Summer Institute for Mathematics at the University of Washington 2011 Solutions

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1. Let x be a positive real number. Prove that

$$\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}\geq 1,$$

where $[x]$ is the integer part of x and $\{x\}$ is the fractional part.

Solution

$$\begin{aligned}
&\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}} \\
= &\sqrt{\frac{[x]}{[x]+\{x\}+\{x\}}}+\sqrt{\frac{\{x\}}{[x]+\{x\}+[x]}} \\
= &\sqrt{\frac{n}{n+2t}}+\sqrt{\frac{t}{2n+t}},
\end{aligned}$$

where $n=[x]$ is a non-negative integer, $t=\{x\}$ lies in the interval $[0,1)$. Note that if $n=0$, we have $t>0$ since $x=n+t>0$, and the sum is 1. Similarly, if $t=0$, we have $n>0$ since $x=n+t>0$, and the sum is also 1. Now, suppose both n,t are non-zero. Let $u=t/n$. Manipulating the fraction, we have

$$\begin{aligned}
\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}} &= \sqrt{\frac{n}{n+2t}}+\sqrt{\frac{t}{2n+t}} \\
&= \sqrt{\frac{n/n}{(n/n)+2(t/n)}}+\sqrt{\frac{t/n}{2(n/n)+(t/n)}} \\
&= \sqrt{\frac{1}{1+2u}}+\sqrt{\frac{u}{2+u}}.
\end{aligned}$$

Let $F(u)$ be the expression above. Square F and we obtain

$$\begin{aligned}
(F(u))^2 &= \left(\sqrt{\frac{1}{1+2u}}+\sqrt{\frac{u}{2+u}}\right)^2 \\
&= \frac{1}{1+2u}+2\sqrt{\frac{1}{1+2u}\cdot\frac{u}{2+u}}+\frac{u}{2+u}
\end{aligned}$$

Since $F(u)$ is the sum of two square roots, it is non-negative. Therefore, $F(u)\geq 1$ is equivalent to $(F(u))^2\geq 1^2=1$, or

$$\frac{1}{1+2u}+2\sqrt{\frac{u}{(1+2u)(2+u)}}+\frac{u}{2+u}\geq 1$$

Note that $u=n/t>0$ (recall that we suppose $n,t>0$), so $(1+2u)(2+u)>0$. Hence we can multiply the above inequality by $(1+2u)(2+u)$ to get an equivalent statement:

$$(2+u)+2\sqrt{u(1+2u)(2+u)}+u(1+2u)\geq(1+2u)(2+u).$$

Grouping the terms, we get

$$2\sqrt{u(1+2u)(2+u)} \geq 3u.$$

Again, both sides of the inequality are positive, so by squaring it, we can get an equivalent statement:

$$4u(1+2u)(2+u) \geq 9u^2.$$

Expanding,

$$8u^3 + 11u^2 + 8u \geq 0,$$

which is true since $u > 0$ and $8u^2 + 11u + 8 = 8(u-1)^2 + 17u \geq 17u > 0$. Since going backward is valid (as argued in each step), the initial inequality is true. ■

Alternative Solution:
As previous solution,

$$
\begin{aligned}
& \sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}} \\
= & \sqrt{\frac{[x]}{[x]+\{x\}+\{x\}}}+\sqrt{\frac{\{x\}}{[x]+\{x\}+[x]}} \\
= & \sqrt{\frac{n}{n+2t}}+\sqrt{\frac{t}{2n+t}},
\end{aligned}
$$

where $n=[x]$ is a non-negative integer, $t=\{x\}$ lies in the interval $[0,1)$. Note that if $n=0$, $t>0$ since $x=n+t>0$, and the sum is 1.
Now, suppose n is a positive integer. Fix n and consider the function $f:[0,1)\to\mathbb{R}$ with $f(t)=\sqrt{\frac{n}{n+2t}}+\sqrt{\frac{t}{2n+t}}$. For $t>0$,

$$
\begin{aligned}
\frac{df}{dt} &= -\sqrt{\frac{n}{(n+2t)^3}}+\frac{1}{2}\sqrt{\frac{1}{t(2n+t)}}-\frac{1}{2}\sqrt{\frac{t}{(2n+t)^3}} \\
&= \frac{-2\sqrt{nt(2n+t)^3}+(2n+t)\sqrt{(n+2t)^3}-t\sqrt{(n+2t)^3}}{2\sqrt{t(n+2t)^3(2n+t)^3}} \\
&= \frac{-2(2n+t)\sqrt{nt(2n+t)}+2n(n+2t)\sqrt{n+2t}}{2(n+2t)(2n+t)\sqrt{t(n+2t)(2n+t)}} \\
&= \frac{n(n+2t)\sqrt{n+2t}-(2n+t)\sqrt{nt(2n+t)}}{(n+2t)(2n+t)\sqrt{t(n+2t)(2n+t)}}
\end{aligned}
$$

Note that $(n+2t)(2n+t)\sqrt{t(n+2t)(2n+t)}>0$ and

$$
\begin{aligned}
(n-t)^3(n+t) &\geq 0 \qquad \text{since } n\geq 1\geq t \\
n^4-2n^3t+2nt^3-t^4 &\geq 0 \\
n^4+6n^3t+12n^2t^2+8nt^3 &\geq 8n^3t+12n^2t^2+6nt^3+t^4 \\
n(n+2t)^3 &\geq t(2n+t)^3 \\
n^2(n+2t)^3 &\geq nt(2n+t)^3\geq 0 \\
n(n+2t)\sqrt{n+2t} &\geq (2n+t)\sqrt{nt(2n+t)} \\
n(n+2t)\sqrt{n+2t}-(2n+t)\sqrt{nt(2n+t)} &\geq 0.
\end{aligned}
$$

Hence, the function is strictly increasing when $t>0$. Thus, $f(t)\geq f(0)=1$ since f is continuous.
Therefore, $\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}\geq 1$. ■

2. A drawer has d more black socks than white socks. Suppose that if two socks are selected at random then the probability that they match is $\frac{1}{2}$. How many socks of each color are there?

Solution

Suppose it has n white socks and $n+d$ black socks. The probability that the first selected sock is white is $\frac{n}{2n+d}$. Given the first sock is white, the probability that the second selected sock is white is $\frac{n-1}{2n+d-1}$. Thus, the probability that the two selected socks are both white is $\frac{n(n-1)}{(2n+d)(2n+d-1)}$. Similarly, the probability that the two selected socks are both black is $\frac{(n+d)(n+d-1)}{(2n+d)(2n+d-1)}$. Since we know the probability that the socks match is $\frac{1}{2}$, we have

$$\begin{aligned}
\frac{n(n-1)+(n+d)(n+d-1)}{(2n+d)(2n+d-1)} &= \frac{1}{2} \\
2n(n-1)+2(n+d)(n+d-1) &= (2n+d)(2n+d-1) \\
2n^2-2n+2n^2+4nd+2d^2-2n-2d &= 4n^2+4nd+d^2-2n-d \\
d^2-d &= 2n
\end{aligned}$$

Therefore, there are $\frac{d(d-1)}{2}$ white socks and $\frac{d(d+1)}{2}$ black socks. ■

3. Prove that

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) < e^2 + \pi^2.$$

SOLUTION

We will first prove that if $x > 1$, then $\log(x-1)\log(x+1) < (\log x)^2$.

Note that $0 < \frac{x^2-1}{x^2} < 1$, so

$$\log \frac{x^2-1}{x^2} < 0$$

as log is strictly increasing. Also, $\log x > 0$ since $x > 1$, so

$$\log x \log \frac{x^2-1}{x^2} < 0.$$

Since $0 < \frac{x-1}{x} < 1 < \frac{x+1}{x}$, we have

$$\log \frac{x-1}{x} < 0 < \log \frac{x+1}{x}$$

as log is strictly increasing, so

$$\log \frac{x-1}{x} \log \frac{x+1}{x} < 0.$$

Since $\log(ab) = \log a + \log b$,

$$\begin{aligned}
\log x \log \frac{x^2-1}{x^2} + \log \frac{x-1}{x} \log \frac{x+1}{x} &< 0 \\
\log x \log \left[\frac{x-1}{x} \cdot \frac{x+1}{x}\right] + \log \frac{x-1}{x} \log \frac{x+1}{x} &< 0 \\
\log x \left(\log \frac{x-1}{x} + \log \frac{x+1}{x}\right) + \log \frac{x-1}{x} \log \frac{x+1}{x} &< 0 \\
(\log x)^2 + \log x \log \frac{x-1}{x} + \log x \log \frac{x+1}{x} + \log \frac{x-1}{x} \log \frac{x+1}{x} &< (\log x)^2 \\
\left(\log x + \log \frac{x-1}{x}\right)\left(\log x + \log \frac{x+1}{x}\right) &< (\log x)^2 \\
\log \left[x \cdot \frac{x-1}{x}\right] \log \left[x \cdot \frac{x+1}{x}\right] &< (\log x)^2 \\
\log(x-1)\log(x+1) &< (\log x)^2,
\end{aligned}$$

as we claimed above.

Note that $e^\pi > e > 1$, so by putting $x = e^\pi$, we have

$$\begin{aligned}
\log(e^\pi - 1)\log(e^\pi + 1) &< (\log(e^\pi))^2 \\
\frac{\log(e^\pi - 1)}{\log e} \frac{\log(e^\pi + 1)}{\log e} &< \left(\frac{\log(e^\pi)}{\log e}\right)^2 \\
\log_e(e^\pi - 1)\log_e(e^\pi + 1) &< (\log_e(e^\pi))^2 = \pi^2
\end{aligned}$$

Similarly, since $\pi^e > \pi > 1$, we have

$$\log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) < e^2.$$

Therefore,

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) < e^2 + \pi^2.$$

■

4. A sequence of integers is defined as follows. Starting with $n = 1$, list all the multiples of n up to n^2. Thus, the sequence starts with the multiples of 1 up to 1, followed by the multiples of 2 up to 4, then the multiples of 3 up to 9, and so on, so that its first few terms are $1, 2, 4, 3, 6, 9, 4, 8, 12, 16$. What is the 2011^{th} term in the sequence?

   SOLUTION
   We claim that the $\frac{n(n+1)}{2}$-th term is n^2. We will prove it by induction.
   For $n = 1$, we know the first term is 1. Suppose the $\frac{n(n+1)}{2}$-th term is n^2. Then the next term is $(n + 1)$, according to the rule of the sequence. Hence, we will have multiples of $(n+1)$ until $(n+1)^2$, which occurs after $(n+1)$' multiples of $(n+1)$, i.e. $(n+1)$ numbers after n^2. Hence, we know the $\frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$-th term is $(n + 1)^2$, as desired. By Principle of Mathematical Induction, the claim is true.
   Since $\frac{63 \cdot 64}{2} = 2016$, we know the 2016-th term is 63^2. Hence, the 2011-th term is $63(63 - 5) = 3654$. ■

5. Let a, b, c be positive real numbers and let $0 < m < \frac{1}{4}$. Prove that at least one of the equations has real roots.

$$\begin{aligned} ax^2 + bx + cm &= 0 \\ bx^2 + cx + am &= 0 \\ cx^2 + ax + bm &= 0 \end{aligned}$$

Solution

Without loss of generality, suppose $a \geq b \geq c > 0$. Then $a^2 \geq bc$, so

$$a^2 - 4bcm = a^2 - (4m)(bc) > a^2 - bc \geq 0.$$

Hence, $cx^2 + ax + bm = 0$ has real roots. Therefore, at least one of the three equations has real roots. ■

6. Let A,B,C be the angles of a triangle. Prove that

$$\sin A + \sin B \sin C \leq \frac{1+\sqrt{5}}{2}.$$

Solution

Recall the sum and product formula:

$$\sin x \sin y = \frac{-1}{2}[\cos(x+y) - \cos(x-y)].$$

Thus,

$$\begin{aligned}
\sin A + \sin B \sin C &= \sin A + \frac{-1}{2}[\cos(B+C) - \cos(B-C)] \\
&= \sin A + \frac{1}{2}[-\cos(180^\circ - A) + \cos(B-C)] \\
&= \sin A + \frac{1}{2}[\cos A + \cos(B-C)] \\
&= \sin A + \frac{\cos A}{2} + \frac{\cos(B-C)}{2} \\
&\leq \sin A + \frac{\cos A}{2} + \frac{1}{2} \\
&= \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\sin A + \frac{1}{\sqrt{5}}\cos A\right) + \frac{1}{2}
\end{aligned}$$

Let $\phi = \sin^{-1}\frac{1}{\sqrt{5}}$. Since $0^\circ < \phi < 90^\circ$, we have $\cos\phi = \sqrt{1-\sin^2\phi} = \frac{2}{\sqrt{5}}$. Observe that

$$\sin(A+\phi) = \sin A\cos\phi + \cos A\sin\phi = \sin A \cdot \frac{2}{\sqrt{5}} + \cos A \cdot \frac{1}{\sqrt{5}}.$$

Thus,

$$\begin{aligned}
\sin A + \sin B \sin C &\leq \frac{\sqrt{5}}{2}\sin(A+\phi) + \frac{1}{2} \\
&\leq \frac{\sqrt{5}}{2} + \frac{1}{2}
\end{aligned}$$

■

Alternative Solution:
Fix A. Let $f(B,C) = \sin B \sin C$. To maximize the sum above, we have to maximize f.

$$f(B,C) = \sin B \sin C = \frac{1}{2}(\cos(B-C) - \cos(B+C)) = \frac{1}{2}(\cos(B-C) - \cos(180^\circ - A))$$

Hence, to maximize f, we have to maximize $\cos(B-C)$. Since $0^\circ < B, C < 180^\circ$, $-180^\circ < B - C < 180^\circ$. In this range, $\cos(B-C)$ is maximized when $B - C = 0$. Thus, we want $B = C$ in order to maximize the expression.

Now we want to maximize $g(A) = \sin A + \sin^2 B = \sin A + \sin^2\left(\frac{180^\circ - A}{2}\right)$. Note that

$$g(A) = \sin A + \sin^2\left(\frac{180^\circ - A}{2}\right) = \sin A + \frac{1 - \cos\left(2 \cdot \frac{180^\circ - A}{2}\right)}{2} = \sin A + \frac{1 - \cos(180^\circ - A)}{2} = \sin A + \frac{1 + \cos A}{2}$$

$$g'(A) = \cos A - \frac{\sin A}{2},$$

which equals 0 iff $\cos A = \frac{\sin A}{2}$, or $\tan A = 2$. Let A_0 be such A in $[0^\circ, 180^\circ]$. We have $\sin A_0 = \frac{2}{\sqrt{5}}$, $\cos A_0 = \frac{1}{\sqrt{5}}$ by Pythagoras' Theorem.

$$g''(A) = -\sin A - \frac{\cos A}{2},$$

so $g''(A_0) = -\frac{\sqrt{5}}{2}$, which shows A_0 is a local maximum for g. Note that $g(A_0) = \frac{1+\sqrt{5}}{2}$, $g(0^\circ) = 1$, $g(180^\circ) = 0$ and A_0 is the only critical point for g in $[0^\circ, 180^\circ]$. This implies that A_0 is the maximum for g when $A \in [0^\circ, 180^\circ]$. Therefore, $\sin A + \sin B \sin C \leq g(A) \leq \frac{1+\sqrt{5}}{2}$. ■

7. Let a, b, c be the length of sides opposite angles A,B,C in triangle ABC. Prove that

$$\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} < \frac{a^2+b^2+c^2}{2abc}.$$

SOLUTION
Recall Sine Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \quad \text{where } R \text{ is the radius of the curcumcircle of triangle } ABC,$$

and Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{(and similar for the other two).}$$

Note that

$$\begin{aligned}
\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} &= \frac{\cos A(1-\sin^2 A)}{a} + \frac{\cos B(1-\sin^2 B)}{b} + \frac{\cos C(1-\sin^2 C)}{c} \\
&= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} - \frac{\sin^2 A\cos A}{a} - \frac{\sin^2 B\cos B}{b} - \frac{\sin^2 C\cos C}{c} \\
&= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} \\
&\quad - \left(\frac{\sin A}{a}\right)\sin A\cos A - \left(\frac{\sin B}{b}\right)\sin B\cos B - \left(\frac{\sin C}{c}\right)\sin C\cos C \\
&= \frac{a^2+b^2+c^2}{2abc} - \left(\frac{\sin A\cos A}{2R} + \frac{\sin B\cos B}{2R} + \frac{\sin C\cos C}{2R}\right) \\
&= \frac{a^2+b^2+c^2}{2abc} - \frac{1}{4R}(\sin 2A + \sin 2B + \sin 2C),
\end{aligned}$$

It remains to show that $\sin 2A + \sin 2B + \sin 2C$ is positive.
Recall the sum and product formulae:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Observe that

$$\begin{aligned}
\sin 2A + \sin 2B + \sin 2C &= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \\
&= 2\sin(180^\circ - C)\cos(A-B) + 2\sin C\cos(180^\circ - (A+B)) \\
&= 2\sin C[\cos(A-B) - \cos(A+B)] \\
&= 4\sin A\sin B\sin C > 0
\end{aligned}$$

since $0^\circ < A, B, C < 180^\circ$. Therefore, the result follows. ■

Alternative Solution:

Let R be the radius of the circumcircle of triangle ABC and $x = b+c-a$, $y = c+a-b$, $z = a+b-c$. Then $a = \frac{y+z}{2}$, $b = \frac{z+x}{2}$, $c = \frac{x+y}{2}$. By triangle inequality, $x, y, z > 0$. Now observe that

$$\begin{aligned}
&(x^2y^2 + xy^3 - y^3z - 2y^2z^2 + x^2z^2 + xz^3 - yz^3) \\
&(y^2z^2 + yz^3 - z^3x - 2z^2x^2 + y^2x^2 + yx^3 - zx^3) \\
&(z^2x^2 + zx^3 - x^3y - 2x^2y^2 + z^2y^2 + zy^3 - xy^3) \quad = \quad 0
\end{aligned}$$

Adding some positive terms to it,

$$\begin{aligned}
(x^2y^2 + xy^2z + xy^3 - y^3z + 2x^2yz + 2xyz^2 + 2xy^2z - 2y^2z^2 + x^2z^2 + xz^3 + xyz^2 - yz^3) & \\
+(y^2z^2 + yz^2x + yz^3 - z^3x + 2y^2zx + 2yzx^2 + 2yz^2x - 2z^2x^2 + y^2x^2 + yx^3 + yzx^2 - zx^3) & \\
+(z^2x^2 + zx^2y + zx^3 - x^3y + 2z^2xy + 2zxy^2 + 2zx^2y - 2x^2y^2 + z^2y^2 + zy^3 + zxy^2 - xy^3) & \quad > \quad 0 \\
(y^2 + 2yz + z^2)(x^2 + xz + xy - yz) + (z^2 + 2zx + x^2)(y^2 + xy + yz - xz) & \\
+(x^2 + 2xy + y^2)(z^2 + xz + yz - xy) & \quad > \quad 0 \\
(y+z)^2(x^2 + xz + xy - yz) + (z+x)^2(y^2 + xy + yz - xz) & \\
+(x+y)^2(z^2 + xz + yz - xy) & \quad > \quad 0 \\
\left(\frac{y+z}{2}\right)^2 \left[-\left(\frac{y+z}{2}\right)^2 + \left(\frac{z+x}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2\right] & \\
+\left(\frac{z+x}{2}\right)^2 \left[\left(\frac{y+z}{2}\right)^2 - \left(\frac{z+x}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2\right] & \\
+\left(\frac{x+y}{2}\right)^2 \left[\left(\frac{y+z}{2}\right)^2 + \left(\frac{z+x}{2}\right)^2 - \left(\frac{x+y}{2}\right)^2\right] & \quad > \quad 0
\end{aligned}$$

Recall that $a = \frac{y+z}{2}$, $b = \frac{z+x}{2}$, $c = \frac{x+y}{2}$. Thus,

$$a^2(-a^2 + b^2 + c^2) + b^2(a^2 - b^2 + c^2) + c^2(a^2 + b^2 - c^2) > 0$$

By Sine Law,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

so $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$. Hence,

$$\begin{aligned}
a^2(-4R^2\sin^2 A + 4R^2\sin^2 B + 4R^2\sin^2 C) + b^2(4R^2\sin^2 A - 4R^2\sin^2 B + 4R^2\sin^2 C) & \\
+c^2(4R^2\sin^2 A + 4R^2\sin^2 B - 4R^2\sin^2 C) & \quad > \quad 0 \\
a^2(-\sin^2 A + \sin^2 B + \sin^2 C) + b^2(\sin^2 A - \sin^2 B + \sin^2 C) & \\
+c^2(\sin^2 A + \sin^2 B - \sin^2 C) & \quad > \quad 0
\end{aligned}$$

Now we use the fact that $\sin^2\theta = 1 - \cos^2\theta$:

$$\begin{aligned}
a^2(-\cos^2 A + \cos^2 B + \cos^2 C - 1) + b^2(\cos^2 A - \cos^2 B + \cos^2 C - 1) & \\
+c^2(\cos^2 A + \cos^2 B - \cos^2 C - 1) & \quad < \quad 0 \\
\cos^2 A(b^2 + c^2 - a^2) + \cos^2 B(c^2 + a^2 - b^2) + \cos^2 C(a^2 + b^2 - c^2) & \quad < \quad a^2 + b^2 + c^2
\end{aligned}$$

By Cosine Law,

$$b^2 + c^2 - a^2 = 2bc\cos A \qquad \text{and similar for other 2.}$$

So,

$$\begin{aligned}
2bc\cos^3 A + 2ca\cos^3 B + 2ab\cos^3 C \quad &< \quad a^2 + b^2 + c^2 \qquad (2) \\
\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} \quad &< \quad \frac{a^2 + b^2 + c^2}{2abc}
\end{aligned}$$

Therefore, the inequality is true. ■

8. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Solution
We will first prove the following lemma:

Lemma. ***(Schur's inequality)*** *For any non-negative numbers p, q, r, we have*

$$p(p-q)(p-r) + q(q-p)(q-r) + r(r-p)(r-q) \geq 0.$$

Proof
Notice that the inequality in the lemma is symmetric w.r.t p, q, r (i.e. by exchanging any of those, the inequality is still preserved). Hence, without loss of generality, we can assume $p \geq q \geq r$. Then

$$\begin{aligned} p(p-q)(p-r) + q(q-p)(q-r) + r(r-p)(r-q) &= (p-q)[p(p-r) - q(q-r)] + r(p-r)(q-r) \\ &= (p-q)(p^2 - q^2 - pr + qr) + r(p-r)(q-r) \\ &= (p-q)^2(p+q-r) + r(p-r)(q-r) \\ &\geq 0 \end{aligned}$$

Thus, the lemma follows.

□

Let x, y, z be positive real numbers s.t. $\frac{x}{y} = a$, $\frac{y}{z} = b$, $\frac{z}{x} = c$. Such x, y, z exists, for instance, by $x = a$, $y = 1$, $z = \frac{1}{b}$. Substitute $p = xy$, $q = yz$, $r = zx$ into the lemma:

$$\begin{aligned} p(p-q)(p-r) + q(q-p)(q-r) + r(r-p)(r-q) &\geq 0 \\ x^2y^2(x-z)(y-z) + y^2z^2(y-x)(z-x) + z^2x^2(z-y)(x-y) &\geq 0 \\ (x^3y^3 + x^2y^2z^2 - x^3y^2z - x^2y^3z) + (y^3z^3 + x^2y^2z^2 - xy^3z^2 - xy^2z^3) & \\ +(z^3x^3 + x^2y^2z^2 - x^3yz^2 - x^2yz^3) &\geq 0 \\ x^3y^2z + x^2y^3z + xy^3z^2 + xy^2z^3 + x^3yz^2 + x^2yz^3 &\leq 3x^2y^2z^2 + x^3y^3 + y^3z^3 + z^3x^3 \end{aligned}$$

Since $x, y, z > 0$, we can divide both sides of the inequality by $x^2y^2z^2$.

$$\begin{aligned} \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z} &\leq 3 + \frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2} \\ a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &\leq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \end{aligned}$$

This proves the desired result. ■

Alternative Solution:
Let x, y, z be positive real numbers s.t. $\frac{x}{y} = a$, $\frac{y}{z} = b$, $\frac{z}{x} = c$ (See previous proof for justification of existance). We first claim that

$$3x^2y^2z^2 + x^3y^3 + y^3z^3 + z^3x^3 - (x^3y^2z + xy^3z^2 + x^2yz^3 + x^3yz^2 + x^2y^3z + xy^2z^3) \geq 0.$$

To see this, observe that the inequality is symmetric w.r.t. x, y, z. Hence without loss of generality, we can suppose $x \geq y \geq z$. Now, $x - y \geq 0$, so

$$\begin{aligned} (x-y)z &\geq 0 \\ xz - yz &\geq 0 \\ xy + xz - yz &> 0. \end{aligned}$$

$y - z \geq 0$, so

$$\begin{aligned} (y-z)(xy + xz - yz) &\geq 0 \\ xy^2 - zy^2 - xz^2 + yz^2 &\geq 0 \\ (x-z)y^2 - (x-y)z^2 &\geq 0 \end{aligned}$$

And, $x^2(y - z) \geq 0$, so

$$\begin{aligned} x^2(y-z)[(x-z)y^2 - (x-y)z^2] &\geq 0 \\ x^2y^2(z-x)(z-y) + x^2z^2(y-x)(y-z) &\geq 0 \end{aligned}$$

Now, $y^2z^2(x - y)(x - z) \geq 0$, so

$$\begin{aligned} y^2z^2(x-y)(x-z) + z^2x^2(y-x)(y-z) + x^2y^2(z-x)(z-y) &\geq 0 \\ (xy^2 - y^3)(xz^2 - z^3) + (yx^2 - x^3)(yz^2 - z^3) + (zx^2 - x^3)(zy^2 - y^3) &\geq 0 \\ 3x^2y^2z^2 + x^3y^3 + y^3z^3 + z^3x^3 - (x^3y^2z + xy^3z^2 + x^2yz^3 + x^3yz^2 + x^2y^3z + xy^2z^3) &\geq 0 \end{aligned}$$

Thus the inequality we claimed earlier is true. Now divide both sides by $x^2y^2z^2$, which is positive, and then substitute $a = \frac{x}{y}$, $b = \frac{y}{z}$, and $c = \frac{z}{x}$:

$$\begin{aligned} 3 + \frac{xz}{y^2} + \frac{xy}{z^2} + \frac{yz}{x^2} &\geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \\ 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} &\geq a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \end{aligned}$$

as desired. ■

(Remark: $y^2z^2(x - y)(x - z) + z^2x^2(y - x)(y - z) + x^2y^2(z - x)(z - y) \geq 0$ is a direct result from Vornicu-Schur Inequality (2007), by letting $f(x) = 1/x^2$. For those who are interested, please refer to `http://www.artofproblemsolving.com/Wiki/index.php/Vornicu-Schur_Inequality`.)