1. Assume that x, y, z are numbers with $x+y+z=1$ and $0<x,y,z<1$. Prove that

$$\sqrt{\frac{xy}{z+xy}}+\sqrt{\frac{yz}{x+yz}}+\sqrt{\frac{zx}{y+zx}}\le\frac{3}{2}$$

Solution:
Since $x+y+z=1$, we can multiply the single term in each denominator by $x+y+z$ and simplify:

$$\sqrt{\frac{xy}{z(x+y+z)+xy}}+\sqrt{\frac{yz}{x(x+y+z)+yz}}+\sqrt{\frac{zx}{y(x+y+z)+zx}}\le\frac{3}{2}$$

$$\sqrt{\frac{xy}{xz+yz+z^2+xy}}+\sqrt{\frac{yz}{x^2+xy+xz+yz}}+\sqrt{\frac{zx}{xy+y^2+yz+zx}}\le\frac{3}{2}$$

$$\sqrt{\frac{xy}{(x+z)(y+z)}}+\sqrt{\frac{yz}{(y+x)(z+x)}}+\sqrt{\frac{zx}{(x+y)(z+y)}}\le\frac{3}{2} \tag{1}$$

The AM-GM inequality for 2 variables tells us that

$$\frac{x_1+x_2}{2}=\sqrt{x_1x_2}$$

If

$$x_1=\frac{x}{x+z},x_2=\frac{y}{y+z}$$

then we have that

$$\sqrt{\frac{xy}{(x+z)(y+z)}}\le\frac{\frac{x}{x+z}+\frac{y}{y+z}}{2}$$

We can repeat this process similarly for the other two terms on the left hand side of (1). This gives us

$$\begin{aligned}\sqrt{\frac{xy}{(x+z)(y+z)}}+\sqrt{\frac{yz}{(y+x)(z+x)}}+\sqrt{\frac{zx}{(x+y)(z+y)}}&\le\frac{\frac{x}{x+z}+\frac{y}{y+z}}{2}+\frac{\frac{y}{y+x}+\frac{z}{z+x}}{2}+\frac{\frac{x}{x+y}+\frac{z}{z+y}}{2}\\&=\frac{\frac{x}{x+z}+\frac{z}{x+z}+\frac{y}{y+z}+\frac{z}{y+z}+\frac{x}{x+y}+\frac{y}{x+y}}{2}\\&=\frac{1+1+1}{2}=\frac{3}{2}\end{aligned}$$

as desired. □

2. A polynomial $p(x)$ satisfies $p(5-x) = p(5+x)$ for all real x. Suppose $p(x) = 0$ has four distinct real roots. Find the sum of these roots.

Solution:
Let r_1 be a root. Then

$$0 = p(r_1) = p(5 - (5 - r_1)) = p(5 + (5 - r_1)) = p(10 - r_1) = 0$$

and we have just discovered another root $r_2 = 10 - r_1$, with the property that $r_1 + r_2 = 10$. Since all roots of this polynomial are distinct, we can repeat the process to find the other pair of roots r_3 and r_4, also with the property that $r_3 + r_4 = 10$. So the sum of all the roots of this polynomial is $10 + 10 = 20$. □

3. A person cashes a check at a bank. By mistake the teller pays the number of cents as dollars and dollars as cents. The person spends \$3.50 before noticing, then later finds that the remaining money is exactly double the amount of the check. What was the amount?

   Solution:
   Define d as the number of dollars in the original check and c be the number of cents ($0 \leq c < 100$); both are integers. Then we know that

$$2(100d + c) + 350 = 100c + d$$

$$200d + 2c + 350 = 100c + d$$

$$98c - 199d = 350$$

$$c = \frac{350 + 199d}{98} = 3 + \frac{56 + 199d}{98} \tag{1}$$

   Since c must be integral, the fraction in (1) must be integral as well. That is equivalent to saying that $199d$ must be congruent to $98 - 56 = 42$, modulo 98. That is,

$$199d \equiv 42 \mod 98 \implies 3d \equiv 42 \mod 98$$

   and so d must be of the form $14 + 98n$, for some integer n. However if $n > 0$ then

$$c = 3 + \frac{56 + 199d}{98} > 100$$

   and so n must be 0, d must be 14, c must correspondingly be 32, and so the value of the original check is \$14.32. □

4. Let a, b, c be integers such that 2005 divides both $ab + 9b + 81$ and $bc + 9c + 81$. Prove that 2005 also divides $ca + 9a + 81$.

Solution:
Since both $ab + 9b + 81$ and $bc + 9c + 81$ are divisible by 2005, their product must be as well. Their product can also be rewritten as

$$(ab + 9b + 81)(bc + 9c + 81) = bc\underbrace{(ab + 9b + 81)}_{0 \mod 2005} + 9b(ac + 9a + 81) + 81\underbrace{(bc + 9c + 81)}_{0 \mod 2005} \equiv 0 \mod 2005$$

So it must be that

$$9b(ac + 9a + 81) \equiv 0 \mod 2005 \implies b(ac + 9a + 81) \equiv 0 \mod 2005$$

since 9 and 2005 are coprime.
We know that $ab + 9b + 81 = b(a+9) + 81$ is divisible by 2005. We claim that b cannot share any factor with 2005. Let's assume it does, say that factor is $n > 1$. Furthermore, let $b(a + 9) + 81 = 2005k$ for some integer k. Denote $b/n' = b'$ and $2005/n = m$, for some integers b', m. Then we have that

$$b(a + 9) + 81 + 2005k$$

$$\underbrace{b'(a + 9)}_{\text{integer}} + 81/n = \underbrace{mk}_{\text{integer}}$$

So $81/n$ must also be an integer. But this is impossible since 81 and 2005 share no common factors except for 1. So we have a contradiction, and thus b shares no factors with 2005.
Therefore, if $b(ac + 9a + 81) \equiv 0 \mod 2005$, and b shares no factor with 2005, then $ac + 9a + 81$ must be divisible by 2005, as desired. □

5. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain in pieces in such a way that each one gets k white beads and m black beads. What is the minimum numbers of cuts that is sufficient? Describe an algorithm to make the cuts.

   Solution:
   Claim: It is possible in 2 cuts.
   Obviously it's not possible in 0 cuts. We'll first show it's not possible in 1. Obviously with just 1 cut, the cut must be made in the middle of the chain so each piece is equal in size. Without loss of generality (WLOG), let $k > m$ and if beads are arranged like so,

   $$\text{BB}\ldots\text{BBWW}\ldots\text{WW—WW}\ldots\text{WW}$$

   where the dash represents the cut, it is clearly insufficient to make just one cut.
   It is now sufficient to show that we can also divide the chain with at most 2 cuts. First we will note the following:

   Of the chain of length $2k + 2m$, take a subsection of length $k + m$ starting from the beginning:

   $$\underbrace{\circ\circ\ldots\circ\circ}_{k+m}\circ\circ\ldots\circ\circ$$

   If this subsection just so happens to take k white beads and m black beads, we are done. If not, assume WLOG than it contains an insufficient amount of white beads, and therefore an excess of black beads. Thus it contains $k - \Delta k$ white beads and $m + \Delta k$ black beads, for some positive integral Δk. Then take the following sequence of subsections:

   $$\underbrace{\circ\circ\ldots\circ\circ}_{k+m}\circ\circ\ldots\circ\circ$$

   $$\circ\underbrace{\circ\circ\ldots\circ\circ}_{k+m}\circ\circ\ldots\circ$$

   $$\circ\circ\underbrace{\circ\circ\ldots\circ\circ}_{k+m}\circ\ldots\circ$$

   $$\ldots$$

   $$\circ\circ\ldots\circ\circ\underbrace{\circ\circ\ldots\circ\circ}_{k+m}$$

   The first subsection contained $k - \Delta k$ white beads, and last contains $2k - (k - \Delta k) = k + \Delta k$ white beads. Now, what is happening as we progress from one subsection to the next? We remove one bead from the set, and add another. Through this process, the amount of white beads can only increase by 1, stay the same, or decrease by 1.
   To recap, we started with $k - \Delta k$ white beads, and through a process of adding 1, subtracting 1, or doing nothing, we finished with $k + \Delta k$ white beads. Thus by discrete continuity, there must have been a subsection with exactly k white beads, and thus exactly m black beads.

   So we can always find a subsection of consecutive beads within the chain that is composed of exactly k white beads and m black beads. So our two cuts simply remove this section for one thief, and the remnants are equally satisfactory for the other. $\square$

6. Find all solutions of the system

$$x + \log\left(x + \sqrt{x^2+1}\right) = y \tag{1}$$

$$y + \log\left(y + \sqrt{y^2+1}\right) = z \tag{2}$$

$$z + \log\left(z + \sqrt{z^2+1}\right) = x \tag{3}$$

Solution:
We can add all the equations together and cancel $x + y + z$ to obtain

$$\log\left(x + \sqrt{x^2+1}\right) + \log\left(y + \sqrt{y^2+1}\right) + \log\left(z + \sqrt{z^2+1}\right) = 0 \tag{4}$$

Now assume x is positive. Then clearly

$$\log\left(x + \sqrt{x^2+1}\right) > \log\left(0 + \sqrt{0+1}\right) = 0$$

So the left hand side of (1) is positive, implying that y is positive as well. Similarly z is positive, since the left hand side of (2) is positive. So if x is positive, x, y, z are all positive.
Now assume x is negative. Replace $x = -m$. Then we can see that

$$\sqrt{m^2+1} - m > \sqrt{m^2} - m = 0$$

as well as

$$\sqrt{m^2+1} - m < \sqrt{(m^2+2m+1} - m = m + 1 - m = 1$$

Thus $\sqrt{m^2+1} - m$ lies between 0 and 1. Thus

$$\log\left(x + \sqrt{x^2+1}\right) = \log\left(\sqrt{m^2+1} - m\right) < 0$$

So by logic similar to what we used above, y is negative since the left hand side of (1) is negative, and consequently z is negative as well. Thus if x is negative, x, y, z are all negative.
So what have we shown? Assuming that x is either positive or negative, the variables are correspondingly all positive, or all negative. But this is clearly impossible, as seen by (4). If they were all positive, the left hand side of (4) would be strictly greater than 0, and similarly strictly less than 0 if all of them were negative. Thus the only possibility is that x is neither positive or negative, thus $x = 0$, and correspondingly $y = z = 0$. So $x = y = z = 0$ is the only solution. $\square$

7. Find all solutions of $\cos(x)\cos(2x)\cos(3x) = 1$.

Solution:

$$\cos(x)\cos(2x)\cos(3x) \leq |\cos(x)||\cos(2x)||\cos(3x)| \leq 1^3 = 1$$

with equality if and only if $|\cos(x)| = |\cos(2x)| = |\cos(3x)| = 1$. So for the three terms $\cos x, \cos 2x, \cos 3x$, there are only two combinations of numbers which satisfy our equation: (1,1,1) and $(-1,-1,1)$.
Note that $\cos(k\pi) = 1$ is 1 exactly when k is an even integer, -1 if k is an odd integer, and some value between the two otherwise.

Case 1: (1,1,1)
In this case, $x, 2x, 3x$ must all be even multiples of π. So k must be any even integer.

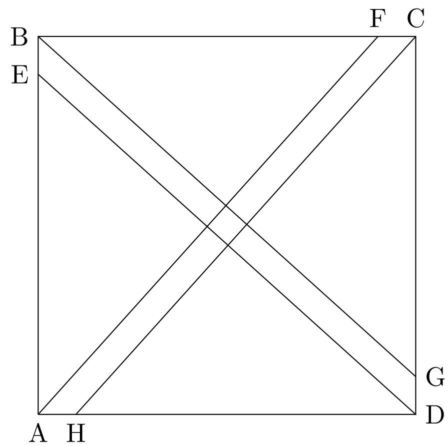
Case 2: $(-1,-1,1)$
In this case, $x, 2x, 3x$ must contains two odd multiples of π and one even one. Therefore k must be any odd integer.

Combining these two results gives us an answer of $x = k\pi$, where k is any integer. □

8. Let $ABCD$ be a unit square and mark off points E, F, G, H successively in sides AB, BC, CD, DA so that $AE = BF = CG = DH = 2011/2012$. Prove that the region that is the intersection of triangles AGB, BHC, CED, DFA is a square with area

$$\frac{1}{2011^2 + 2012^2}$$

Solution:



We can place this square on the coordinate plane, with A at the origin and C at $1, 1$. Then we have the lines

$$DE : y = -\frac{2011}{2012}x + \frac{2011}{2012}$$
$$BG : y = -\frac{2011}{2012}x + 1$$
$$AF : y = \frac{2012}{2011}x$$
$$CH : y = \frac{2012}{2011} - \frac{1}{2011}$$

We have that $BG \parallel DE$, $AF \parallel CH$, and $BG \perp AF$, so the region of intersection is at least a rectangle. However, it is not hard to see that this region is in fact a square, as the sides of this rectangle are equal to the distance between the two sets of lines. These distances are equal since the diagram possess rotational symmetry. Hence, to find the area of the desired square, we calculate the side length, and square it. We will approach this result in 3 steps:

(1) Find the point of intersection between DE and AF.

We are finding the intersection between the lines

$$y = \frac{2012}{2011}x \ , \ y = -\frac{2011}{2012}x + \frac{2011}{2012}$$

Setting y's equal gives

$$\frac{2012}{2011}x = -\frac{2011}{2012}x + \frac{2011}{2012}$$
$$x = \frac{\frac{2011}{2012}}{\frac{2011}{2012} + \frac{2012}{2011}} = \frac{2011^2}{2011^2 + 2012^2}$$

which gives

$$y = \frac{2012}{2011}x = \frac{2011 \times 2012}{2011^2 + 2012^2}$$

(2) Find the point of intersection between BG and AF.
We are finding the intersection between the lines

$$y = \frac{2012}{2011}x \ , \ y = -\frac{2011}{2012}x + 1$$

Setting y's equal gives

$$\frac{2012}{2011}x = -\frac{2011}{2012}x + 1$$

$$x = \frac{1}{\frac{2011}{2012} + \frac{2012}{2011}} = \frac{2011 \times 2012}{2011^2 + 2012^2}$$

which gives

$$y = \frac{2012}{2011}x = \frac{2012^2}{2011^2 + 2012^2}$$

(3) Calculate the distance between these points.
We are finding the distance between the points

$$\left(\frac{2011^2}{2011^2 + 2012^2}, \frac{2011 \times 2012}{2011^2 + 2012^2}\right) \ , \ \left(\frac{2011 \times 2012}{2011^2 + 2012^2}, \frac{2012^2}{2011^2 + 2012^2}\right)$$

To simplify calculation, we will factor the denominator out of both coordinates and multiply it back at the end. So our distance is equal to

$$\begin{aligned} D &= \frac{1}{2011^2 + 2012^2}\sqrt{(2011^2 - 2011 \times 2012)^2 + (2011 \times 2012 - 2012^2)^2} \\ &= \frac{1}{2011^2 + 2012^2}\sqrt{(-2011)^2 + (-2012)^2} \\ &= \frac{1}{\sqrt{2011^2 + 2012^2}} \end{aligned}$$

And since D is the side length of our square, its area is

$$\frac{1}{2011^2 + 2012^2}$$

as desired. □