2010 Problems

1. The happy Sunday vacationer got into his rented boat and headed up the river – against the direction of its flow – with the motor wide open. He had traveled upriver one mile when his hat blew off. Unconcerned, he continued his trip. Ten minutes later, he remembered that his return railroad ticket was under the hat band. Turning around immediately, he headed downriver with the motor still wide open and recovered his hat exactly at his starting point. How fast was the river flowing?

2. Suppose the medians of a triangle are proportional to the corresponding sides. Prove the triangle is equilateral.

3. The positive integers from 1 to 100 are arranged in some random order along a circle. The sum of every three consecutively arranged numbers is calculated. Prove that there exist two such sums whose difference is at least 3.

4. Prove that
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\cdots+\frac{1}{2009}-\frac{1}{2010}=\frac{1}{1006}+\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2010}.$$

5. Evaluate
$$\cos^2 1^\circ+\cos^2 2^\circ+\cdots+\cos^2 90^\circ.$$

6. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers. Suppose this sequence has the property that $a_{a_n}+a_n=2n$ for all $n\geq 1$. Prove that $a_n=n$ for all n.

7. Suppose that M is the midpoint of side AB of square $ABCD$. Let P and Q be the points of intersection of the line MD with the circle with center M and radius MA, where P is inside the square and Q is outside the square. Prove that
$$\frac{PB}{PA}=\frac{1+\sqrt{5}}{2}.$$

8. Let a, b, c be positive real numbers so that $a+b+c=1$. Prove that
$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\geq 64.$$