Summer Institute for Mathematics at the University of Washington

2011 Problems

1. Let x be a positive real number. Prove that

$$\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}} \geq 1,$$

where $[x]$ is the integer part of x and $\{x\}$ is the fractional part.

2. A drawer has d more black socks than white socks. Suppose that if two socks are selected at random then the probability that they match is $1/2$. How many socks of each color are there?

3. Prove that

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) < e^2 + \pi^2.$$

4. A sequence of integers is defined as follows. Starting with $n = 1$, list all the multiples of n up to n^2. Thus, the sequence starts with the multiples of 1 up to 1, followed by the multiples of 2 up to 4, then the multiples of 3 up to 9, and so on, so that its first few terms are $1, 2, 4, 3, 6, 9, 4, 8, 12, 16$. What is the 2011th term in the sequence?

5. Let a, b, c be positive real numbers and let $0 < m < \frac{1}{4}$. Prove that at least one of the following equations has real roots.

$$ax^2 + bx + cm = 0$$
$$bx^2 + cx + am = 0$$
$$cx^2 + ax + bm = 0.$$

6. Let A, B, C be the angles of a triangle. Prove that

$$\sin A + \sin B \sin C \leq \frac{1+\sqrt{5}}{2}.$$

7. Let a, b, c be the length of sides opposite angles A, B, C in triangle ABC. Prove that

$$\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} < \frac{a^2 + b^2 + c^2}{2abc}.$$

8. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$