Generalized Nash Equilibrium From a Robustness Perspective in Variational Analysis

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Abstract

Generalized Nash equilibrium is updated by a formulation in terms of extended-real-valued functions being minimized by the agents. Tools of variational analysis are employed to investigate how a local equilibrium, focused on local solutions in the minimization, might respond to perturbations of the parameters on which the agents' problems depend. Stability properties in this setting of competitive multiple-agent optimization are developed and contrasted with those already known for single-agent optimization in understanding solution robustness.

However, the purpose of the investigation goes beyond these results, in themselves, to the deeper issue of whether the concept of Nash equilibrium is adequate for the constructive modeling of agent interactions. To be meaningful, an equilibrium needs to be well-posed in reflecting natural tendencies and circumstances instead of something fragile and ephemeral.

Unfinished business is indicated with unanswered questions, and a mathematical landscape is thereby revealed where much more remains to be explored.

Keywords: *competitive optimization, game theory, generalized Nash equilibrium, variational analysis, solution robustness, tilt stability, full stability, variational convexity.*

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1 Introduction

In the original framework of Nash equilibrium with agents $k = 1, \ldots, N$, there are strategy sets C_k and functions f_k of $x = (x_1, \ldots, x_N) \in C_1 \times \cdots \times C_N$. An equilibrium is an $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_N)$ such that, for each k, $\bar{x}_k$ minimizes $f_k(x_k, \bar{x}_{-k})$ subject to $x_k \in C_k$.[2] A *generalized* Nash equilibrium allows C_i to depend on x_{-k}. But this distinction, while useful for technical reasons in the past, seems antiquated and artificial today, when constraints are often represented by letting ∞-values signal infeasibility.

In concentrating on situations where the agents' problems are in the category of finite-dimensional "continuous" optimization, we can jump to supposing that the interests of agent k are captured by an extended-real-valued function f_k on all of $\mathbb{R}^n := \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ and think of an equilibrium as an instance of $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_N)$ such that, for each k, $\bar{x}_k$ minimizes $f_k(x_k, \bar{x}_{-k})$ over $\mathbb{R}^n$. But why insist on a *global* minimum, which could pose an insurmountable obstacle for an agent? In natural concept, shouldn't an equilibrium be *local*, or perhaps merely reflect a sort of *stationarity* in a conflict of forces or interests?

Whatever the perspective, new challenges emerge in this extended framework for which the traditional methodology of game theory is likely inadequate, but which might be resolved with the help of the finite-dimensional variational analysis available now for instance in [12]. The purpose of this paper is to point out key issues and make some initial progress, while articulating facts and concepts that others could take up in making further progress.

An important question, of course, is how to know when an equilibrium of theory an extended kind is guaranteed to exist. Standard existence theorems in game theory largely rely on fixed-point theorems like that of Kakutani, in which some possibly set-valued mapping takes a compact convex set into itself. Is there a way of adapting such methodology to the proposed new setting? Another important question is how an equilibrium might be determined through sequential interactions of the agents or a numerical algorithm. This could also be a route to establishing existence constructively.

The question we will mainly work toward answering here, though, is how to assess the *robustness* of an *existing* equilibrium. This has its own importance, but it also relates strongly to the question of computations. If an equilibrium is so delicate that it falls apart irrevocably under a tiny shift in underlying data, such as could come from numerical errors in representation, what hope would there be in computing it, or that agents might naturally arrive at it? Knowledge about robustness could also influence approaches to existence, which might be tuned to equilibrium of a superior quality.

This brings out a deeper reason behind our investigation. Is the concept of Nash equilibrium even a "good" concept for the mathematical modeling of agent interactions? In our view, stability issues pose a key test for that. It's crucial therefore to get a clear picture of their strengths and weaknesses in this context. From that picture, it can be hoped that local existence or prevalence, at least, can be constructively established. If not, the concept might lose its appeal and need reworking with added features that counter volatility.

To address robustness, we formally introduce a general parameter vector $p \in \mathbb{R}^d$ and suppose

$$\text{each agent has an extended-real-valued function } \varphi_k \text{ of } (x, p) \in \mathbb{R}^n \times \mathbb{R}^d \tag{1.1}$$

that is proper (not $\equiv \infty$ and never having value $-\infty$) and lsc (lower semicontinuous). We define a *local* generalized Nash equilibrium for $\bar{p}$ to be an $\bar{x}$ such that

$$\varphi_k(x_k, \bar{x}_{-k}, \bar{p}) \text{ has a local minimum in } x_k \text{ at } \bar{x}_k, \ \forall k, \tag{1.2}$$

[2] Here x_{-k} is, as usual in Nash theory, the vector obtained when x_k is deleted from $x = (x_1, \ldots, x_N)$. It's customary to sometimes write $f_k(x_k, x_{-k})$ instead of $f_k(x_1, \ldots, x_N)$ for notational convenience, without really meaning that the vector components $x_1, \ldots, x_N$ of x have been permuted.

where of course the minimization is implicitly over the (not necessarily closed) set

$$C_k(\bar{x}_{-k},\bar{p}) = \{\, x_k \,|\, \varphi_k(x_k,\bar{x}_{-k},\bar{p}) < \infty \}.$$

In terms of the general subgradients of variational analysis [12, Definition 8.3],[3] we immediately have at our disposal, as necessary for (1.2), the first-order "stationarity" conditions

$$0 \in \partial_{x_k}\varphi_k(\bar{x}_k,\bar{x}_{-k},\bar{p}),\ \forall k, \tag{1.3}$$

which we regard as describing a *variational* generalized Nash equilibrium for $\bar{p}$. Of course, convexity of $\varphi_k(x_k,x_{-k},p)$ in x_k would turn the necessity in (1.3) into sufficiency. Then from both angles we would have in fact a *global* generalized Nash equilibrium. But Nash theory ought to be able to accommodate nonconvex optimization problems for the agents as well. For that, second-order sufficient conditions for local optimality in variational analysis could be added to the first-order conditions in (1.3) in order to guarantee (1.2). Anyway, the stability analysis of equilibrium could be centered on the variational *solution mapping* associated with (1.3), namely

$$S_0(p) := \{\, x \,|\, 0 \in \Phi(x,p)\},\ \ \text{where}\ \ \Phi(x,p) = (\partial_{x_1}\varphi_1(x,p),\ldots,\partial_{x_N}\varphi_N(x,p)). \tag{1.4}$$

In a graphically localized sense [2], might S_0 be single-valued, even Lipschitz continuous? Might it then have one-sided directional derivatives governed by formulas in terms of the functions φ_k?

Among the many motivations for our efforts is the robustness issue already brought up. Might an equilibrium described by (1.2), as reflected in (1.3), be seriously disrupted by a slightest shift of $\bar{p}$ to some nearby p? However, there are also issues of interest at a higher level. We could be dealing with a hierarchical set-up, where a "principal agent" is selecting p in an upper problem of optimization which depends on the outcome of that selection in the equilibrium response of the "underling" agents $k = 1,\ldots,N$.[4] We could even, in that vein, be looking at a problem of *control* where a time-dependent choice of $p(t)$ is envisioned as inducing an equilibrium trajectory $x(t)$.

In a previous paper [8], we studied the local version of classical Nash equilibrium that corresponds here in specialization to taking

$$\varphi_k(x_k,x_{-k},p) = f_k(x_k,x_{-k},p) + \delta_{C_k}(x_k)\ \text{ for } f_k \text{ in } \mathcal{C}^2 \text{ and } C_k \text{ closed convex.} \tag{1.5}$$

The variational description in (1.3) comes out then, in terms of normal cones to the sets C_k, as the collection of variational inequalities

$$-\nabla_{x_k} f_k(\bar{x}_k,\bar{x}_{-k},\bar{p}) \in N_{C_k}(\bar{x}_k),\ \forall k, \tag{1.6}$$

or equivalently as the single variational inequality

$$-(\nabla_{x_1} f_1(\bar{x},\bar{p}),\ldots,\nabla_{x_N} f_N(\bar{x},\bar{p})) \in N_{C_1\times\cdots\times C_N}(\bar{x}). \tag{1.7}$$

Results for parameterized variational inequalities in the book [2] were applied in this setting in [8].

An obvious next stage would be to replace the fixed C_k in (1.5) by a set $C_k(x_{-k},p)$ designated by constraints of nonlinear programming type,

$$\begin{gathered} C_k(x_{-k},p) = \{\, x_k \in X_k \,|\, F_k(x_k,x_{-k},p) \in K_k\}\ \text{ for a closed convex set } X_k \subset I\!R^{n_k}, \\ \text{a closed convex cone } K_k \subset I\!R^{m_k}, \text{ and a } \mathcal{C}^2 \text{ mapping } F_k : I\!R^n \times I\!R^d \to I\!R^{m_k}, \end{gathered} \tag{1.8}$$

[3]For φ proper and lsc, $v \in \partial\varphi(x)$ if there exist sequences $v^\nu \to v$ and $x^\nu \to x$ with $\varphi(x^\nu) \to \varphi(x)$ such that $\varphi(x) \geq \varphi(x^\nu) + v^\nu{\cdot}[x - x^\nu] + o(|x - x^\nu|)$.

[4]Recently, a local generalized Nash model for this was taken up in [1].

and investigate when

$$f_k(x_k, \bar{x}_{-k}, \bar{p}) \text{ has a local minimum in } x_k \in C_k(\bar{x}_{-k}, \bar{p}) \text{ at } \bar{x}_k,\ \forall k. \tag{1.9}$$

Then, with seeming simplicity, the variational description analogous to (1.6) has the form

$$-\nabla_{x_k} f_k(\bar{x}_k, \bar{x}_{-k}, \bar{p}) \in N_{C_k(\bar{x}_{-k},\bar{p})}(\bar{x}_k),\ \forall k, \tag{1.10}$$

while the one in (1.7) becomes

$$-(\nabla_{x_1} f_1(\bar{x}, \bar{p}), \ldots, \nabla_{x_N} f_N(\bar{x}, \bar{p})) \in N_{C_1(\bar{x}_{-1},\bar{p})\times\cdots\times C_N(\bar{x}_{-N},\bar{p})}(\bar{x}). \tag{1.11}$$

The catch, though, is that, unless $C_k(\bar{x}_{-k}, \bar{p})$ is a *convex* set, the normal cone in (1.10) is no longer the one of convex analysis but rather the more broadly defined one of variational analysis [12]. The condition on $\bar{x}_k$ in (1.10) is then a nontraditional "extended" variational inequality. And even with convexity of every $C_k(\bar{x}_{-k}, \bar{p})$, the joint condition in (1.11) lies outside usual mathematical territory.

Nonetheless, there is a well known work-around in the face of these difficulties that relies on an appeal to Lagrange multipliers. The Lagrangian function associated with minimizing $f_k(x_k, x_{-k}, p)$ subject to constraining x_k to the set $C_k(x_{-k}, p)$ in (1.8) is

$$\begin{aligned}&L_k(x_k, y_k, x_{-k}, p) = f_k(x_k, x_{-k}, p) + y_k{\cdot}F_k(x_k, x_{-k}, p) \text{ for}\\ &(x_k, y_k) \in X_k \times Y_k, \text{ where } Y_k \text{ is the convex cone polar to } K_k.\end{aligned} \tag{1.12}$$

Under the basic constraint qualification for (1.8) that[5]

$$\left.\begin{array}{c} -\nabla_{x_k} F_k(\bar{x}_k, \bar{x}_{-k}, \bar{p})^* y_k \in N_{X_k}(\bar{x}_k) \\ \text{with } y_k \in N_{K_k}(F_k(\bar{x}_k, \bar{x}_{-k}, \bar{p}) \end{array}\right\} \implies y_k = 0, \tag{1.13}$$

there is the formula (cf. [12, 6.14]) that

$$\begin{aligned}&-\nabla_{x_k} f_k(\bar{x}_k, \bar{x}_{-k}, \bar{p}) \in N_{C_k(\bar{x}_{-k},\bar{p})}(\bar{x}_k)\\ &\quad\iff \left\{\begin{array}{l} \exists \bar{y}_k \in N_{K_k}(F_k(\bar{x}_k, \bar{x}_{-k}, \bar{p}) \text{ such that} \\ -\nabla_{x_k} f_k(\bar{x}_k, \bar{x}_{-k}, \bar{p}) - \nabla_{x_k} F_k(\bar{x}_k, \bar{x}_{-k}, \bar{p})^* \bar{y}_k \in N_{X_k}(\bar{x}_k) \end{array}\right.\\ &\quad\iff (-\nabla_{x_k} L_k(\bar{x}_k, \bar{y}_k, \bar{x}_{-k}, \bar{p}), \nabla_{y_k} L_k(\bar{x}_k, \bar{y}_k, \bar{x}_{-k}, \bar{p})) \in N_{X_k\times Y_k}(\bar{x}_k, \bar{y}_k),\end{aligned} \tag{1.14}$$

where the final condition is a variational inequality of *standard* type, since $X_k \times Y_k$ is a convex set.

This brings us to the notion of a *Lagrangian* generalized Nash equilibrium as described by

$$(-\nabla_{x_k} L_k(\bar{x}_k, \bar{y}_k, \bar{x}_{-k}, \bar{p}), \nabla_{y_k} L_k(\bar{x}_k, \bar{y}_k, \bar{x}_{-k}, \bar{p})) \in N_{X_k\times Y_k}(\bar{x}_k, \bar{y}_k),\ \forall k, \tag{1.15}$$

which in the manner of passing from (1.10) to (1.11) can likewise be written as a single variational inequality of standard type over the product of the sets $X_k \times Y_k$. Adopting the Lagrangian model (1.15), as an acceptable substitute for the equilibrium in (1.9) or (1.10) under (1.8), requires a philosophical adjustment. The game must be viewed in terms of the "strategy" of agent k consisting not just a locally optimal solution $\bar{x}_k$, but $(\bar{x}_k, \bar{y}_k)$, where $\bar{y}_k$ is a Lagrange multiplier vector associated with that solution. Of course, in circumstances where $\bar{y}_k$ is uniquely determined by $\bar{x}_k$, the strategy distinction falls away.

Philosophy aside, the fact that a Lagrangian generalized Nash equilibrium takes the form of a parameterized variational inequality of standard type opens the way for stability analysis it in the manner undertaken for classical Nash equilibrium in [8]. That would be valuable to pursue, for greater understanding of equilibrium as in (1.9) with respect to constraints as in (1.8). But our focus instead will be on how the broader model in (1.2) and (1.3) may successfully be handled by other techniques than those from [2] utilized in [8]. We will get back to that in Section 3 after laying some groundwork in Section 2 about what is known for single-agent optimization.

[5] The transpose of a matrix A is denoted here by A^*.

2 Stability of a local minimum

Basic to our aims is a solid understanding of when a locally optimal solution to minimizing a function that depends on parameters is "stable" with respect to that dependence. We can't hope to make progress with perturbations of a local generalized Nash equilibrium as in (1.2) without first knowing how a local minimum of $\varphi_k(x_k, x_{-k}, p)$ with respect to x_k might behave in its dependence on (x_{-k}, p).

Fortunately, we can draw on a lot of theory that is already in place for the general case in some $I\!R^n$ of local minimizers in x of

$$\varphi(x,u) - v{\cdot}x \;\text{ for proper lsc function } \varphi \text{ on } I\!R^n \times I\!R^m, \tag{2.1}$$

where the vector $u \in I\!R^m$ has a broad role like our earlier p, to be studied around a nominal $\bar{u}$ such as $\bar{u} = 0$, but the vector v expressly gives *tilt* perturbations to be studied around $\bar{v} = 0$. Those tilt perturbations could obviously be absorbed into the specification of u, but long experience in sensitivity analysis in optimization has shown that their role is so fundamental to developments of theory that an explicit treatment is essential.

In fact, we can best begin by looking at tilt perturbations v in the absence of parameterization by a general u. Consider a proper lsc function f on $I\!R^n$ and a point $\bar{x}$ at which f has a local minimum, where accordingly $0 \in \partial f(\bar{x})$. A good question to ask is what the effect on local minimization might be when $f(x)$ is perturbed to $f(x) - v{\cdot}[x - \bar{x}]$ for a vector $v \neq 0$ giving a small "tilt" that doesn't affect the function value at $\bar{x}$ itself. Will $\bar{x}$ be shifted to a nearby unique $x(v)$? The following precisely formulated property enters the discussion.

Definition 2.1 (tilt stability of a local minimum [7])**.** *A local minimizer $\bar{x}$ of f is tilt stable if there is a neighborhood $\mathcal{X}$ of $\bar{x}$ along with a neighborhood $\mathcal{V}$ of $v = 0$ such that the mapping*

$$M : v \in V \;\mapsto\; \operatorname*{argmin}_{x\in\mathcal{X}} \Big\{ f(x) - v{\cdot}[x - \bar{x}] \Big\} = \operatorname*{argmin}_{x\in\mathcal{X}} \Big\{ f(x) - v{\cdot}x \Big\} \tag{2.2}$$

is single-valued and Lipschitz continuous on $\mathcal{V}$ with $M(0) = \bar{x}$.

This turns out to be intimately related to another property in variational analysis which may initially seem puzzling but lies behind the answers to many questions.

Definition 2.2 (variational convexity [9])**.** *The function f is variationally convex at a point $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ if there are exist $\delta > 0$ and neighborhoods $\mathcal{X}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{v}$, as well as a proper lsc convex function h, such that for some choice of*

$$\mathcal{G} = \{\, (x,v) \in \mathcal{X} \times \mathcal{V} \,|\, f(x) \leq f(\bar{x}) + \delta \} \;\text{ with } \delta > 0,\ \mathcal{X} \times \mathcal{V} \text{ a nbhd. of } (\bar{x},\bar{v}), \tag{2.3}$$

not only $\mathcal{G} \cap \operatorname{gph} \partial h = \mathcal{G} \cap \operatorname{gph} \partial f$, but also $h(x) = f(x)$ for all (x,v) in that common intersection. Variational strong convexity has h not just convex but strongly convex.

In essence, such variational convexity means that, in a local sense, the behavior of f and its subgradients can't be distinguished from the behavior associated with a convex function. Here, our interest centers on *strong* variational convexity in the case of $\bar{v} = 0$ and its important connection with tilt stability, together with with characterizations by a locally uniform quadratic growth condition and by subgradient strong monotonicity. For the full statement of this, recall that $\bar{v}$ is a *regular* subgradient of f at $\bar{x}$ in [12] when[6]

$$f(x) \geq f(\bar{x}) + \bar{v}{\cdot}[x - \bar{x}] + o(|x - \bar{x}|), \tag{2.4}$$

which is equivalent by [12, 8.5] to the existence of a function $g \leq f$ with $g(\bar{x}) = f(\bar{x})$ such that g is differentiable at $\bar{x}$ and $\nabla g(\bar{x}) = \bar{v}$.

[6]The canonical norm of x in $I\!R^n$ is denoted by $|x|$.

Theorem 2.3 (criteria for tilt stability [9]). *The following properties are equivalent. They ensure tilt stability, and with it, for (x,v) close enough to $(\bar{x},0)$, the sufficiency as well as the necessity of the first-order condition $v\in\partial f(x)$ for x to be a local minimizer of $f(x)-v{\cdot}x$:*

(a) *f is variationally strongly convex at $\bar{x}$ for the subgradient $\bar{v}=0$.*

(b) *$\exists\,\sigma>0$ and $\mathcal{G}$ as in (2.3) such that*

$$v\in\partial f(x),\ (x,v)\in\mathcal{G}\quad\Longrightarrow\quad f(x')\geq f(x)+v{\cdot}[x'-x]+\frac{\sigma}{2}|x'-x|^2,\ \forall x'\in\mathcal{X}. \tag{2.5}$$

(c) *$\bar{v}$ is a regular subgradient of f at $\bar{x}$, and $\exists\,\sigma>0$ and $\mathcal{G}$ as in (2.3) such that*

$$(x'-x){\cdot}(v'-v)\geq\sigma|x'-x|^2\ \text{ for all }\ (x,v),\,(x',v')\in\mathcal{G}, \tag{2.6}$$

Although the particular $\mathcal{G}$ might need adjustment in passing between these equivalent properties, the same values of σ serve in (b) and (c) and correspond to the reciprocals of the Lipschitz constants that can serve for the mapping (2.2) in tilt stability.

In passing from minimizing $f(x)$ to minimizing $\varphi(x,u)$ in x with u as a general parameter, there is good reason, from the facts just laid out, to keep the influence of tilt parameters v on solution stability directly in view.

Definition 2.4 (full stability of a local minimum [3]). *With respect to the parameterization by u, a local minimizer $\bar{x}$ of $\varphi(x,\bar{u})$ is a fully stable local minimum if there are neighborhoods $\mathcal{X}$ of $\bar{x}$, $\mathcal{U}$ of $\bar{u}$ and $\mathcal{V}$ of $\bar{v}=0$, such that the mapping*

$$M:(u,v)\in\mathcal{U}\times\mathcal{V}\ \mapsto\ \operatorname*{argmin}_{x\in\mathcal{X}}\Big\{\,\varphi(x,u)-v{\cdot}[x-\bar{x}]\Big\} \tag{2.7}$$

is single-valued and Lipschitz continuous on $\mathcal{U}\times\mathcal{V}$ with $M(\bar{u},0)=\bar{x}$.

Of course, the full stability in the definition entails the single-valuedness and Lipschitz continuity of the mapping

$$M_0:u\in\mathcal{U}\ \mapsto\ \operatorname*{argmin}_{x\in\mathcal{X}}\varphi(x,u)\ \text{ with }\ M_0(\bar{u})=\bar{x}, \tag{2.8}$$

since $M_0(u)=M(u,0)$. It's mathematically difficult, though, to put together a toolbox for verifying such stability just in u without dealing somehow also with v.

On the other hand, in incorporating tilt stability, full stability not only asks for it in the minimization of $\varphi(x,u)$ in x for $u=\bar{u}$, but for a uniform version of it as u ranges over a neighborhood $\mathcal{U}$ of $\bar{u}$. Tilt stability is itself required in this way to be stable with respect to perturbations of $\bar{u}$. But not only tilt stability; all the properties for it in Theorem 2.3, such as variational strong convexity, must likewise then hold "uniformly" for the functions $\varphi(\cdot,u)$ when $u\in\mathcal{U}$ at their minimizers. In particular as well, from Theorem 2.3,

$$\begin{array}{l}\text{for } (x,u,v) \text{ close enough to } (\bar{x},\bar{u},0)\text{, the condition } v\in\partial_x\partial\varphi(x,u) \text{ is}\\ \text{sufficient, as well as necessary, for } x \text{ to be a local minimizer in (2.8).}\end{array} \tag{2.9}$$

A criterion for full stability is available from [3] in terms of coderivatives of the mapping $\partial_x\varphi:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{R}^n$. By definition [12, 8G], the graph of the coderivative mapping

$$D^*[\partial_x\varphi](x,u\,|\,v):\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^m\ \text{ at } (x,u) \text{ for }\ v\in\partial_x\varphi(x,u) \tag{2.10}$$

consists of all (v',x',u') such that $(x',u',-v')$ belongs to the normal cone to $\operatorname{gph}\partial_x\varphi$ at (x,u,v).

Theorem 2.5 (coderivative criterion for full stability [3])**.** *Let $\varphi(x,u) = h(H(x,u))$ for a $\mathcal{C}^2$ mapping $H : I\!R^n \times I\!R^m \to I\!R^q$ and proper lsc convex function h on $I\!R^q$. Suppose the first-order condition $0 \in \partial_x\varphi(\bar{x},\bar{u})$ holds along with the constraint qualification*

$$z \in \partial h(H(\bar{x},\bar{u})),\ \nabla_x H(\bar{x},\bar{u})^* z = 0 \quad\Longrightarrow\quad z = 0. \tag{2.11}$$

Then, for $\bar{x}$ to be a fully stable local minimizer of $\varphi(x,\bar{u})$, it is both necessary and sufficient that
(a) *$(\omega,\eta) \in D^*[\partial_x\varphi](\bar{x},\bar{u}\,|\,0)(\xi),\ \xi \neq 0 \quad\Longrightarrow\quad \omega{\cdot}\xi > 0$, and*
(b) *$(0,\eta) \in D^*[\partial_x\varphi](\bar{x},\bar{u}\,|\,0)(0) \quad\Longrightarrow\quad \eta = 0$.*

This specializes [3, Theorem 2.3] by combining it with the result in [3, Proposition 2.2] about "strong amenability," which concerns the availability of a description of φ as a composite function of the kind we have introduced here in the theorem's statement. Functions φ fitting this pattern are widespread in optimization. The "generalized nonlinear programming" studied in [11], for instance, has $\varphi(x,u) = f_0(x) + g(F(x)+u)$. Ordinary nonlinear programming is the case of that where g is the indicator of a standard constraint cone K, a product of intervals $(-\infty,0]$ and $[0,0] = \{0\}$.

A strong point about Theorem 2.5 is that, with such great sweep, it identifies what is both necessary and sufficient for the full stability of a local minimizer. However, in order to apply it, an exact or at least estimating-type of formula is needed for the coderivatives (2.10) of $\partial_x\varphi$ in terms of the h and H in the composition.

We furnish next an example of an exact such formula from [4, Theorem 3.1] that's valid under a restriction beyond the constraint qualification in (2.11). That constraint qualification already yields for (x,u) near $(\bar{x},\bar{v})$ the subgradient formula

$$\partial_x\varphi(x,u) = \{\, v = \nabla_x H(x,u)^* z \,|\, z \in \partial h(H(x,u)) \,\}, \tag{2.12}$$

where in general more than one z could yield the same v. In replacing the constraint qualification (2.11) by the stronger *full* rank condition

$$\nabla H(\bar{x},\bar{u})^* z = 0 \quad\Longrightarrow\quad z = 0, \ \text{ i.e., } \ \operatorname{rank} \nabla H(\bar{x},\bar{u}) = q, \tag{2.13}$$

which propagates by continuity to $\nabla H(x,u)$ for (x,u) close to $(\bar{x},\bar{u})$, we ensure there is exactly one z for each v in (2.12). The coming formula utilizing this reduces the questions about coderivates of $\partial_x\varphi$ to questions about coderatives of h, for which we utilize the second-order subdifferential notation

$$\partial^2 h(w\,|\,z) = D^*[\partial h](w\,|\,z) \ \text{ for } \ z \in \partial h(w). \tag{2.14}$$

Theorem 2.6 (full rank coderivative formula [4])**.** *Let $\varphi(x,u) = h(H(x,u))$ for a $\mathcal{C}^2$ mapping $H : I\!R^n \times I\!R^m \to I\!R^q$ and proper lsc convex function h on $I\!R^q$. Suppose that the first-order condition $0 \in \partial_x\varphi(\bar{x},\bar{u})$ holds and $\nabla H(\bar{x},\bar{u})$ satisfies the full rank condition (2.13). Let*

$$h_0(x,u) = z_0{\cdot}H(x,u) \ \textit{ for the unique } z_0 \textit{ at } (x,u,v) = (\bar{x},\bar{u},0) \textit{ in (2.12).} \tag{2.15}$$

Then

$$\begin{aligned}&(\omega,\eta) \in D^*[\partial_x\varphi](\bar{x},\bar{u}\,|\,0)(\xi) \quad\Longleftrightarrow\quad \exists\,\zeta \in \partial^2 h(H(\bar{x},\bar{u})\,|\,z_0)(\,\nabla_x H(\bar{x},\bar{u})\xi\,)\\ &\textit{with } \omega = \nabla^2_{xx}h_0(\bar{x},\bar{u})\xi + \nabla_x H(\bar{x},\bar{u})^*\zeta,\ \ \eta = \nabla^2_{xu}h_0(\bar{x},\bar{u})\xi + \nabla_u H(\bar{x},\bar{u})^*\zeta.\end{aligned} \tag{2.16}$$

Plugging this into Theorem 2.5 to obtain a necessary and sufficient condition for full stability under the full rank condition, we see that the condition in (a) turns into a sort of strong positive-definiteness,

$$\left.\begin{array}{c} 0 < \xi{\cdot}\nabla^2_{xx}h_0(\bar{x},\bar{u})\xi + \xi{\cdot}\nabla_x H(\bar{x},\bar{u})^*\zeta \\ \forall \zeta \in \partial^2 h(H(\bar{x},\bar{u})\,|\,z_0)(\,\nabla_x H(\bar{x},\bar{u})\xi\,) \end{array}\right\} \ \text{when } \ \xi \neq 0, \tag{2.17}$$

while the condition in (b) then comes down to just

$$\nabla_u H(\bar{x},\bar{u})^*\zeta = 0 \text{ for all } \zeta \in \partial^2 h(H(\bar{x},\bar{u})\,|\,z_0)(0). \tag{2.18}$$

It remains still to work out second-order subdifferentials of h that appear in these conditions, but there are plenty of formulas for that available in [5]. There, and further in [6], more can be found about full stability than in the original paper [3].

As may have been noticed, Theorem 2.5 appears to offer something more about pure tilt stability than was recorded in Theorem 2.3. Pure tilt stability resurfaces when

$$\varphi(x,u) = f(x), \quad D^*[\partial_x\varphi](x\,|\,u) = \partial^2 f(x\,|\,\bar{u}). \tag{2.19}$$

Then (b) of Theorem 2.5 departs, and we are left with its partner (a) in the form that

$$\omega \in \partial^2 f(\bar{x}\,|\,0)(\xi),\ \xi \neq 0 \implies \omega{\cdot}\xi > 0. \tag{2.20}$$

Couldn't this criterion be added to Theorem 2.3, say as (d) in the list of properties there? No, the equivalence of (2.20) is only available for functions f less general than the ones in Theorem 2.3. In [7], it was achieved under an assumption on f of "continuous prox-regularity." We have avoided getting into the technicalities of that here in favor of structure as in Theorem 2.5 which supports the extra assumption, while being easier to appreciate. In the setting of (2.19), this asks for f to be of the composite form $f(x) = h(H(x))$ for a proper lsc convex function h and $\mathcal{C}^2$ mapping H, and for $\bar{x}$ with $0 \in \partial f(\bar{x})$ to satisfy the constraint qualification that no nonzero $z \in \partial h(H(\bar{x}))$ has $\nabla H(\bar{x})^* z = 0$.

3 Stability of a local equilibrium

Back in the extended formulation of generalized Nash equilibrium introduced in (1.1), we were looking in particular at $\bar{x}_k$ being a local minimizer of $\varphi_k(x_k,\bar{x}_{-k},\bar{p})$, an instance of parameterized minimization in x_k having (x_{-k},p) as parameter vector. In taking up the study of equilibrium stability, in the light now of the theory in Section 2, we have clear guidance for dealing with this. We ought naturally to concentrate on the case of a *strong* local equilibrium, where

$$\bar{x}_k \text{ is a fully stable minimizer of } \varphi_k(x_k,\bar{x}_{-k},\bar{p}),\ \forall k, \tag{3.1}$$

in the sense of Definition 2.4 and parameter $u_k = (x_{-k},p)$, for which we have, in line with (2.9), that

$$\begin{array}{l} \text{for } (x_k,x_{-k},p,v_k) \text{ close enough to } (\bar{x}_k,\bar{x}_{-k},\bar{p},0), \text{ the condition} \\ v_k \in \partial_{x_k}\varphi_k(x_k,x_{-k},p) \text{ is sufficient, as well as necessary, for} \\ x_k \text{ to be a local minimizer of } \varphi_k(x_k,x_{-k},p) - v_k{\cdot}[x_k-\bar{x}_k]. \end{array} \tag{3.2}$$

A local equilibrium (1.2) is then the same as a variational equilibrium (1.3), and we can rely on what variational analysis might tell us about the solution mapping S_0 in (1.4) the Φ behind it, with

$$(v_1,\ldots,v_N) \in \Phi(x_1,\ldots,x_N,p) \iff v_k \in \partial_{x_k}\varphi_k(x_k,x_{-k},p),\ \forall k. \tag{3.3}$$

But with considerations of tilt stability looming, the solution mapping S_0 in (1.4) should clearly now be upgraded to

$$S(p,v) = \{\, x \,|\, v \in \Phi(x,p)\}. \tag{3.4}$$

To set a goal, we can begin by formulating for generalized Nash equilibrium in our extended sense the analogue of full stability in optimization.

Definition 3.1 (full stability of a local equilibrium)**.** *In parameterization by p, a local equilibrium $\bar{x}$ as in (1.2) for $\bar{p}$ is fully stable if there are neighborhoods $\mathcal{P}$ of $\bar{p}$, $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ of $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_N$ of $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_N) = (0, \ldots, 0)$, such that the mapping*

$$M : (p, v) \in \mathcal{P} \times \mathcal{V} \mapsto \left\{ x \,\middle|\, x_k \in \operatorname*{argmin}_{x_k \in \mathcal{X}_k} \{ \varphi_k(x_k, x_{-k}, p) - v_k \cdot [x_k - \bar{x}_k] \}, \, \forall k \right\} \tag{3.5}$$

is single-valued and Lipschitz continuous on $\mathcal{P} \times \mathcal{V}$ with $M(\bar{p}, 0) = \bar{x}$.

For a strong local equilibrium (3.1), accompanied by (3.2) and at the center of our attention, M is the graphical localization of the solution mapping S in (3.4) obtained by

$$\operatorname{gph} M = [\mathcal{P} \times \mathcal{V} \times \mathcal{X}] \cap \operatorname{gph} S \tag{3.6}$$

Again, the tilt parameters could be set to 0, and we would have a mapping $M_0 : p \in \mathcal{P} \mapsto x = M(p, 0)$ localizing the original solution mapping S_0 in (1.4), with $\operatorname{gph} M_0 = [\mathcal{P} \times \mathcal{X}] \cap \operatorname{gph} S_0$. But the effects of tilts need direct scrutiny from the perspective of equilibrium, just as they did in optimization.

It might be imagined that the full stability in Definition 3.1 would encompass full stability in the agents' optimization problems, but that's not true. That agent-wise stability in x_k concerns responses to shifts (p, v) away from $(\bar{p}, 0)$ while x_{-k} is fixed at $\bar{x}_{-k}$. In the equilbrium setting, such a shift requires an adjustment from $\bar{x}$ to x that involves all the agents' strategies simultaneously.

By the same token, the uniqueness in local optimality that accompanies full stability for the agents' problems in (3.5), which allows the "$x_k \in$ argmin" there to be replaced by "$x_k =$ argmin," can't be expected to guarantee in (3.5) that M itself is single-valued. The minimization problems aren't independent; they interact with each other, and that's of course the essence of Nash equilibrium.

Putting p perturbations aside temporarily and focusing entirely on v perturbations will help in getting a better handle on this complexity.

Definition 3.2 (tilt stability of a local equilibrium)**.** *A local equilibrium $\bar{x}$ as in (1.2) for $\bar{p}$ is tilt stable if there neighborhoods $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ of $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_N$ of $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_N) = (0, \ldots, 0)$, such that the mapping*

$$\bar{M} : v \in \mathcal{V} \mapsto \left\{ x \,\middle|\, x_k \in \operatorname*{argmin}_{x_k \in \mathcal{X}_k} \{ f_k(x_k, x_{-k}) - v_k \cdot [x_k - \bar{x}_k] \}, \, \forall k \right\} \text{ for } f_k(x) = \varphi_k(x, \bar{p}) \tag{3.7}$$

is single-valued and Lipschitz continuous on $\mathcal{V}$ with $\bar{M}(0) = \bar{x}$.

An elementary example in this framework will bring out the key issues we face beyond those already confronted when only single-agent optimization was involved. Suppose the functions f_k on $\mathbb{R}^{n_k}$ in (3.7) are $\mathcal{C}^2$. An equilibrium then, in variational terms, corresponds to having

$$\nabla_{x_k} f_k(\bar{x}_k, \bar{x}_{-k}) = 0, \, \forall k, \tag{3.8}$$

with each $\bar{x}_k$ being a *tilt stable* local minimizer if and only if

$$\nabla^2_{x_k x_k} f_k(\bar{x}_k, \bar{x}_{-k}) \text{ is positive-definite, } \forall k. \tag{3.9}$$

The relation between x and v in (3.7) reduces to $\nabla_{x_k} f_k(x_k, x_{-k}) = v_k$, which can be expressed by

$$J(x) = v \text{ for } J(x) = (\nabla_{x_1} f_1(x), \ldots \nabla_{x_N} f_N(x)). \tag{3.10}$$

Tilt stability hinges on whether the $\mathcal{C}^2$ mapping $J : \mathbb{R}^n \to \mathbb{R}^n$ has, in localization around the pair $(\bar{x}, 0)$ in its graph, a single-valued inverse that is Lipschitz continuous. Of course, for this question we have a precise answer in the classical inverse function theorem. Tilt stability corresponds to the nonsingularity of the matrix

$$\nabla J(\bar{x}) = \Big[\nabla^2_{x_k x_j} f_k(\bar{x})\Big]_{k=1,j=1}^{N,N} \quad \text{with positive-definite blocks on the diagonal.} \tag{3.11}$$

The diagonal positive-definiteness, arising from the assumed full stability in the minimization of each agent separately, is obviously not enough to ensure nonsingularity of ∇J, *unless the off-diagonal terms are comparatively "small."* In game terms, that "smallness" could have the interpretation that, in the minimization of $f_k(x_k, x_{-k})$ in x_k, the influence of x_{-k} as a parameter isn't overly disruptive. This is a significant insight which we'll later return to.

If the single-valuedness even in plain tilt stability of an equilibrium is elusive, what might be the fallback for a relaxation of the concepts in Definitions 3.1 and 3.2? The Aubin property, which is a graphically localized version of set-valued Lipschitz continuity could come to the rescue. That basic property in variational analysis is examined from all sides in [12, 9F]. Instead of explaining it here, starting with its formal definition, we'll take a shortcut and formulate it directly in our application setting in terms of a condition known from [12, 9.37] to be equivalent to the one in the formal definition:[7]

$$\begin{array}{l}\text{the mapping } M \text{ in (3.5) is Aubin continuous around } (\bar{p},0) \text{ for } \bar{x}\in M(\bar{p},0) \text{ if}\\ \text{the function } (x,p,v)\mapsto \operatorname{dist}(x, M(p,v)) \text{ is Lipschitz continuous around } (\bar{x},\bar{p},0),\end{array} \tag{3.12}$$

and likewise,

$$\begin{array}{l}\text{the mapping } \bar{M} \text{ in (3.7) is Aubin continuous around 0 for } \bar{x}\in \bar{M}(0) \text{ if}\\ \text{the function } (x,v)\to \operatorname{dist}(x, \bar{M}(v)) \text{ is Lipschitz continuous around } (\bar{x},0).\end{array} \tag{3.13}$$

It's easy to see that, under single-valuedness, Aubin continuity becomes Lipschitz continuity.

Definition 3.3 (near tilt or full stability of a local equilibrium)**.** *By an equilibrium $\bar{x}$ being nearly fully stable will be meant that it has the relaxed version of the property in Definition 3.1 where single-valuedness is relinquished and M is just required to be Aubin continuous around $(\bar{p},0)$ for $\bar{x}\in M(\bar{p},0)$. Similarly for an equilibrium being nearly tilt stable as a relaxation of Definition 3.2.*

Such near stability without single-valuedness is nevertheless a powerful attribute. It guarantees that, under small shifts in parameters, an equilibrium always, at least, has a nearby replacement, moreover with the degree of shift in equilibrium bounded proportionally to the degree of shift in the parameters. A great virtue of Aubin continuity, moreover, is its coderivative characterization by the Mordukhovich criterion [12, 9.40]. We appeal to that next.

Theorem 3.4 (coderivative criterion for nearly full stability)**.** *A strong local equilibrium $\bar{x}$ for $\bar{p}$, as in (3.1), is nearly fully stable if and only if the mapping Φ in (3.3) satisfies*

$$(0,\pi)\in D^*\Phi(\bar{x},\bar{p}\,|\,0)(\omega) \quad\Longrightarrow\quad \pi=0,\ \omega=0. \tag{3.14}$$

Proof. The claim is that (3.14) is necessary and sufficient for the Aubin continuity in Definition 3.3, which the Mordukhovich criterion characterizes by

$$D^*M(\bar{p},0\,|\,\bar{x})(0) = \{(0,0)\}, \tag{3.15}$$

[7] The distance between a point x and a set C is denoted by $\operatorname{dist}(x,C)$.

as long as the graph of M is locally closed around $(\bar{p}, 0, \bar{x})$. We know that local closedness holds for this kind of equilibrium through (3.2), which relates M and Φ by (3.6) in terms of the mapping S in (3.4). This relationship means that, locally around $\bar{x}$, $\bar{p}$ and $\bar{v} = 0$, we have for graphs and their associated normal cones:

$$\begin{array}{lll} (p,v,x) \in \operatorname{gph} M & \iff & (x,p,v) \in \operatorname{gph} \Phi, \text{ and therefore} \\ (\pi,\omega,\xi) \in N_{\operatorname{gph} M}(p,v,x) & \iff & (\xi,\pi,\omega) \in N_{\operatorname{gph} \Phi}(x,p,v). \end{array} \tag{3.16}$$

The graph of the coderative mapping $D^*M(\bar{p}, 0 \,|\, \bar{x})$ consists of all $(-\xi, \pi, \omega)$ such that (π, ω, ξ) belongs to the first normal cone in (3.16), whereas the graph of the coderivative mapping $D^*\Phi(\bar{x}, \bar{p} \,|\, 0)$ consists of all $(-\omega, \xi, \pi)$ such that (ξ, π, ω) belongs to the second normal cone in (3.16). Thus,

$$(\pi,\omega) \in D^*M(\bar{p}, 0 \,|\, \bar{x})(-\xi) \quad \iff \quad (\xi,\pi) \in D^*\Phi(\bar{x}, \bar{p} \,|\, 0)(-\omega). \tag{3.17}$$

On this basis, with inconsequential changes of signs, the condition in (3.15) can be identified with the one in (3.14). □

Although an exact formula for the coderivatives in Theorem 3.4 is difficult to obtain in general circumstances, we can offer an estimate under a constraint qualification.

Theorem 3.5 (coderivative estimate)**.** *The inclusion*

$$D^*\Phi(\bar{x}, \bar{p} \,|\, 0)(\omega_1, \ldots, \omega_N) \;\subset\; D^*[\partial_{x_1}\varphi_1](\bar{x}, \bar{p} \,|\, 0)(\omega_1) + \cdots + D^*[\partial_{x_N}\varphi_N](\bar{x}, \bar{p} \,|\, 0)(\omega_N) \tag{3.18}$$

holds under the constraint qualification that

$$\left.\begin{array}{l} (\xi^k, \pi^k) \in D^*[\partial_{x_k}\varphi_k](\bar{x}, \bar{p} \,|\, 0)(0), \;\forall k \\ (\xi^1, \pi^1) + \cdots + (\xi^N, \pi^N) \;=\; (0,0) \end{array}\right\} \Longrightarrow \quad (\xi^k, \pi^k) = (0,0) \;\forall k. \tag{3.19}$$

Proof. This will be derived from the rule in [12, 10.41] for the coderivatives of a mapping that is the sum of other mappings by construing the formula (3.3) for Φ to mean

$$\Phi(x,p) = \Phi_1(x,p) + \cdots + \Phi_N(x,p) \;\text{ for }\; \Phi_k(x,p) = (\ldots, 0, \partial_{x_k}\varphi_k(x,p), 0, \ldots). \tag{3.20}$$

The coderivatives of the mappings Φ_k are given by

$$(\xi,\pi) \in D^*\Phi_k(x,p \,|\, v_1, \ldots, v_N)(\omega_1, \ldots, \omega_N) \quad \iff \quad (\xi,\pi) \in D^*[\partial_{x_k}\varphi_k](x,p \,|\, v_k)(\omega_k). \tag{3.21}$$

Two assumptions in [12, 10.41] need to be verified in order to confirm (3.19) through (3.21). Both are concerned with the possibilities for putting together "supervectors"[8]

$$(v^1, \ldots, v^N) \;\text{ with }\; v^k = (v^k_1, \ldots, v^k_N) \in \Phi_k(x,p), \;\sum_{k=1}^N v^k = v = (v_1, \ldots, v_N), \tag{3.22}$$

but here the only choice for a given v is each $v^k = (\ldots, 0, v_k, 0, \ldots)$, and that simplifies everything. The first assumption in [12, 10.41] is a local boundedness condition on the mapping that takes (x,p,v) into the set of supervectors $(v^1, \ldots, v^N)$ in (3.22). It trivializes under the simplification. The second assumption is a constraint qualification on the coderivatives on the left in (3.21) which the simplification reduces to (3.19). □

[8] We used superscripts k for vectors associated with agent k that aren't, like x_k, just in $I\!R^{n_k}$.

Corollary 3.6 (sufficient condition for nearly full stability). *Under the constraint qualification in (3.19), a sufficient condition for the criterion (3.14) for nearly full stability to hold is*

$$(\xi^k,\pi^k)\in D^*[\partial_{x_k}\varphi_k](\bar{x},\bar{p}\,|\,0)(\omega_k),\ \sum\nolimits_k \xi^k=0 \quad\Longrightarrow\quad \sum\nolimits_k \pi^k=0,\ \omega_k=0,\ \forall k, \tag{3.23}$$

and then also $\pi^k=0$, $\forall k$.

Proof. The condition in (3.23) comes directly from applying the inclusion in (3.18). But once it is known that $\xi^k=0$ and $\omega_k=0$, it follows from assuming (3.19) that $\pi^k=0$, too. □

This sufficient condition for nearly full stability can be worked out in more detail when each

$$\begin{array}{l}\varphi_k(x,p)=h_k(H_k(x,p))\ \text{ for a } \mathcal{C}^2 \text{ mapping } H_k\\ \text{and a proper lsc convex function } h_k \text{ on } I\!\!R^{q_k}.\end{array} \tag{3.24}$$

If each $\nabla H_k(\bar{x},\bar{p})$ has rank q_k, the coderivative formula provided by Theorem 2.6 can be invoked. Lots more cases covered by (3.24) can be explored further. For instance, we might have

$$h_k(H_k(x,p))=g_k(x,p)+\delta_C(G(x,p)),$$

so that the problem of agent k is to

$$\text{minimize }\ g_k(x_k,x_{-k})\ \text{ in } x_k \text{ subject to }\ G(x_k,x_{-k},p)\in C.$$

This has the interpretation that the agents must share common resources, such as a budget in money or some other good. Agent k is limited to the amounts left untouched by all the other agents. Stability criteria specifically tailored to such circumstances could be derived.

What can be said, finally, about how any of these stability observations might help in actually finding a local equilibrium? There is good reason to be dissatisfied with applications of game theory in which the designated "solution" is too delicate to withstand tiny perturbations in model parameters. This has already been noted as both an impediment to numerical methodology and a possible flaw in concept. However, an even bigger issue that needs to be faced is the frequent lack of a dynamical mechanism for the agents to intereact with each other and thereby ultimately reach an equilibrium, presumably one that exhibits some stability, not fragility. Remedying that lack is a topic too big to get into here, but when might one equilibrium likely be more "attractive" than another from this perspective?

The example discussed after Definiton 3.2 suggests something to investigate. A key property there is the diagonal positive-definiteness in (3.12) and its potential for keeping interagent influences from getting out of hand. The square matrix $\nabla J(\bar{x})$ isn't symmetric, but we can ask about positive-definiteness of its *symmetric* part, $\frac{1}{2}[\nabla J(\bar{x})+\nabla J(\bar{x})^*]$. That positive-definiteness can be adopted as the signal that the block-diagonal holds sway over the rest. It guarantees that $\nabla J(\bar{x})$ is nonsingular and therefore that the solution mapping in this example, the inverse of the $\mathcal{C}^1$ mapping J, is itself locally single-valued and $\mathcal{C}^1$, hence locally Lipschitz continuous, thus giving tilt stability. But there is more. The positive-definiteness of the symmetric part of $\nabla J(\bar{x})$ is also the condition ensuring that the $\mathcal{C}^1$ mapping J is *strongly monotone* on a neighborhood $\mathcal{X}$ of $\bar{x}$:

$$\exists\,\sigma>0\ \text{ such that }\ [J(x')-J(x)]\cdot[x'-x]\geq\sigma|x'-x|^2\ \text{ for }\ x',x\in\mathcal{X}. \tag{3.25}$$

Strong monotonicity is a fundamental property in variational analysis that extends beyond smooth mappings to the general theory set-valued mappings from a space $I\!\!R^n$ into itself; see [12, Chapter 12].

It's important in particular in algorithmic schemes for solving $0 \in T(\bar{x})$ for a mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where refers locally to

$$\begin{array}{l} \exists\sigma > 0 \text{ and a neighborhood } \mathcal{X} \times \mathcal{V} \text{ of } (\bar{x}, 0) \in \operatorname{gph} T \text{ such that} \\ [x' - x]\cdot[v' - v] \geq \sigma|x' - x|^2,\ \forall (x, v),\ (x', v') \in [\mathcal{X} \times \mathcal{V}] \cap \operatorname{gph} T. \end{array} \tag{3.26}$$

Such strong monotonicity[9] is *maximal* if there is no other mapping T', likewise strongly monotone in $\mathcal{X} \times \mathcal{V}$, such that $[\mathcal{X} \times \mathcal{V}] \cap \operatorname{gph} T'$ is strictly larger than $[\mathcal{X} \times \mathcal{V}] \cap \operatorname{gph} T$. For a single-valued continuous mapping as in (3.25), maximality is automatic. A key fact is that *under maximal strong mononoticity of T, its inverse T^{-1}, with $\bar{x} \in T^{-1}(0)$, has a graphical localization around $(0, \bar{x})$ that is single-valued and Lipschitz continuous.*

Some of this has already played out in our discussion of tilt stability in general. Strong monotonicity appears in characterization (d) of Theorem 2.3, where it again turns out automatically to be maximal. The full stability concept in Definition 2.4 brings with it, through that characterization, the local strong monotonicity of the set-valued mapping $x \mapsto \partial_x \varphi(x, u)$ holding in a uniform sense for u near $\bar{u}$. In this section, where we have concentrated on the strong form of local equilibrium in (3.1), it holds that way for the mappings $x_k \mapsto \partial_{x_k} \varphi_k(x_k, x_{-k}, p)$ around $(\bar{x}_{-k}, \bar{p})$.

Now, though, we have a bigger picture in which the relation $v \in \Phi(x, p)$ is featured as the general parameterized replacement for the smooth relation $v = J(x)$ in the special example. What would be the consequence of strong monotonicity showing up there? In looking for an answer to that, we need to be cautious about an aspect of generalized Nash equilibrium that hasn't, until now, needed attention. *An equilibrium, and all the properties we have articulated for it so far, remain unchanged of each of the functions φ_k is rescaled to $\lambda_k \varphi_k$ with $\lambda_k > 0$,* as simply a change in units in which the objectives of the agents are calibrated. Such rescaling could, however, make or break strong monotonicity. To account for this, we introduce the notation

$$\Lambda\Phi : (x, p) \mapsto \Big(\lambda_1 \partial_{x_1} \varphi_1(x, p), \ldots, \lambda_N \partial_{x_N} \varphi_N(x, p)\Big) \ \text{ for } \ \Lambda = (\lambda_1, \ldots, \lambda_N),\ \lambda_k > 0. \tag{3.27}$$

Definition 3.7 (strong monotonity of a local equilibrium)**.** *A strong local equilibrium (3.1) is strongly monotone at $\bar{p}$ if, for Φ in (3.3) with $0 \in \Phi(\bar{x}, \bar{p})$, and some choice if Λ as in (3.27), there are neighborhoods $\mathcal{X} \times \mathcal{V}$ and $\mathcal{P}$ of $\bar{p}$ where the mappings $x \mapsto \Lambda\Phi(x, p)$ are maximally strong monotone, all with the same $\sigma > 0$.*

This is satisfied for instance in the example following Definition 3.2 for $\Lambda = (1, 1, \ldots, 1)$ if the positive-definiteness of the diagonal blocks in (3.11) sufficiently dominates the effects of the off-diagonal blocks. As there, the property in Definition 3.7 can thus be interpretated as describing circumstances where the local optimization in the agents' subproblems is not so feeble as to be easily disrupted without recourse by small changes in p or the strategies of the other agents.

Theorem 3.8 (full stability via strong monotonicity)**.** *For a strong local equilibrium (3.1) that is strongly monotone, near full stability implies full stability.*

Proof. The strong monotonicity assumed locally for $x \mapsto \Phi(x, p)$ makes its localized inverse $v \mapsto M(p, v)$ be single-valued and Lipschitz continuous. Then $M(p, v)$ must locally be single-valued as a function of (p, v), so the Aubin continuity with respect to (p, v) in the definition of nearly full stability turns into the Lipschitz continuity in the definition full stability. □

Besides the advantages flowing directly from Theorem 3.8, a strongly monotone equilibrium might be much more open to being located by an iterative scheme that starts from out-of-equilibrium strategies x_k^0 that aren't too far away. This could utilize a numerical procedure like the proximal point

[9]Plain, not necessarily strong, monotonicity would replace the quadratic bound on the right just by 0.

algorithm, which is specifically designed for finding a "zero" of a maximal monotone mapping and can succeed even in localized execution as in [10], but hopefully with an adaptation that may be interpreted as based on steps performed by the agents themselves. Or it might be manifested in how agents react to each other's tentative moves in competitive optimization and, in that way, also help to confirm the validity of an equilibrium model in some practical application. There are so many interesting topics that await future research.

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