UTILITY, GENERALIZED ENTROPY AND MEASURES OF LIABILITY

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LECTURE 4

Integral Functionals

$(\Omega, \mathcal{F}, P)$ = some probability space

A closed-set-valued mapping $S : \Omega \to R^n$ is **measurable** when

$\{\omega \mid S(\omega) \cap C\} \in \mathcal{F}$ for all closed sets $C \subset R^n$

A function $f : R^n \times \Omega \to \bar{R}$ is a **normal integrand** when

$f(x, \omega)$ is lsc in x and $S : \omega \to \operatorname{epi} f(\cdot, \omega)$ is measurable

Consequence: $f(x(\omega), \omega)$ is measurable when $x(\omega)$ is measurable

Conjugacy on paired spaces $\mathcal{L}_n^p(\Omega, \mathcal{F}, P)$ and $\mathcal{L}_n^q(\Omega, \mathcal{F}, P)$

For a normal integrand f, the integral functional

$$I_f(x(\cdot) = E\{f(x(\cdot), \cdot)\} = \int_\Omega f(x(\omega), \omega) dP(\omega)$$

is (with minor assumption) well-defined for $x(\cdot) \in \mathcal{L}_n^p(\Omega, \mathcal{F}, P)$, and

$I_f^* = I_{f^*}$ on $\mathcal{L}_n^q(\Omega, \mathcal{F}, P)$, $\quad I_f^{**} = I_{f^{**}}$ on $\mathcal{L}_n^p(\Omega, \mathcal{F}, P)$

Note: I_f is **convex** when $f(x, \omega)$ is convex in x, and then

$v(\cdot) \in \partial I_f(x(\cdot)) \iff v(\omega) \in \partial f(x(\omega), \omega)$ almost surely

Utility Maximization in Finance

Instruments: $i = 0, 1, \ldots, m$ with returns X_i, risk-free for $i = 0$

prices π_i with $\pi_0 = 1$, rates of return $r_i = X_i/\pi_i - 1$, r_0 constant

$Y_i = X_i/[1 + r_0] - \pi_i$ gives net return in present money

Portfolios: weights ξ_i yielding $\sum_{i=0}^m \xi_i X_i$ at cost $\sum_{i=0}^m \xi_i \pi_i$, or

in present money yielding $\sum_{i=1}^m \xi_i Y_i + w$ from investment w

Monetary utility, normalized:

$u(x) =$ the amount of present money deemed acceptable in lieu of receiving the future amount $[1 + r_0]x$

u is concave, nondecreasing, with $u(0) = 0$, $u(x) \leq x$

Utility maximization problem

maximize $E\{u(\sum_{i=1}^m \xi_i Y_i + w)\}$ over $\xi = (\xi_1, \ldots, \xi_m)$

$\mathcal{U}(X) = E\{u(X)\}$ assesses present worth of future gain $[1 + r_0]X$

Reformulation to Minimization in Loss Context

$v(x) = -u(-x) =$ the **liability** exposure associated with x

$=$ the amount of present money deemed necessary as compensation for losing $[1+r_0]x$ in the future

v is convex, nondecreasing, with $v(0)=0$, $v(x) \geq x$

Liability minimization problem

minimize $E\{v(\sum_{i=1}^m \xi_i[-Y_i] - w)\}$ over $\xi = (\xi_1,\ldots,\xi_m)$

$\mathcal{V}(X) = E\{v(X)\} = I_v(X) =$ **integral** functional on $\mathcal{L}^p(\Omega,\mathcal{F},P)$

$\mathcal{V}$ is convex, nondecreasing, with $\mathcal{V}(0)=0$, $\mathcal{V}(X) \geq EX$

Conjugate: $\mathcal{V}^*(Q) = I_{v^*}(Q) = E\{v^*(Q)\}$ on $\mathcal{L}^q(\Omega,\mathcal{F},P)$

$\mathcal{V}^*$ is convex, $\mathcal{V}^*(Q) \geq 0$, $\mathcal{V}^*(1)=0$, and $\mathcal{V}^*(Q) < \infty \Rightarrow Q \geq 0$

Insurance interpretation: $\mathcal{V}(X)$ is the **premium** to be charged (relative to v) for covering the uncertain future loss $[1+r_0]X$

Lagrangian and Dual Problem

$$\mathcal{V}(X) = E\{v(X)\}, \qquad \mathcal{V}^*(Q) = E\{v^*(Q)\}$$

Lagrangian for the minimization problem:

$$L(\xi_1, \ldots, \xi_m; Q) = E\{(\textstyle\sum_{i=1}^m \xi_i[-Y_i] + [-w])Q\} - \mathcal{V}^*(Q)$$

Derivation of the dual objective:

$$\begin{aligned} g(Q) &= \inf_{\xi_1,\ldots,\xi_m} L(\xi_1, \ldots, \xi_m; Q) \\ &= [-w]EQ - \mathcal{V}^*(Q) \ \text{ if } Q \geq 0 \text{ and } E[Y_i Q] = 0, \end{aligned}$$

but $= -\infty$ otherwise

Dual problem

maximize $[-w]EQ - E\{v^*(Q)\}$ subject to
$Q \geq 0$ and $E[Y_i Q] = 0$ for $i = 1, \ldots, m$

$-w =$ the money extracted from the market in the present
for taking on the future losses associated with $\sum_{i=0}^m \xi_i[-X_i]$

Application of Duality Criteria

These primal and dual problems fit the extended Fenchel format:

$$(\mathcal{P}) \qquad \textit{minimize}\big\{\langle c,\xi\rangle + h(\xi) + k(b - A\xi)\big\},$$
$$(\mathcal{D}) \qquad \textit{maximize}\big\{\langle b,Q\rangle - k^*(Q) - h^*(A^*Q - c)\big\},$$

with $\xi \in \boldsymbol{R}^m$ and $Q \in \mathcal{L}^q$, paired with $\mathcal{L}^p$, $p < \infty$, by taking

$$c = 0,\ h \equiv 0,\ h^* = \delta_0,\ k = \mathcal{V},\ k^* = \mathcal{V}^*,\ b = -w,$$
$$A : \xi \to \textstyle\sum_{i-1}^m \xi_i Y_i, \qquad A^* : Q \to (\, E[Y_1Q], \ldots, E[Y_mQ]\,)$$

Criteria to be specialized:

$$b \in \mathrm{int}[A(\mathrm{dom}\, h) + \mathrm{dom}\, k], \qquad c \in \mathrm{int}[A^*(\mathrm{dom}\, k^*) - \mathrm{dom}\, h^*]$$

Duality theorem

(a) $\inf(\mathcal{P}) = \max(\mathcal{D})$ if $-w \in \mathrm{int}\,\big\{X \in \mathcal{L}^p \,\big|\, E\{v(X)\} < \infty\big\}$

(b) $\min(\mathcal{P}) = \sup(\mathcal{D})$ if
$0 \in \mathrm{int}\,\big\{\,(\,E[Y_1Q], \ldots, E[Y_mQ]\,) \,\big|\, Q \in \mathcal{L}^q,\ E\{v^*(Q)\} < \infty\big\}$

It is possible also to work with $X \in \mathcal{L}^\infty$ and $Q \in (\mathcal{L}^\infty)^*$. Further analysis then relates the results to known conditions in finance.

Valuations of Liability Generalized

functionals $\mathcal{V}(X)$, not just of form $I_v(X)$, for potential losses X

Liability measures

Call $\mathcal{V}$ a **measure of liability** if: (V1) $\mathcal{V}(0) = 0$, $\mathcal{V}(X) \geq EX$,
(V2) $\mathcal{V}$ convex, (V3) $\mathcal{V}$ nondecreasing, (V4) $\mathcal{V}$ lsc

Conjugate characterization:

$\mathcal{V}^*$ convex, lsc, $\mathcal{V}^*(Q) \geq 0$, $\mathcal{V}^*(1) = 0$, $\mathcal{V}^*(Q) < \infty \Rightarrow Q \geq 0$

Consider a **trade-off:** minimize $C + \mathcal{V}(X - C)$ over $C \in \mathbb{R}$

charge C up front, reducing uncertain future losses accordingly

Derivation of associated risk measure and entropy

(a) $\mathcal{R}(X) = \min_C \left\{ C + \mathcal{V}(X - C) \right\}$ is a coherent measure of risk

(b) $\mathcal{R}^*(Q) = \mathcal{V}^*(Q)$ if $EQ = 1$, but $\mathcal{R}^*(Q) = \infty$ otherwise

$\mathcal{R}^*(Q)$ is thus an **entropy** functional $\mathcal{I}(Q)$, $-\mathcal{I}(Q) =$ the entropy

Minimization of Portfolio Risk

$\mathcal{V}$ = measure of liability, $\mathcal{R}$ = associated risk, $\mathcal{I} = \mathcal{R}^*$ entropy

$$\mathcal{R}(\textstyle\sum_{i=1}^m \xi_i[-Y_i] - w) = \mathcal{R}(\sum_{i=1}^m \xi_i[-Y_i]) - w$$

Portfolio risk minimization problem

minimize $\mathcal{R}(\,\sum_{i=1}^m \xi_i[-Y_i]\,)$ over $\xi = (\xi_1, \ldots, \xi_m)$

Lagrangian function:

$$L(\xi_1, \ldots, \xi_m; Q) = E\{\,\textstyle\sum_{i=1}^m \xi_i[-Y_i]Q\} - \mathcal{I}(Q)$$
$$= \textstyle\sum_{i=1}^m \xi_i E\{[-Y_i]Q\} - \mathcal{V}^*(Q) \text{ if } Q \geq 0,\ EQ = 1$$

but $= -\infty$ otherwise

Corresponding dual problem in entropy

maximize $-\mathcal{I}(Q)$ subject to $E[Y_i Q] = 0$ for $i = 1, \ldots, m$

$\Rightarrow$ Q is a **risk neutral** probability density, $Q = dP^*/dP$

an "entropic distance" of P^* from the nominal P is minimized

Averslty in Liability Valuation

Call a liability measure $\mathcal{V}$ **averse** if $\mathcal{V}(X) > EX$ when $X \neq 0$

Associated measures of error and deviation

Let $\mathcal{V}$ be an averse measure of liability, and let $\mathcal{R}(X)$ be the associated measure of risk, $\mathcal{R}(X) = \min_C \left\{ C + \mathcal{V}(X - C) \right\}$

(a) $\mathcal{R}(X)$ is an **averse measure of risk** and coherent
(b) $\mathcal{E}(X) = \mathcal{V}(X) - EX$ is a **measure of error**
(c) $\mathcal{D}(X) = \min_C \left\{ \mathcal{E}(X - C) \right\}$ agrees with $\mathcal{D}(X) = \mathcal{R}(X - EX)$

Integral functional case: $\mathcal{V}(X) = E\{v(X)\}$

v convex, nondecreasing, with $v(0) = 0$, $v(x) \geq x$

$\mathcal{V}(X) = E\{v(X)\}$ is averse when $v(x) > x$ for $x \neq 0$
$\mathcal{E}(X) = E\{\varepsilon(X)\}$ for the function $\varepsilon(x) = v(x) - x$

CVaR Revisited

Consider the liability measure $\mathcal{V}(X) = E\{v(X)\}$ and associated error measure $\mathcal{E}(X) = E\{\varepsilon(X)\} = E\{v(X) - X\}$, deviation measure $\mathcal{D}(X) = \min_C \{\mathcal{E}(X - C)\}$ and coherent risk measure $\mathcal{R}(X) = \min_C \{C + \mathcal{V}(X - C)\}$ in the case of

$v(x) = (1-\alpha)^{-1} \max\{x, 0\}$ (averse), with

$\varepsilon(x) = v(x) - x = [(1-\alpha)^{-1} - 1] \max\{x, 0\} + \max\{-x, 0\}$

where $0 < \alpha < 1$, so that $(1-\alpha)^{-1} > 1$. Then

- **(a)** $\mathcal{V}(X) = (1-\alpha)^{-1} E[X_+]$
- **(b)** $\mathcal{E}(X) = [(1-\alpha)^{-1} - 1] E[X_+] + E[X_-]$ Koenker-Basset error
- **(c)** $\mathcal{R}(X) = \mathrm{CVaR}_\alpha(X)$
- **(d)** $\mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX)$;

For "utility" version of this, see paper of Ben-Tal and Teboulle

Some References

[1] R. T. Rockafellar (1998), *Variational Analysis*, Springer-Verlag
Chapter 14 (for issues of measurability)

[2] R. T. Rockafellar (1971), "Integrals which are convex functionals, II," *Pacific Journal of Mathematics* 39, 439–469
conjugates on $(\mathcal{L}^\infty)*$ are covered as well

[3] R. T. Rockafellar (1976), "Integral functionals, normal integrands and measurable selections," in *Nonlinear Operators and the Calculus of Variations*, L. Waelbroeck (ed.), Lecture Notes in Math. 543, Springer-Verlag, 157-207
most now in [1], except weak compactness characterization

[4] A. Ben-Tal, M. Teboulle (2007), "An old-new concept of convex risk measures: the optimized certainty equivalent," *Mathematical Finance* 17, 449–476.