Summer Institute for Mathematics at the University of Washington

2012 Problems

1. Assume that x, y, z are numbers with $x + y + z = 1$ and $0 < x, y, z < 1$. Prove that

$$\sqrt{\frac{xy}{z+xy}} + \sqrt{\frac{yz}{x+yz}} + \sqrt{\frac{zx}{y+zx}} \leq \frac{3}{2}.$$

2. A polynomial $p(x)$ satisfies $p(5-x) = p(5+x)$ for all real x. Suppose $p(x) = 0$ has four distinct real roots. Find the sum of these roots.

3. A person cashes a check at a bank. By mistake the teller pays the number of cents as dollars and dollars as cents. The person spends $3.50 before noticing, then later finds that the remaining money is exactly double the amount of the check. What was the amount?

4. Let a, b, c be integers such that 2005 divides both $ab+9b+81$ and $bc+9c+81$. Prove that 2005 also divides $ca + 9a + 81$.

5. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain in pieces in such a way that each one gets k white beads and m black beads. What is the minimum number of cuts that is sufficient? Describe an algorithm to make the cuts.

6. Find all solutions of the system

$$x + \log(x + \sqrt{x^2+1}) = y, \quad (1)$$
$$y + \log(y + \sqrt{y^2+1}) = z, \quad (2)$$
$$z + \log(z + \sqrt{z^2+1}) = x. \quad (3)$$

7. Find all solutions of $\cos x \cdot \cos 2x \cdot \cos 3x = 1$.

8. Let $ABCD$ be a unit square and mark off points E, F, G, H successively in sides AB, BC, CD, DA so that $AE = BF = CG = DH = \frac{2011}{2012}$. Prove that the region that is the intersection of triangles AGB, BHC, CED, DFA is a square with area

$$\frac{1}{2011^2 + 2012^2}.$$