NOTES ON LINEAR INDEPENDENCE FOR VECTORS OF FUNCTIONS

Section 7.4 in the book is perhaps the most important in this course, but it is given a rather short treatment of the subject. The goal of these notes is to provide more details and further examples in order to help you study.

1. Linear independence

In these notes, we deal with vectors of length $n$ whose entries are functions of one variable $t$ that are defined on some open interval $(\alpha, \beta) = \{t \in \mathbb{R} | \alpha < t < \beta\}$.

Usually we have a finite collection of such things, and such a collection will be denoted by $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(r)}(t)$.

This notation is meant to indicate that we have $r$ vectors of functions, and each of the vectors on the list looks like

$$
\mathbf{x}^{(j)}(t) = \begin{pmatrix}
x_{1j}(t) \\
x_{2j}(t) \\
\vdots \\
x_{nj}(t)
\end{pmatrix},
$$

where each entry $x_{ij}(t)$ is a function in the variable $t$ defined on $(\alpha, \beta)$.

**Example 1.1.** Let $n = 2$ and $r = 3$. Then

$$
\begin{pmatrix}
t \\
t^2
\end{pmatrix},
\begin{pmatrix}
1 \\
e^t
\end{pmatrix},
\begin{pmatrix}
1/t \\
0
\end{pmatrix}
$$

is a collection of vectors of functions defined on the interval $(0, +\infty)$. Here, $x_{11}(t) = t$, $x_{12}(t) = 1$, $x_{22}(t) = e^t$, etc.

**Definition 1.2.** The vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t)$ are said to be linearly independent if the only way to write

$$
\lambda_1 \mathbf{x}^{(1)}(t) + \lambda_2 \mathbf{x}^{(2)}(t) + \cdots + \lambda_r \mathbf{x}^{(r)}(t) = \mathbf{0}
$$

(where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are scalars in $\mathbb{R}$) is with $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

It is important to note that the vector on the right in the above expression is the zero vector of functions, that is, each of its entries is the function that assigns to each $t \in (\alpha, \beta)$ the scalar $0 \in \mathbb{R}$. 

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Example 1.3. The vectors
\[
\begin{pmatrix} t \\ t^2 \end{pmatrix}, \begin{pmatrix} 1 \\ e^t \end{pmatrix}
\]
are linearly independent. To see this, suppose that there is an equation
\[
\lambda_1 \begin{pmatrix} t \\ t^2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
where \(\lambda_1\) and \(\lambda_2\) are scalars. We need to show that \(\lambda_1 = \lambda_2 = 0\). Note that the above equation must hold for all \(t\). In particular, it holds for \(t = 1\). Evaluating at \(t = 1\) gives us the equation
\[
\lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
This may be written as the matrix equation
\[
\begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Because the matrix
\[
\begin{pmatrix} 1 & 1 \\ 1 & e \end{pmatrix}
\]
is non-singular (its determinant is non-zero), we see that the only solution to the matrix equation is \(\lambda_1 = \lambda_2 = 0\).

Example 1.4. The vectors
\[
\begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}, \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} 3e^{-t} \\ 0 \end{pmatrix}
\]
are linearly dependent. Indeed, we have
\[
3 \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix} - 6 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + 1 \begin{pmatrix} 3e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Sometimes it is necessary to talk about what happens at a particular value of \(t_0 \in (\alpha, \beta)\).

Definition 1.5. Let \(t_0 \in (\alpha, \beta)\) be a fixed real number. The vectors \(\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t)\) are said to be linearly independent at \(t_0\) if the vectors \(\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \ldots, \mathbf{x}^{(r)}(t_0)\) are linearly independent as vectors in \(\mathbb{R}^n\). We say that \(\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t)\) are pointwise linearly independent if they are linearly independent at every value of \(t_0 \in (\alpha, \beta)\).

Here are two facts that one should convince one’s self of.

- If \(\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t)\) are linearly dependent, then they are always pointwise linearly dependent.
- If \(\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t)\) are linearly independent at some point \(t_0 \in (\alpha, \beta)\), then they must be linearly independent.
In fact, these are equivalent statements. Caution should be taken, however, as their converses do not hold.

**Example 1.6.** This is an example of a collection of vectors that is linearly independent but pointwise linearly dependent.

The vectors

\[
\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \begin{pmatrix} t \\ t^2 \end{pmatrix}
\]

are linearly independent. Indeed, if

\[
\lambda_1 \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} + \lambda_2 \begin{pmatrix} t \\ t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

then evaluating at \( t = 0 \) we have

\[
\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

so that \( \lambda_1 = 0 \). On the other hand, evaluating at \( t = 1 \) yields

\[
\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Both rows give us the equation \( \lambda_1 + \lambda_2 = 0 \). From this and \( \lambda_1 = 0 \) from above, one concludes that \( \lambda_1 = \lambda_2 = 0 \) so that the vectors are indeed linearly independent.

A common misconception is that these vectors must then be pointwise linearly independent. In fact, they are not linearly independent at any point \( t_0 \in \mathbb{R} \), i.e., they are pointwise linearly dependent! To see why, pick any point \( t_0 \in \mathbb{R} \) and suppose that

\[
\lambda_1 \begin{pmatrix} 1 \\ t_0 \\ t_0^2 \end{pmatrix} + \lambda_2 \begin{pmatrix} t_0 \\ t_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Again, this may be written as the matrix equation

\[
\begin{pmatrix} 1 & t_0 \\ t_0 & t_0^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The \( 2 \times 2 \) matrix on the left has determinant zero, so it is singular. This means that there is a non-zero solution \((\lambda_1, \lambda_2)^t\) to this equation.

The reason the above phenomenon happens is that there is a linear dependence relation at each point \( t_0 \in \mathbb{R} \), but the coefficients \( \lambda_1, \lambda_2 \) cannot be chosen to simultaneously satisfy the above equation for all values of \( t_0 \). This latter condition is what it really means for a collection of vectors to be linearly dependent.
2. The Wronskian

Some of the above information can be dealt with directly via a special function.

**Definition 2.1.** Let $x^{(1)}(t), \ldots, x^{(n)}(t)$ be a collection of $n$ vectors of $n$ functions, i.e., this is the case $r = n$ from the previous section. We may form an $n \times n$ matrix of functions $X(t)$ by placing the vector $x^{(j)}(t)$ in the $j$th column. In other words, we have

$$X(t) = \begin{pmatrix} x^{(1)}(t) & x^{(2)}(t) & \cdots & x^{(n)}(t) \end{pmatrix}.$$ 

Using the notation from the previous section, this means that

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}.$$ 

The **Wronskian** of $x^{(1)}(t), \ldots, x^{(n)}(t)$ is defined to be the function

$$W[x^{(1)}, \ldots, x^{(n)}](t) = \det X(t).$$ 

The properties of determinants ensure that this is indeed a function of $t$. For simplicity, when the collection of vectors is obvious from the context, we simply use $W(t)$ to denote its Wronskian.

**Example 2.2.** The Wronskian of the vectors in Example 1.3 is

$$W(t) = \det X(t) = \begin{vmatrix} t & 1 & \varepsilon^t \\ t^2 & e^t & t e^t - t^2 \end{vmatrix} = t e^t - t^2.$$ 

**Proposition 2.3.** The vectors $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly independent at $t_0 \in (\alpha, \beta)$ if and only if $W(t_0) \neq 0$.

**Proof.** The vectors $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly independent at the point $t_0$ if and only if the equation

$$\lambda_1 x^{(1)}(t_0) + \cdots + \lambda_n x^{(n)}(t_0) = 0$$

has only the zero solution $\lambda_1 = \cdots = \lambda_n = 0$. Writing this as a matrix equation, that happens if and only if

$$\begin{pmatrix} x_{11}(t_0) & \cdots & x_{1n}(t_0) \\ \vdots & \ddots & \vdots \\ x_{n1}(t_0) & \cdots & x_{nn}(t_0) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a unique solution, which happens if and only if the $n \times n$ matrix on the left is non-singular, which happens if and only if its determinant is non-zero. But this matrix is precisely $X(t_0)$. In other words, $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly independent at $t = t_0$ if and only if $W(t_0) = \det X(t_0) \neq 0$. 

**Corollary 2.4.** If there exists a point $t_0 \in (\alpha, \beta)$ for which $W[x^{(1)}, \ldots, x^{(n)}](t_0) \neq 0$, then $x^{(1)}(t), \ldots, x^{(n)}(t)$ are linearly independent.
Proof. By the proposition, the condition $W(t_0) \neq 0$ is equivalent to $x^{(1)}(t), \ldots, x^{(n)}(t)$ being linearly independent at $t_0$. But if these vectors are linearly independent at $t_0$, then they must be linearly independent (by a statement in the previous section which you were asked to prove).

A cautionary example is that given in Example 1.6. There, the Wronskian is zero at every value of $t_0$, which automatically tells us that the vectors are pointwise linearly dependent. On the other hand, they were shown to be linearly independent. This illustrates that, although the Wronskian may vanish for a collection of vectors, they might still be linearly independent.

The real value of the Wronskian will be illustrated in the following section.

3. Homogeneous systems of linear differential equations

Our goal is to understand solutions to the homogeneous system of equations

\[
\begin{pmatrix}
x'_1 \\
x'_2 \\
\vdots \\
x'_n
\end{pmatrix} = \begin{pmatrix}
p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\
p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t)
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix},
\]

where the functions $p_{ij}(t)$ are assumed to be continuous on some open interval $(\alpha, \beta)$.

Under these hypotheses, we know that if $x^0 = (x^0_1, x^0_2, \ldots, x^0_n)^t$ is any vector in $\mathbb{R}^n$ and $t_0$ is a chosen point in $(\alpha, \beta)$, then there exists a unique solution $x(t)$ to the equation (1) satisfying the condition that $x(t_0) = x^0$.

For convenience, we rewrite the equation (1) as $x' = P(t)x$.

Proposition 3.1.

1. If $x^{(1)}(t)$ and $x^{(2)}(t)$ are solutions to $x' = P(t)x$, then so is $x^{(1)}(t) + x^{(2)}(t)$.
2. If $x(t)$ is a solution to $x' = P(t)x$ and $c \in \mathbb{R}$ is any scalar, then $cx(t)$ is also a solution.

In summary, if $x^{(1)}(t), \ldots, x^{(r)}(t)$ are solutions to $x' = P(t)x$ and $c_1, \ldots, c_r \in \mathbb{R}$ are any scalars, then

\[c_1x^{(1)}(t) + \cdots + c_rx^{(r)}(t)\]

is also a solution.

The previous proposition (which you should prove!) shows, in effect, that the set of solutions to $x' = P(t)x$ forms what is called a vector space, that is, a set endowed with an addition operation and scalar multiplication satisfying some usual properties. One can show (in fact we will) that this is a finite dimensional vector space. For such a vector space, one can always find a basis, i.e., a linearly independent spanning set. Our job will be to determine the dimension of the vector space of solutions to $x' = P(t)x$, which is defined to be the number of vectors in any of its bases.
Proposition 3.2. If \( x^{(1)}(t), \ldots, x^{(n)}(t) \) are pointwise linearly independent solutions to \( x' = P(t)x \) (note that there are precisely \( n \) vectors here), then any solution to \( x' = P(t)x \) can be written uniquely as a linear combination
\[
c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t)
\]
with scalars \( c_1, \ldots, c_n \in \mathbb{R} \).

Proof. Let \( \phi(t) \) be any solution to \( x' = P(t)x \). We must find scalars \( c_1, \ldots, c_n \in \mathbb{R} \) such that \( \phi(t) = c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t) \). To this end, pick your favourite value \( t_0 \) in the interval \((\alpha, \beta)\) and consider \( \phi(t_0) \), which is a vector of scalars that we write as
\[
\phi(t_0) = x^0 = \begin{pmatrix} x_0^0 \\ x_0^1 \\ \vdots \\ x_0^n \end{pmatrix}.
\]

We first wish to find a solution to \( x' = P(t)x \) of the form
\[
x(t) = c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t)
\]
that satisfies the initial condition \( x(t_0) = x^0 \). This is equivalent to solving the equation
\[
c_1x^{(1)}(t_0) + \cdots + c_nx^{(n)}(t_0) = x^0
\]
for \( c_1, c_2, \ldots, c_n \). But this is equivalent to solving the system of linear equations
\[
c_1x_{11}(t_0) + \cdots + c_nx_{1n}(t_0) = x_1^0 \\
\vdots \\
c_1x_{n1}(t_0) + \cdots + c_nx_{nn}(t_0) = x_n^0
\]
for the variables \( c_1, \ldots, c_n \). This has a unique solution if and only if the (scalar) matrix
\[
\begin{pmatrix}
x_{11}(t_0) & \cdots & x_{1n}(t_0) \\
\vdots & \ddots & \vdots \\
x_{n1}(t_0) & \cdots & x_{nn}(t_0)
\end{pmatrix}
\]
is non-singular, i.e., if and only if its determinant is non-zero. But this is just the matrix \( X(t_0) \) defined in the previous section, so its determinant is \( W(t_0) \), which is non-zero since \( x^{(1)}(t), \ldots, x^{(n)}(t) \) are pointwise linearly independent. This shows that there is a unique solution \( c_1, \ldots, c_n \) for the equation \( c_1x^{(1)}(t_0) + \cdots + c_nx^{(n)}(t_0) = x^0 \).

What we have shown so far is that, given any solution \( \phi(t) \) to \( x' = P(t)x \), there always exists a solution to \( x' = P(t)x \) having the form \( c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t) \) that satisfies the same initial conditions that \( \phi(t) \) does, i.e., such that
\[
c_1x^{(1)}(t_0) + \cdots + c_nx^{(n)}(t_0) = \phi(t_0).
\]
But since the functions $p_{ij}(t)$ are continuous on the interval $(\alpha, \beta)$, we know that there exists a unique solution to $x' = P(t)x$ satisfying these initial conditions. This forces us to have
\[ c_1x^{(1)}(t) + \cdots + c_nx^{(n)}(t) = \phi(t), \]
which is what we needed to show. \qed

The previous result motivates the following definition.

**Definition 3.3.** If $x^{(1)}(t), \ldots, x^{(n)}(t)$ are pointwise linearly independent solutions to $x' = P(t)x$, then we say that they form a fundamental set of solutions to $x' = P(t)x$.

To keep things in context, we have shown that if there exists a fundamental set $x^{(1)}(t), \ldots, x^{(n)}(t)$ of solutions to $x' = P(t)x$, then any solution to $x' = P(t)x$ can be expressed as a linear combination of these vectors. Our next goal is to show that there always exists some fundamental set of solutions. We require a result that tells us more about the Wronskian of such vectors.

**Lemma 3.4.** If $x^{(1)}(t), \ldots, x^{(n)}(t)$ are any solutions to $x' = P(t)x$, then their Wronskian $W(t)$ is either the zero function on $(\alpha, \beta)$, or $W(t_0) \neq 0$ for all $t_0 \in (\alpha, \beta)$.

*Proof.* The proof given in the book is good enough. \qed

The previous proposition tells us is that if $x^{(1)}(t), \ldots, x^{(n)}(t)$ are solutions to $x' = P(t)x$ and we can find a point $t_0 \in (\alpha, \beta)$ at which they are linearly independent, then they must be pointwise linearly independent throughout $(\alpha, \beta)$. This is a very strong condition! It allows us to conclude the following.

**Proposition 3.5.** There always exists a fundamental set of solutions to the equation $x' = P(t)x$.

*Proof.* For each $1 \leq j \leq n$, let $e_j$ be the $j$th standard basis element in $\mathbb{R}^n$. In other words, $e_j$ is the column vector of length $n$ that has 1 in the $j$th row and zero everywhere else. For example, we have
\[
  e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{etc.}
\]

Fixing our favourite value of $t_0 \in (\alpha, \beta)$, we know that since the functions $p_{ij}(t)$ are continuous on $(\alpha, \beta)$, there exists for each $1 \leq j \leq n$ a unique solution $x^{(j)}(t)$ to the equation $x' = P(t)x$ satisfying the initial condition $x^{(j)}(t_0) = e_j$. We claim that the collection $x^{(1)}(t), \ldots, x^{(n)}(t)$ constructed in this way is a fundamental set of solutions to $x' = P(t)x$. 

To see this, let’s compute their Wronskian at \( t_0 \). We have
\[
W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0) = \det \left( \begin{array}{ccc} \mathbf{x}^{(1)}(t_0) & \cdots & \mathbf{x}^{(n)}(t_0) \end{array} \right) \\
= \det (\mathbf{e}_1 \cdots \mathbf{e}_n) \\
= \det I_n \\
= 1.
\]
Here, \( I_n \) denotes the \( n \times n \) identity matrix. Because \( W(t_0) \neq 0 \), the lemma tells us that \( W(t) \neq 0 \) for any value of \( t \in (\alpha, \beta) \). In other words, the vectors \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) are pointwise linearly independent on \((\alpha, \beta)\), which means that they form a fundamental set of solutions to \( \mathbf{x}' = P(t) \mathbf{x} \).

There is something that is not stated in the book but should be. Remember that, in general, the Wronskian is rather poor at telling us whether or not a set of vectors \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(r)}(t) \) is linearly dependent. But if \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) is a collection of solutions to \( \mathbf{x}' = P(t) \mathbf{x} \), then it really does tell us everything about their linear dependence/independence.

**Proposition 3.6.** Let \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) be a collection of solutions to the equation \( \mathbf{x}' = P(t) \mathbf{x} \). Then exactly one of the following holds.

1. The Wronskian of \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) is non-zero (thus it is never zero at any point in \((\alpha, \beta)\)), and in which case they form a fundamental set of solutions and are linearly independent.

2. Their Wronskian of \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) is the zero function, in which case they must be linearly dependent.

**Remark 3.7.** The only new information here is part (2). Recall that, in general, the Wronskian being zero does not always mean that a collection of vectors is linearly dependent. (See Example 1.6.)

**Proof of Proposition 3.6.** Suppose that the Wronskian \( W(t) \) of \( \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t) \) is identically zero and let \( t_0 \) be your favourite value in the interval \((\alpha, \beta)\). The fact that \( W(t_0) = 0 \) tells us that these vectors are linearly dependent at \( t_0 \); in other words, the vectors \( \mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(n)}(t_0) \) in \( \mathbb{R}^n \) must be linearly dependent. This means that at least one of them, say \( \mathbf{x}^{(j)}(t_0) \), must be a linear combination of the others. We may therefore write
\[
\mathbf{x}^{(j)}(t_0) = \sum_{k \neq j} c_k \mathbf{x}^{(k)}(t_0)
\]
for some scalars \( c_k \in \mathbb{R} \), where \( k \) runs through all values \( 1 \leq k \leq n \) for which \( k \neq j \). But this shows that the vectors \( \mathbf{x}^{(j)}(t) \) and \( \sum_{k \neq j} c_k \mathbf{x}^{(k)}(t) \) are solutions to \( \mathbf{x}' = P(t) \mathbf{x} \) satisfying the same initial conditions at the point \( t = t_0 \). Because the functions \( p_{ij}(t) \) are continuous on \((\alpha, \beta)\), we know that there is only one such solution. This forces us to have
\[
\mathbf{x}^{(j)}(t) = \sum_{k \neq j} c_k \mathbf{x}^{(k)}(t).
\]
This shows that $x^{(j)}(t)$ is a linear combination of the other vectors in the list

$$x^{(1)}(t), \ldots, x^{(n)}(t),$$

so the collection must be linearly dependent.

\[\square\]

**Corollary 3.8.** If $x^{(1)}(t), \ldots, x^{(n)}(t)$ are solutions to the equation $x' = P(t)x$, then they form a fundamental set of solutions if and only if they are linearly independent. In other words, for solutions to equations of the form $x' = P(t)x$, the notions of linear independence and pointwise linear independence coincide.

**Proof.** If $x^{(1)}(t), \ldots, x^{(n)}(t)$ are pointwise linearly independent, then they are automatically linearly independent. On the other hand, if they are linearly independent, then by Proposition 3.6, their Wronskian is never zero on $(\alpha, \beta)$, so they are pointwise linearly independent.

The previous proposition is very powerful. In the future, we will be concerned with finding all solutions (i.e., a fundamental set of solutions) to the equation $x' = P(t)x$ in the case where each function $p_{ij}(t)$ is a scalar function. If we find some set of solutions $x^{(1)}(t), \ldots, x^{(n)}(t)$ to this equation, we will therefore need to verify that it is indeed a fundamental set. Proposition 3.6 tells us that this is true if and only if the Wronskian $W[x^{(1)}, \ldots, x^{(n)}](t)$ is not the zero function.

### 4. Summary

The following summarises the salient points of these notes.

- You should know the definition of linear independence for a collection of vectors of functions.
- You should know the definition of linear independence at a point $t_0 \in (\alpha, \beta)$, and the definition of pointwise linear independence throughout the interval $(\alpha, \beta)$.
- You should know how to show whether or not a given collection of vectors satisfies the above properties.
- You should know how to compute the Wronskian of a collection of $n$ vectors of functions having length $n$.
- You should understand that if $x^{(1)}(t), \ldots, x^{(n)}(t)$ is any collection of vectors (not necessarily solutions to a differential equation), then the Wronskian really only tells you that if $W(t)$ is not identically zero on $(\alpha, \beta)$, then the collection is linearly independent.
- You should know the definition of a fundamental set of solutions to the equation $x' = P(t)x$.
- You should understand a bit about why fundamental sets of solutions always exist.
- You should know that if $x^{(1)}(t), \ldots, x^{(n)}(t)$ are solutions to $x' = P(t)x$, then their Wronskian $W(t)$ is either the zero function on $(\alpha, \beta)$ or $W(t_0) \neq 0$ at all points $t_0 \in (\alpha, \beta)$. 
• You should know that if $x^{(1)}(t), \ldots, x^{(n)}(t)$ are solutions to $x' = P(t)x$, then they form a fundamental set of solutions if and only if their Wronskian is not the zero function.