings and prices, which are not known to be supported by market activities in any ordinary sense, but depend rather on the intervention of a clearing house agent.

To get the prices p_j^W in Table 2, we proceeded as follows. In the Cobb-Douglas setting of Example 1 with its coefficients β_{ij} and initial holding x_{ij}^0, the final holdings $\tilde{x}_{ij}$ and relative prices $\tilde{p}_j$ in the Walras model solve a system of linear equations coming via algebraic manipulations from agent utility maximization and supply-demand balance. In the notation $\tilde{\beta}_{ij} = \beta_{ij} / \sum_{k=0}^n \beta_{ik}$, this system combines the traditional price normalization equation $\sum_{j=1}^n \tilde{p}_j = 1$ with

$$\begin{aligned} \tilde{x}_{ij} - \tilde{\beta}_{ij} \left(\sum_{j=1}^n \tilde{p}_j x_{ij}^0 \right) &= \tilde{\beta}_{ij} x_{i0}^0, \; i = 1, \ldots, m, \; j = 0, \ldots, n \\ \tilde{p}_j \sum_{i=1}^m x_{ij}^0 - \sum_{k=1}^n \tilde{p}_k \left(\sum_{i=1}^m \tilde{\beta}_{ij} x_{il}^0 \right) &= \sum_{i=1}^m \tilde{\beta}_{ij} x_{i0}^0, \; j = 0, \ldots, n. \end{aligned}$$

The money-denominated prices in Table 2 are derived from this by

$$\bar{p}_j^W = \frac{\tilde{p}_j}{\tilde{p}_0} \text{ for } j - 1, \ldots, n. \tag{5.3}$$

Bigger experiment in the same setting. In order to test scalability of the trading algorithm, we also applied it to a larger version of Example 1 with $n + 1 = 100$ goods and $m = 10$ agents. Despite the much greater scope of activity, an equilibrium was again achieved on all runs, which took a median of 59345 iterations within 711 seconds of execution time.

The next compututional example is aimed differently. Instead of scaling upward, we downsize while confronting the algorithm with a challenge. from another angle.

Example 2. *This concerns an economy like the one in Example 1 but with only $n+1 = 3$ goods and $m = 3$ agents, and, most importantly, with the initial holdings of the agents hugely out of balance. The Cobb-Douglas exponents and initial holdings are indicated in Table 3. Note that each of the three goods is held at first mostly by just one of the three agents.*

Agent	Parameter	$j = 0$	$j = 1$	$j = 2$
$i = 1$	β_1	0.60	0.15	0.15
	x_1^0	10	10	10
$i = 2$	β_2	0.01	0.85	0.04
	x_2^0	2	8	80
$i = 3$	β_3	0.01	0.09	0.80
	x_3^0	2	80	8

Table 3: Example 2 utility parameters β_{ij}, and initial holdings x_{ij}^0

For this example we report in Table 4 the equilibrium price outcomes of two different runs $\nu = 1, 2$. The associated final holdings of the agents are shown in Table 5. Despite the initial differences, the results are quite consistent with each other. The corresponding prices and holdings coming from the Walras model for the nonmonetary goods are very different, however. This appears all the more to cast doubt on that model's ability to reflect genuine market forces.

	$\bar{p}^1$	$\bar{p}^2$	$\bar{p}^W$
$j = 1$	0.0475	0.0460	0.4921
$j = 2$	0.0494	0.0484	0.4614

Table 4: Equilibrium prices $\bar{p}^\nu$ for Example 2

	$\bar{x}_{10}^\nu$	$\bar{x}_{20}^\nu$	$\bar{x}_{30}^\nu$	$\bar{x}_{11}^\nu$	$\bar{x}_{21}^\nu$	$\bar{x}_{31}^\nu$	$\bar{x}_{12}^\nu$	$\bar{x}_{22}^\nu$	$\bar{x}_{32}^\nu$
$\nu = 1$	13.97	0.01	0.02	73.58	21.33	3.09	70.66	0.96	26.37
$\nu = 2$	13.97	0.01	0.02	75.95	19.08	2.96	72.13	0.85	25.01
Walrus	13.02	0.48	0.50	6.62	82.23	9.16	7.06	4.13	86.82

Table 5: Final holdings $\bar{x}^\nu$ for Example 2

Our final example has a different character. It employs utility functions not of Cobb-Douglas type that allow agents to reduce their holdings of non-money goods to 0, if they so desire. As

Trial	Agent	Price threshold good 1		Price threshold good 2	
		$p_{i1}^{0,\nu}$	$\bar{p}_{i1}^{\nu}$	$p_{i2}^{0,\nu}$	$\bar{p}_{i2}^{\nu}$
$\nu = 0$	$i = 1$	31.4643	18.9509	21.3957	18.1378
$\nu = 0$	$i = 2$	2.5298	18.9507	3.2888	18.1379
$\nu = 1$	$i = 1$	31.4643	19.6394	25.1714	17.5170
$\nu = 1$	$i = 2$	2.5298	19.6393	2.5298	17.5172
$\nu = 2$	$i = 1$	31.4643	17.4484	28.9471	15.7335
$\nu = 2$	$i = 2$	2.5298	21.0987	1.7709	15.7335

Table 7: Initial and final prices for Example 3

an aid in setting up these utility functions we make use of the fact that at any stage of trading the holdings $x_{ij} \geq 0$ of the agents satisfy

$$\sum_{i=1}^{m} x_{ij} = s_j, \text{ where } s_j = \sum_{i=1}^{m} x_{ij}^0 \text{ (total supply of good } j\text{), hence } x_{ij} \in [0, s_j]. \tag{5.4}$$

Example 3. *This concerns an economy with just $m = 2$ agents and $n+1 = 3$ goods. Each agent i has a separable utility function $u_i(x_i) = \sum_{j=0}^{2} u_{ij}(x_{ij})$ in which the term for the money good has the form $u_{i0}(x_{i0}) = x_{i0}^{\alpha_i}$ for some $\alpha_i \in (0,1)$. The terms for the non-money goods $j = 1, 2$, have instead the form*

$$u_{ij}(x_{ij}) = a_{ij}x_{ij} - \frac{1}{2}b_{ij}x_{ij}^2 \text{ for } x_{ij} \in [0, s_j], \text{ with } a_{ij} > 0,\ b_{ij} > 0,\ \frac{a_{ij}}{b_{ij}} > s_j, \tag{5.5}$$

where s_j is the total supply of good j as in (5.4).[13] *The values of the utility parameters are indicated in Table 6. Note that for good $j = 2$ a shift is introduced between the two agents of an amount μ^ν that depends on the run index ν,*

	$p_{i1}^{0,\nu}$	$\bar{p}_{i1}^{\nu}$	$p_{i2}^{0,\nu}$	$\bar{p}_{i2}^{\nu}$
$\nu = 0, i = 1$	31.4643	18.9509	21.3957	18.1378
$\nu = 0, i = 2$	2.5298	18.9507	3.2888	18.1379
$\nu = 1, i = 1$	31.4643	17.4484	28.9471	15.7335
$\nu = 1, i = 2$	2.5298	21.0987	1.7709	15.7335

Table 6: Utility parameters and initial holdings in Example 3, where $\mu^\nu = 3\nu$

Note that this family of utility functions has marginal utility going to ∞ as money goes to 0, but not as other goods go to 0. The sets of admissible holdings X_i in (2.3) come out therefore as

$$X_i = \left\{ x_i \,\middle|\, x_{i0} > 0,\ x_{i1} \geq 0,\ x_{i2} \geq 0 \right\}. \tag{5.6}$$

[13] The conditions in (5.5) ensure that u_{ij} is a strongly concave increasing quadratic function on the interval $[0, s_j]$ and has positive slope still at s_j itself. It is mathematically possible then to build an extension beyond s_j that is strongly concave, increasing and continuously twice differentiable, so as to end up with a utility function u_i that fits our general assumptions. But because of (5.4), that extension can be left implicit.

This puts us outside the domain of the guaranteed convergence of the trading algorithm in Theorem 3, which depended on assumption (4.1), but we wanted to see anyway what might happen computationally in such circumstances.

Following the rules in the previous sections, the price thresholds $p_{ij}(x_i)$ in (2.8) can be explicitly computed as

$$p_{ij}(x_i) = \frac{a_{ij} - b_{ij}x_{ij}}{\alpha_i x_{i0}^{\alpha_i - 1}}, \quad j = 1, 2 \tag{5.7}$$

we can also compute the solution to the seller's problem accirdubg ti (2.14),

$$\begin{aligned} \xi_j^+(x_i, \pi_j) &= \mathrm{argmax}_{\xi_j} \left\{ \, u_i(x_i + \xi_j[\pi_j, -1]) \,\middle|\, x_i + \xi_j[\pi_j, -1] \in X_i \, \right\} \\ &= \underset{0 \le \xi_j \le x_{ij}}{\mathrm{argmax}} \left\{ \, (x_{i0} + \xi_j \pi_j)^{\alpha_i} + a_{ij}(x_{ij} - \xi_j) - \frac{1}{2} b_{ij}(x_{ij} - \xi_j)^2 \, \right\} \end{aligned} \tag{5.8}$$

and, correspondingly the buyer's problem from (2.20)

$$\begin{aligned} \xi_j^-(x_i, \pi_j) &= \mathrm{argmax}_{\xi_j} \left\{ \, u_i(x_i - \xi_j[\pi_j, -1]) \,\middle|\, x_i - \xi_j[\pi_j, -1] \in X_i \, \right\} \\ &= \underset{0 \le \xi_j < \frac{x_{i0}}{\pi_j}}{\mathrm{argmax}} \left\{ \, (x_{i0} - \xi_j \pi_j)^{\alpha_i} + a_{ij}(x_{ij} + \xi_j) - \frac{1}{2} b_{ij}(x_{ij} + \xi_j)^2 \, \right\}. \end{aligned} \tag{5.9}$$

We ran three instances of this example in which agent 1 starts with none of good $j = 1$, agent 2 starts with almost no money, and the initial amount of good $j = 2$ shifts from agent 1 to agent 2 in dependence on the run index, $\nu = 0$, 1, 2. The holdings and prices on termination are reported in Table 7.

Two different behaviors are seen. On the one hand, in runs $\nu = 0$, 1, our algorithm converged to an equilbrium within an average of 174 iterations and 0.512 seconds, despite initial holdings not all being positive and utility functions not satisfying assumption (4.1). However, for trial $\nu = 2$, we observed a lack of convergence of the price thresholds for good $j = 1$, as can be seen in Figure 1; this is the good which agent 1 had none of at the start. *The failure to reach an equilibrium in this instance indicates that, for economies with admissible holdings not necessarily positive, as allowed by their utility functions not fitting with assumption (4.1), something more needs to be added to the trading scheme to avoid stagnation.*

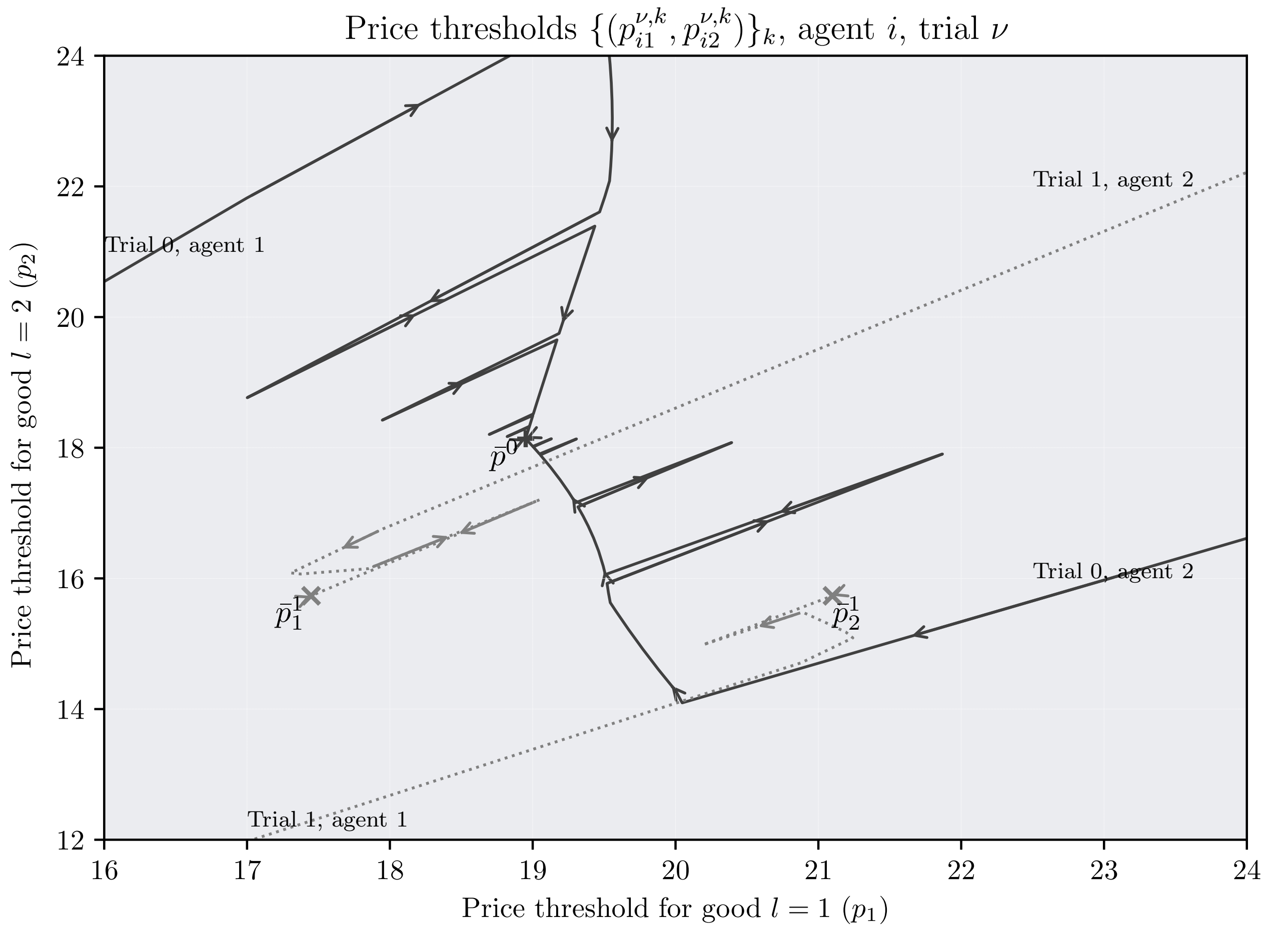

Figure 1: Trial 0. Successful example, despite violation of assumption (4.1). Equilibrium reached as the price threshold curve for agent 1, $\{p_1^{0,k}\}$, met the curve for agent 2, $\{p_2^{0,k}\}$. Trial 1. Failed example in the absense of (4.1). Both curves remained far from each other even though price thresholds for good 2 came to coincide, ($\bar{p}_{1,2}^0 = \bar{p}_{2,2}^1$).

The bilateral trade dynamics between the agents is depicted below in Figure 2, where we plot the Edgeworth box for the nonmonetary pair of goods in the economy.

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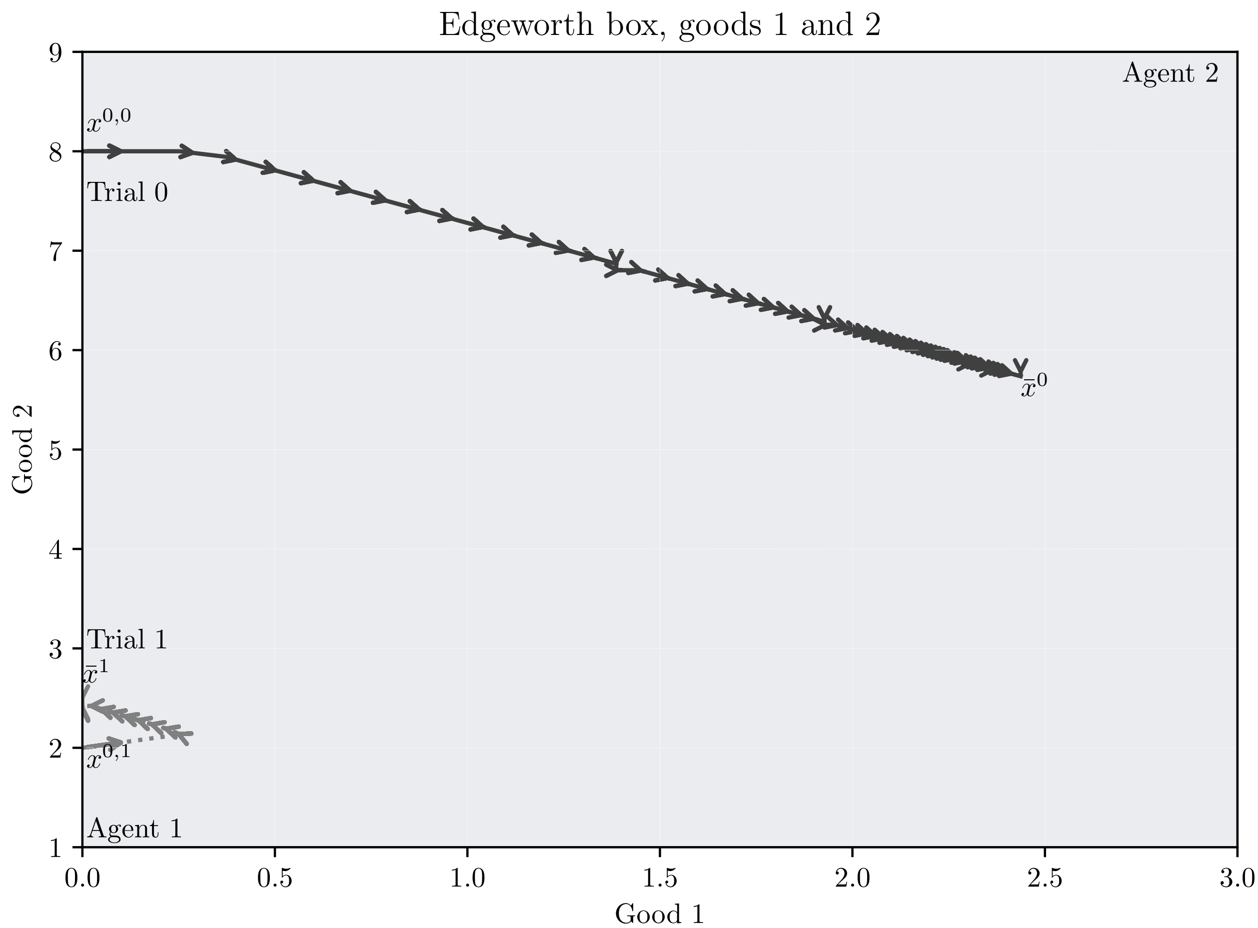

Figure 2: Trial 0. Starting from an initial endowment in the upper left region of the box, the agents bilaterally trade until they reach equilibrium allocations ($\bar{x}^0$). Trial 1. Agents enter the economy with allocations in the lower left corner of the box, but bilateral exchanges lead them to a boundary point, which does not correspond to an equilibrium.