Summer Institute for Mathematics at the University of Washington 2010 Solutions

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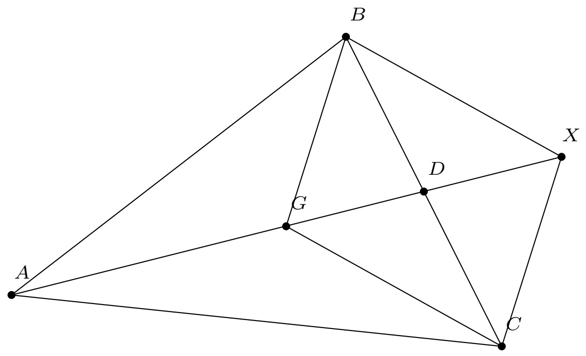
1. The happy Sunday vacationer got into his rented boat and headed up the river - against the direction of its flow - with the motor wide open. He had traveled upriver one mile when his hat blew off. Unconcerned, he continued his trip. Ten minutes later, he remembered that his return railroad ticket was under the hat band. Turning around immediately, he headed downriver with the motor still wide open and recovered his hat exactly at his starting point. How fast was the river flowing?

   Solution
   Suppose the river was flowing at the speed x miles per minute, and the net speed of the vacationer's boat with the motor wide open was y miles per minute.
   The time elapsed between his hat blew off and he recovering his hat is the time when the hat traveled along the river for 1 mile. Hence, the time elapsed is $\frac{1}{x}$ minute.
   The vacationer traveled against the river for 10 minutes, so he traveled for $10(y-x)$ miles against the river. Then it took him $\frac{1}{x} - 10$ minutes to get back to the starting point, so he traveled $(y+x)(\frac{1}{x} - 10)$ miles along the river. In this $\frac{1}{x}$ minutes, his net distance traveled is in fact 1 mile along the river, so $1 = (y+x)(\frac{1}{x} - 10) - 10(y-x)$. Solve it and we obtain $x = \frac{1}{20}$, so the river was flowing at $\frac{1}{20}$, or 0.05, miles per minute, or 3 miles per hour. ■

2. Suppose the medians of a triangle are proportional to the corresponding sides. Prove that the triangle is equilateral.

SOLUTION



Let ABC be the triangle, D, E, F be the midpoints of BC, CA, AB respectively, G be the centroid. Extend the line AGD so that $GD = DX$.

Since G is the centroid and AD is a median, $AG = 2GD = GD + DX = GX$. Hence,

$$\frac{GX}{BC} = \frac{AG}{BC}.$$

By our assumption, we know that

$$\frac{AD}{BC} = \frac{BE}{CA} = \frac{CF}{AB}.$$

Since G is the centroid, $AG = \frac{2}{3}AD$, $BG = \frac{2}{3}BE$, $CG = \frac{2}{3}CF$. Thus,

$$\begin{aligned}\frac{GX}{BC} = \frac{AG}{BC} = \frac{\frac{2}{3}AD}{BC} &= \frac{\frac{2}{3}BE}{CA} = \frac{BG}{CA} \\ &= \frac{\frac{2}{3}CF}{AB} = \frac{CG}{AB}.\end{aligned}$$

Since $GD = DX$ and $BD = DC$, $BGCX$ is a parallelogram. So $CG = XB$ and

$$\frac{XG}{BC} = \frac{BG}{AC} = \frac{CG}{AB} = \frac{BX}{AB}.$$

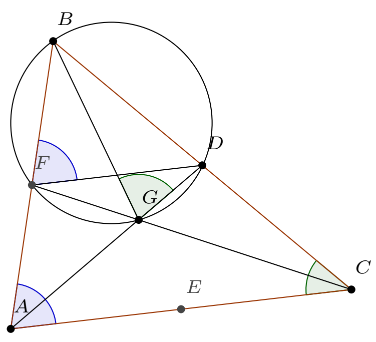
Since the three sides of ΔXBG and ΔBAC are proportional, we have $\Delta XBG \sim \Delta BAC$. Thus,

$$\angle BGD = \angle BGX = \angle BCA.$$

Similarly,

$$\angle CGE = \angle CAB, \qquad \angle AGF = \angle ABC.$$

Since $\angle AGF = \angle ABC = \angle FBD$, $BFGD$ is a cyclic quadrilateral.



Since $\angle BGD$ and $\angle BFD$ are angles in the same segment, they are equal. Thus,

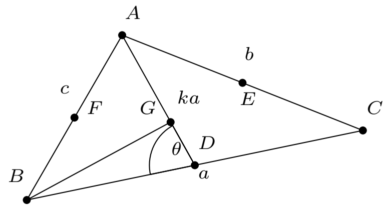
$$\angle ACB = \angle BGD = \angle BFD.$$

Since BD is parallel to AC, we have

$$\angle ACB = \angle BFD = \angle BAC.$$

Similarly, $\angle ABC = \angle ACB$, so ΔABC is equilateral. ■

Alternative Solution



Let ABC be the triangle, D, E, F be the midpoints of BC, CA, AB respectively, G be the centroid, a, b, c be the lengths of BC, CA, AB respectively, and ka, kb, kc be the lengths of the corresponding medians ($k > 0$ is a real number). Let θ be the angle ADB.

By Cosine Law on triangle ABD and triangle ACD,

$$\begin{aligned} c^2 &= \frac{a^2}{4} + k^2a^2 - ka^2\cos\theta, \\ b^2 &= \frac{a^2}{4} + k^2a^2 - ka^2\cos(180^\circ - \theta) = \frac{a^2}{4} + k^2a^2 + ka^2\cos\theta \end{aligned}$$

So

$$2(b^2 + c^2) = a^2(4k^2 + 1)$$

Similarly,

$$2(c^2 + a^2) = b^2(4k^2 + 1), \qquad 2(a^2 + b^2) = c^2(4k^2 + 1)$$

Sum all the three equaions up, we have

$$\begin{aligned} 4(a^2 + b^2 + c^2) &= (a^2 + b^2 + c^2)(4k^2 + 1) \\ 4k^2 = 3, & \qquad k = \frac{\sqrt{3}}{2} \end{aligned}$$

Since G is the centroid, $\frac{BG}{GE} = 2$, so $BG = \frac{2kb}{3} = \frac{b}{\sqrt{3}}$. Similarly, $GD = \frac{ka}{3} = \frac{a}{2\sqrt{3}}$. By Cosine Law on triangle GDB,

$$\begin{aligned} \frac{b^2}{3} &= \frac{a^2}{4} + \frac{a^2}{12} - \frac{2a^2}{\sqrt{3}}\cos\theta \\ b^2 &= a^2(1 - 2\sqrt{3}\cos\theta) \end{aligned}$$

By Cosine Law on triangle ADC,

$$\begin{aligned} b^2 &= \frac{a^2}{4} + \frac{3a^2}{4} + \frac{\sqrt{3}a^2}{2}\cos\theta \\ b^2 &= a^2\left(1 + \frac{\sqrt{3}}{2}\cos\theta\right) \end{aligned}$$

Thus,

$$\begin{aligned} a^2(1 - 2\sqrt{3}\cos\theta) &= a^2\left(1 + \frac{\sqrt{3}}{2}\cos\theta\right) \\ \left(\frac{\sqrt{3}}{2} + 2\sqrt{3}\right)\cos\theta &= 0 \\ \angle ADB = \theta &= 90^\circ \end{aligned}$$

So $AD \perp BC$. Similarly, $BE \perp CA$ and $CF \perp AB$, so triangle ABC is equilateral. ■

3. The positive integers from 1 to 100 are arranged in some random order along a circle. The sum of every three consecutively arranged numbers is calculated. Prove that there exist two such sums whose difference is at least 3.

SOLUTION

We will prove it by contradiction. Suppose any two such sums have difference at most 2. Let $a_1 = 1$ and a_n be the n^{th} number after 1, counting clockwisely.
By assumption, $|(a_2 + a_3 + a_4) - (a_1 + a_2 + a_3)| \leq 2$, so $|a_4 - a_1| \leq 2$, i.e. a_4 is 2 or 3. Similarly, a_{98} is 2 or 3. Without loss of generality (which means the other case(s) can be done in the exact same way), we suppose $a_4 = 2$, and $a_{98} = 3$.
Now we claim that $a_{3n+1} = 2n$ and $a_{101-3n} = 2n + 1$ for $1 \leq n \leq 33$. We will prove it by Strong Induction. The base case is true by assumption.
Suppose the claim is true for all $1 \leq k < n$. Consider the set $S = \{1, 2, 3, \cdots, 2n - 1\}$. These numbers are associated with a_l for some $l = 3m + 1$ or $101 - 3m$, where $1 \leq m < n$, by induction assumption. Clearly, $3n + 1 \neq 3m + 1$ and $101 - 3n \neq 101 - 3m$ for all $1 \leq m < n$. Also, $3a + 1$ is 1 modulo 3, where $101 - 3b$ is 2 modulo 3, so $3n + 1 \neq 101 - 3m$ and $101 - 3n \neq 3m + 1$ for all $1 \leq m < n$. Hence a_{3n+1} and a_{101-3n} cannot be in the set.
By assumption, $2 \geq |(a_{3n-1} + a_{3n} + a_{3n+1}) - (a_{3n-2} + a_{3n-1} + a_{3n})| = |a_{3n+1} - a_{3(n-1)+1}| = |a_{3n+1} - (2n-2)|$, so $a_{3n+1} \in \{2n-4, 2n-3, 2n-2, 2n-1, 2n\}$. However, the first four values are in S, so a_{3n+1} must be $2n$. Similarly, $2 \geq |(a_{101-3n} + a_{100-3n} + a_{99-3n}) - (a_{100-3n} + a_{99-3n} + a_{98-3n})| = |a_{101-3n} - a_{101-3(n-1)}| = |a_{101-3n} - (2n-1)|$, so $a_{101-3n} \in \{2n-3, 2n-2, 2n-1, 2n, 2n+1\}$. However, the first 3 values are in S, and $a_{3n+1} = 2n$, so a_{101-3n} must be $2n + 1$.

Hence, the claim is true by Strong Induction. By applying the claim to $n = 32$ and 33, we have $a_{97} = 64$, $a_5 = 65$, $a_{100} = 66$, $a_2 = 67$. Consider a_3. By assumption we have $2 \geq |(a_1 + a_2 + a_3) - (a_{100} + a_1 + a_2)| = |a_3 - a_{100}| = |a_3 - 66|$, so $a_3 \in \{64, 65, 66, 67, 68\}$. However, from above we see the first four numbers are associated with other a_n's, so $a_3 = 68$. Now, $|(a_{100} + a_1 + a_2) - (a_2 + a_3 + a_4)| = |(66 + 1 + 67) - (67 + 68 + 2)| = 3$, contradicting our assumption that the difference is at most 2.
Therefore, there exist two such sums with difference at least 3. ■

4. Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2009}-\frac{1}{2010}=\frac{1}{1006}+\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2010}.$$

Solution

$$\begin{aligned}
& 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2009}-\frac{1}{2010} \\
= \;& 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2009}+\frac{1}{2010}-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2010}\right) \\
= \;& 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2009}+\frac{1}{2010}-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1005}\right) \\
= \;& \frac{1}{1006}+\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2010}
\end{aligned}$$

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5. Evaluate

$$\cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ.$$

Solution

Note that $\cos(2n)^\circ = 2\cos^2 n^\circ - 1$, so $\cos^2 n^\circ = \frac{1+\cos(2n)^\circ}{2}$. Hence,

$$\begin{aligned}
& \cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ \\
= \; & \frac{\cos 2^\circ + 1}{2} + \frac{\cos 4^\circ + 1}{2} + \cdots + \frac{\cos 180^\circ + 1}{2} \\
= \; & \frac{1}{2}(\cos 2^\circ + \cos 4^\circ + \cdots + \cos 180^\circ) + 45
\end{aligned}$$

Note that $\sin(2k+1)^\circ - \sin(2k-1)^\circ = 2\cos(2k)^\circ \sin 1^\circ$, so $\cos(2k)^\circ = \frac{1}{2\sin 1^\circ}(\sin(2k+1)^\circ - \sin(2k-1)^\circ)$, and

$$\begin{aligned}
& \cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ \\
= \; & \frac{1}{4\sin 1^\circ}(2\sin 1^\circ \cos 2^\circ + 2\sin 1^\circ \cos 4^\circ + \cdots + 2\sin 1^\circ \cos 180^\circ) + 45 \\
= \; & \frac{1}{4\sin 1^\circ}[(\sin 3^\circ - \sin 1^\circ) + (\sin 5^\circ - \sin 3^\circ) + \cdots + (\sin 181^\circ - \sin 179^\circ)] + 45 \\
= \; & \frac{\sin 181^\circ - \sin 1^\circ}{4\sin 1^\circ} + 45 = \frac{-2\sin 1^\circ}{4\sin 1^\circ} + 45 = \frac{89}{2}
\end{aligned}$$

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Alternatively, note that $\cos\theta = \sin(90^\circ - \theta)$, so

$$\begin{aligned}
& \cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ \\
= \; & (\cos^2 1^\circ + \cos^2 89^\circ) + (\cos^2 2^\circ + \cos^2 88^\circ) + \cdots + (\cos^2 44^\circ + \cos^2 46^\circ) + \cos^2 45^\circ \\
= \; & (\cos^2 1^\circ + \sin^2 1^\circ) + (\cos^2 2^\circ + \sin^2 2^\circ) + \cdots + (\cos^2 44^\circ + \sin^2 44^\circ) + \frac{1}{2} \\
= \; & 44 + \frac{1}{2} = \frac{89}{2}
\end{aligned}$$

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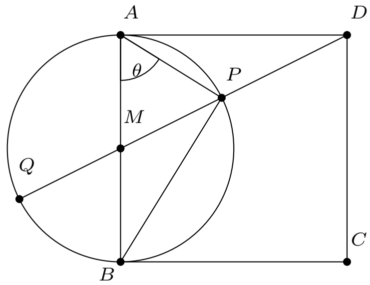
6. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers. Suppose this sequence has the property that $a_{a_n} + a_n = 2n$ for all $n \geq 1$. Prove that $a_n = n$ for all n.

   Solution
   We will prove it by Strong Induction.
   Base case: For $n = 1$, we have $a_{a_1} + a_1 = 2$. Since a_{a_1} and a_1 are positive integers, $a_{a_1} \geq 1$ and $a_1 \geq 1$, so $a_{a_1} + a_1 \geq 2$, equality holds iff $a_{a_1} = a_1 = 1$. Therefore, a_1 must be 1.
   Inductive Step: Suppose $a_k = k$ for all $1 \leq k < n$.
   If $a_n < n$, then $a_{a_n} = a_n$ by induction assumption. Then $2n = a_{a_n} + a_n = 2a_n$, implying $a_n = n$, contradicting our assumption that $a_n < n$. Hence, $a_n \geq n$.
   If $a_n > n$, then by $a_{a_n} + a_n = 2n$ we have $a_{a_n} < n$. By Induction assumption, we have $a_{a_{a_n}} = a_{a_n} < n$. Hence, $2n > a_{a_{a_n}} + a_{a_n} = 2a_n > 2n$, a contradiction. Thus, $a_n \leq n$. So we force $a_n = n$.
   Therefore, by Strong Induction, $a_n = n$ for all n. ■

7. Suppose that M is the midpoint of side AB of square $ABCD$. Let P and Q be the points of intersection of the line MD with the circle with center M and radius MA, where P is insider the square and Q is outside the square. Prove that

$$\frac{PB}{PA} = \frac{1+\sqrt{5}}{2}.$$

SOLUTION



Let $\theta = \angle PAB$. Since MA, MP are radii of the circle, $MA = MP$, so $\angle AMP = 180^\circ - 2\theta$. Consider the right triangle DAM. We have

$$\tan(180^\circ - 2\theta) = \frac{AD}{AM} = \frac{AB}{AM} = 2, \text{ so}$$

$$-2 = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Thus, $\tan^2\theta - \tan\theta - 1 = 0$. Since $0^\circ < \theta < 90^\circ$, we have $\tan\theta > 0$, so $\tan\theta = \frac{1+\sqrt{5}}{2}$.
Since AB is the diameter of the circle, $\angle APB = 90^\circ$, so

$$\frac{PB}{PA} = \tan\theta = \frac{1+\sqrt{5}}{2}.$$

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8. Let a, b, c be positive real numbers so that $a+b+c=1$. Prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64.$$

Solution
Since $1=a+b+c$, we have

$$\begin{aligned}
& \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \\
= & \left(1+\frac{a+b+c}{a}\right)\left(1+\frac{a+b+c}{b}\right)\left(1+\frac{a+b+c}{c}\right) \\
= & \frac{64}{abc}\left(\frac{a+a+b+c}{4}\right)\left(\frac{b+a+b+c}{4}\right)\left(\frac{c+a+b+c}{4}\right)
\end{aligned}$$

Note that AM-GM inequality states that for any $x, y, z, w \geq 0$, we have

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}.$$

So,

$$\begin{aligned}
& \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \\
= & \frac{64}{abc}\left(\frac{a+a+b+c}{4}\right)\left(\frac{b+a+b+c}{4}\right)\left(\frac{c+a+b+c}{4}\right) \\
\geq & \frac{64}{abc}\left(\sqrt[4]{a^2bc}\right)\left(\sqrt[4]{ab^2c}\right)\left(\sqrt[4]{abc^2}\right) \qquad \text{(AM-GM Inequality)} \\
= & \frac{64}{abc} \cdot (abc) = 64
\end{aligned}$$

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