Relaxing the matroid axioms: h-vectors

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Based on joint works with Ernest Chong, Steven Klee and Jeremy Martin.
Brief Outline

- First part: introduce the big picture. Some motivation, general ideas and consequences
- Second part: go into one example of how to use the axioms and what kind of information we get.
Laplacians of simplicial complexes

- $\Delta$: $(d - 1)$-dimensional simplicial complex.
- $C_\bullet(\Delta)$: simplicial chain complex over $\mathbb{R}$ with differential $\partial$.
- $\delta$: The adjoint to $\partial$ in the faces basis (transpose).
- $L_k = \partial \delta + \delta \partial : C_k(\Delta) \to C_k(\Delta)$: The $k$-th Laplacian of $\Delta$. Self adjoint.

Theorem (Reiner-Kook-Stanton, Reiner-Duval, Duval)
The eigenvalues of the Laplacians of both matroids and shifted complexes are integral and satisfy the same recurrence.

ISSUE: Proofs are radically different. Why so similar? Having integral spectrum is an extremely rare property.
A question

Question (Duval-Reiner)

Is there a class of simplicial complexes that contains both matroid (independence) complexes and shifted complexes that explains the similarities?

**Goal:** Construct a class that explains many similarities. We expect a class of simplicial complexes that behaves almost like matroids, but contains pure shifted simplicial complexes.
Approach

Relax several matroid cryptomorphisms larger classes of simplicial complexes. Intersect such larger classes. Interesting cryptomorphisms that we can successfully relax:

- Independence
- Exchange axiom
- Circuit axiom
- Rank semimodularity
- Greedy algorithm

In addition there is a duality theory.

Open question: Is there an analogue of the matroid polytope?
Main setting

Work with simplicial complexes on an ordered vertex set!

- $(E, \Delta)$: Finite ordered set $E$, simplicial complex $\Delta$ whose vertices are in $E$. (Allow loops)
- $\Delta$ is always assumed to be pure and $(d-1)$-dimensional.
- Adopt language from matroid theory: Independents = faces, Bases = Facets, Circuits = minimal non-faces.
- $\text{rank}(A) = \text{rank of } A \subseteq E = \text{size of maximal face contained in } A.$
Brief definition

**Definition**

\((E, \Delta)\) is shifted if \(F\) is a face, \(j \in F\) and \(i < j\) is such that \(i \notin F\) then \((F - \{j\}) \cup \{i\} \in \Delta\).

We will discuss a neat way to think about pure shifted simplicial complexes later!
The axioms: summary

- **Independence (WIA):** If $I_1$ and $I_2$ are independent, and there is $I \subseteq I_1 \cap I_2$ such that $I_1 \setminus I_2$ is a subset of the first lexicographic basis of $\text{link}_\Delta(I)$ and $|I_1| > |I_2|$, then there is $b \in I_1 \setminus I_2$ such that $I_2 \cup \{b\} \in \Delta$.

- **Exchange (WEA):** $B_1$, $B_2$ bases. If $b_1 \in B_1 \setminus B_2$ and $b_1 > \max(B_2 \setminus B_1)$ there exists $b_2 \in B_2 \setminus B_1$ with $(B_1 - b_1) \cup \{b_2\} \in \Delta$.

- **Circuit (WCA):** $C_1$, $C_2$ circuits. If $c \in C_1 \cap C_2$ and $c < \max(C_1 \triangle C_2)$, there is circuit $C_3 \subseteq (C_1 \cup C_2) - c$.
Part II

**Rank (WRA):** If $X \subset E$, $x < y \in E - X$, $y > \max(X)$, 
\[ \text{rank}(X) = \text{rank}(X \cup x) = \text{rank}(X \cup y) \] then \[ \text{rank}(X) = \text{rank}(X \cup \{x, y\}). \]

**Greedy algorithm (WGA):** If $w$ is a weight on $E$ such that the minimal lex facet is a minimal weight facet, then the greedy algorithm chooses a minimal weight basis.
Independece of the axioms and separation of matroid properties

Each such axiom generates a different class of complexes.

Which axioms imply which matroid properties? Examples include:

- **WIA**: $B_0$ first lex basis. If $\text{rank}(X) = |X \cap B_0|$, then $\Delta|_X$ is pure.

- **WEA**: vertex decomposability, shellability and theory of internal activity (to be discussed more carefully later)

- **WEA**: Hopf algebra of WEA complexes. Contains natural isomorphic copy the Hopf algebra of matroids. Refines some of the fundamental invariants from the theory, e.g the Billera-Jia-Reiner quasisymmetric function.
More properties

Some more properties.

- **WCA**: gives external activity theory and (partial) fundamental circuit theory.
- **WRA**: allows for a reconstruction of the poset of flats in terms of a (partial) closure operator.
- **WEA + WCA**:
  - Tutte Polynomial universal wrt to (ordered) deletion-contraction.
  - Interpretation of the Tutte Polynomial in terms of internal and external activities.
  - nbc-complex is shellable and h-vector is a Tutte evaluation.
Other goal:

Shifted complexes are in many ways easier than matroids: they have more structure!

**Idea:** Transfer matroid problems into problems about shifted complexes, solve the simpler problem and hope it serves as a guideline for a general solution.
Conjecture (Stanley 77)

The $h$-vector of the independence complex of a matroid is a pure $O$-sequence.

Explanation of the conjecture: $f_i(\Delta) =$ number of faces of dimension $i$. Let $(h_0, \ldots, h_d)$ be given by the relation:

$$h(\Delta, t) = \sum_{j=0}^{d} h_j t^j = \sum_{j=0}^{d} f_{j-1} t^j (1 - t)^{d-j}$$

The conjecture predicts existence of family of monomials (multicomplex) closed under divisibility such that all maximal monomials have the same degree and there are exactly $h_i$ monomials of degree $i$. 
A note about shifted complexes

- $(d-1)$ dimensional shifted complexes with at most $n$ vertices are in bijection with order ideals in Young’s Lattice fitting in a $d \times (n-d)$ box.

- A facet with vertices $n \geq v_d > \cdots > v_1 \geq 1$, corresponds to the partition $n - d \geq \lambda_d \geq \cdots \geq \lambda_1 \geq 0$ with $\lambda_i = v_i - i$.

- **Theorem: (Klivans 02)** Shifted matroids are principal order ideals under this correspondence.
Example

Figure: The poset of facets of a shifted complex
Internal Activity for WEA complexes

From now on we work with complexes satisfying WEA. $B$ is a basis.

- We say $b$ is internally passive if there is $b' < b$ such that $(B - b) \cup \{b'\}$ is a facet.
- $IP(B)$ is the set of internally passive elements of $B$.
- (Björner for matroids 80’s)(S. 15+)

$$\sum_{j=0}^{d} h_j x^j = \sum_{B} x^{|IP(B)|}$$
Figure: The internally passive elements in blue.
Let $A$ be the smallest lexicographic basis. $F$ face disjoint of $A$. Consider $(A, \Gamma_F)$, $\Gamma_F := \{G \subseteq A \mid G \cup F \in \Delta\}$. It satisfies WEA.

**Theorem (Klee, S. 15, S. 15+)**

$B$ basis, $B - A = F$ and $G = A \cap B$. Then

$$IP(B) = F \cup IP(G, \Gamma_F)$$

Implies the following:

**Corollary (Klee, S. 15)**

$$h(\Delta, x) = \sum_{F \subset (E \setminus A) \text{ face}} x^{|F|} h(\Gamma_F, x)$$
A conjecture

This suggests a combinatorial approach to constructing multi complexes by induction.

**Conjecture (Klee, S. 14)**

For a WEA complexes there is a combinatorial way to construct a multicomplex consistent with the previous $h$-vector decomposition. In the matroid case the multicomplex should be pure.

**Theorem (Klee, S. 15)**

- For rank $d$ complexes it suffices to prove the conjecture for complexes with $|E| \leq 2d - 1$.
- Using the previous and a computer we can verify Stanley’s conjecture for rank 4 matroids.

**Remark:** The order on the set really matters for the proof to work.
For shifted complexes

**Theorem (S. 15+)**

Such monomials can be constructed for shifted complexes and provides an alternative proof of Stanley’s conjecture for shifted (Schubert) matroids.

To solve it we use Young’s lattice and a fun game in diagrams. The original proof for shifted matroids is due to Schweig.
Figure: Support of the intended monomial in red and the internally passive elements in blue.
Constructing the monomial

- **Facet** $\iff$ **partition** $\lambda = \lambda_d \geq \cdots \geq \lambda_1$ that fits into a $d \times (n - d)$ box.
- **Monomial**: $m_{\lambda}$.
- **Variables of** $m_{\lambda} = \{\lambda_i + i\} - [d]$. (Variables determined by Durfree square)
- **The degree of** $m_{\lambda} = \text{number of rows of } \lambda$.
- **Remove roads of Durfree square. Get Ferrers board inside** $(d - d_{\lambda}) \times d_{\lambda}$. Construct inductively for it, shift variables, paste and multiply by support.
Figure: The monomials corresponding to each facet
Some remarks

- The poset structure of the lattice is the key to seeing that the approach works. Replace Young’s lattice by Gale order. Removing top facets leaves matroid theory, but preserves several axioms!
- We seem to need a strong enough inductive hypothesis!
A question

Question

Is this construction of monomials already in the literature? Is it relevant for any of the algebraic/geometric structures that are indexed by partitions?
THANK YOU VERY MUCH