Summer Institute for Mathematics at the University of Washington

2013 Problems

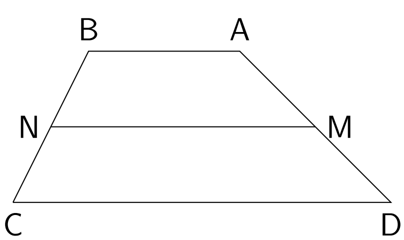
1. Jack and Jill agreed to a duel that would take place between noon and 1:00 pm. The rules are such that each duellist need only wait five minutes for his or her opponent to show up. Each duellist can leave with honor after waiting five minutes. Assuming each duellist does not appear until sometime between noon and 1:00 pm and leaves after five minutes or at 1:00 pm, whichever comes first, what is the probability of a duel taking place? Assume all arrival times of the duellists are equally likely and independent of each other.

2. How many distinct integers are there in the set

$$\left\lfloor \frac{1^2}{2013} \right\rfloor, \left\lfloor \frac{2^2}{2013} \right\rfloor, \ldots, \left\lfloor \frac{2013^2}{2013} \right\rfloor,$$

   where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let M, N be points on the lines AD and BC, respectively, such that $MN \parallel AB$.



   Prove that

$$DC \cdot MA + AB \cdot MD = MN \cdot AD.$$

4. The numbers 2^{2013} and 5^{2013} are written out in base 10 (decimal) notation one after another to create a single number. How many digits are there in the number? You must give an algebraic proof of your answer. The output of a calculator will not be accepted.

5. Prove that the roots of the polynomial $p(x) = a_{10}x^{10} + a_9x^9 + \ldots a_3x^3 + 3x^2 + 2x + 1$ are not all real.

6. Tom, Mary, and Jerry play three rounds of a game. At the conclusion of each round, the loser has to give each other player enough money to triple the holdings of that other player. It is permissible for a player to go into debt at some stage of the game (iou). The successive losers of the three rounds are Tom, then Mary, and finally Jerry. Each player ends with \$27. How much money did each person start with?

7. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}.$$

8. How many roots are there of the equation $\sin x = \dfrac{x}{100}$, where x is measured in radians?