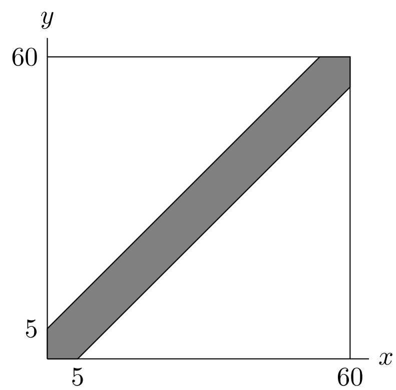
1. Jack and Jill agreed to a duel that would take place between noon and 1:00 pm. The rules are such that each duelist need only wait five minutes for his or her opponent to show up. Each duelist can leave with honor after waiting five minutes. Assuming each duelist does not appear until sometime between noon and 1:00 pm and leaves after five minutes or at 1:00 pm, whichever comes first, what is the probability of a duel taking place? Assume all arrival times of the duelists are equally likely and independent of each other.

   Solution:
   We can identify the problem with a coordinate plane. Let the x-axis denote the minutes after noon that Jack arrives, and let the y-axis denote that of Jill. Depending on Jack's arrival, Jill must either arrive at most five minutes before or after him. We can plot the desired region as shown:



   In the shaded region, Jill arrives within five minutes of Jack, which is the necessary and sufficient condition for the duel to take place. Thus the probability that a duel will take place is the ratio of the area of the desired region to the area of the total region. Clearly the area of the total region is $60^2 = 3600$. The area of the shaded region equals $3600 - 2(55^2/2) = 575$. Thus the probability is $575/3600 = 23/144$. $\square$

2. How many distinct integers are there in the set

$$\left\lfloor \frac{1^2}{2013} \right\rfloor, \left\lfloor \frac{2^2}{2013} \right\rfloor, \dots, \left\lfloor \frac{2013^2}{2013} \right\rfloor$$

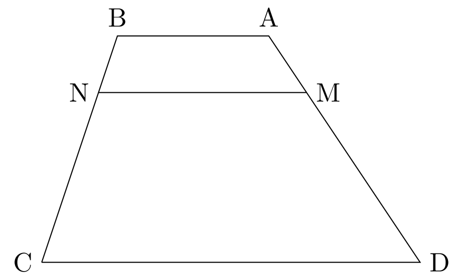
where $\lfloor x \rfloor$ denotes the greatest integers less than or equal to x.

Solution:
Let $a_n = \lfloor n^2/2013 \rfloor$. Additionally, let $D_n = n^2/2013 - (n-1)^2/2013$. We see that if $D_n \geq 1$, then a_n and a_{n-1} will be different integers. However, we can rewrite D_n as

$$\frac{n^2}{2013} - \frac{(n-1)^2}{2013} = \frac{2n-1}{2013}$$

So for $n \geq 1007$, $D_n \geq 1$, so a_n will be a distinct integer. So we already have at least $2013 - 1007 + 1 = 1007$ distinct integers in our set.
Now, $a_{1006} = 502$, so there are at most 503 distinct nonnegative integers for $n < 1007$ (the numbers $0 - 502$). However, since $D_n < 1$ for all $n < 1007$, it is impossible for any integer below 502 to be "skipped". Thus every nonnegative integer not greater than 502 is represented, and in total we have $503 + 1007 = 1510$ distinct integers. $\square$

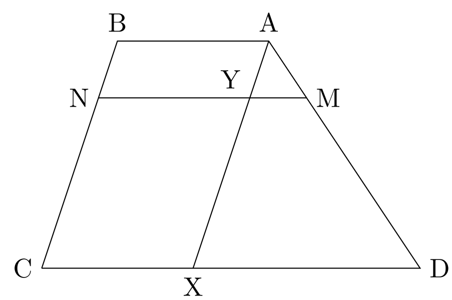
3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let M, N be points on the lines AD and BC, respectively, such that $MN \parallel AB$.



Prove that

$$DC \cdot MA + AB \cdot MD = MN \cdot AD$$

Solution:
WLOG assume $AD \geq BC$. Draw another line and label:



Let $CD = a, AB = b, MN = m, AM = c$, and $MD = d$. Then $CX = a - b$ and $NY = m - b$.
We can see that triangles YAM and XAD are similar, thus we have that

$$\frac{m-b}{a-b} = \frac{c}{c+d} \tag{1}$$

We want to show that $ac + bd = mc + md$. But we can just reduce (1) to find

$$(m-b)(c+d) = c(a-b)$$

$$mc + md - bc - bd = ac - bc$$

$$md + mc = ac + bd$$

as desired. □

4. The numbers 2^{2013} and 5^{2013} are written out in base 10 (decimal) notation one after another to create a single number. How many digits are there in the number? You must give an algebraic proof of your answer. The output of a calculator will not be accepted.

   Solution:
   The number of digits in n is $\log n + 1$, where log here is in base 10. Thus, 2^{2013} has $\lfloor \log 2^{2013} \rfloor + 1$ digits and 5^{2013} has $\lfloor \log 5^{2013} \rfloor + 1$ digits. Together, they have

$$\begin{aligned}
\lfloor 2013 \log 2 \rfloor + \lfloor 2013 \log 5 \rfloor + 2 &= \lfloor 2013 \log 2 \rfloor + \lfloor 2013 \log 5 \rfloor + \{2013 \log 2\} + \{2013 \log 5\} \\
&\quad - \{2013 \log 2\} - \{2013 \log 5\} + 2 \\
&= 2013(\log 2 + \log 5) - \{2013 \log 2\} - \{2013 \log 5\} + 2 \\
&= 2 + 2013 \log 10 - (\{2013 \log 2\} + \{2013 \log 5\}) \\
&= 2015 - (\{2013 \log 2\} + \{2013 \log 5\})
\end{aligned}$$

   where $\{x\}$ denotes the fractional part of x, $0 \leq \{x\} < 1$. Our result must be an integer, so $\{2013 \log 2\} + \{2013 \log 5\}$ must also be an integer. But since it is equal to the sum of two fractional parts, each less than 1, the result can only be 0 or 1. However, if the sum were 0, both of $\{\log 2^{2013}\}$ and $\{\log 5^{2013}\}$ must be 0, which is impossible since neither 2^{2013} nor 5^{2013} are divisible by 10. Thus the sum of the fractional parts is 1, and the total number of digits is $2015 - 1 = 2014$. $\square$

5. Prove that the roots of the polynomial

$$p(x) = a_{10}x^{10} + a_9x^9 + \ldots + a_3x^3 + 3x^2 + 2x + 1$$

are not all real.

Solution:
Since $p(x)$ is a tenth degree polynomial, we know it has 10 roots, say $r_1 \ldots r_{10}$, not necessarily distinct. By Vieta's formulas, we have that

$$r_1r_2 \ldots r_{10} = 1 \tag{2}$$

$$r_1r_2 \ldots r_9 + r_1r_2 \ldots r_8r_{10} + \ldots + r_2r_3 \ldots r_{10} = -2 \tag{3}$$

$$r_1r_2 \ldots r_8 + r_1r_2 \ldots r_7r_9 + \ldots + r_3r_4 \ldots r_{10} = 3 \tag{4}$$

We can divide (2) and (3) by $r_1r_2 \ldots r_{10} = 1$:

$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_{10}} = -2 \tag{5}$$

$$\frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \ldots + \frac{1}{r_9r_{10}} = 3 \tag{6}$$

We can square (4) and subtract twice (5) to obtain

$$\left(\frac{1}{r_1}\right)^2 + \left(\frac{1}{r_2}\right)^2 + \ldots + \left(\frac{1}{r_{10}}\right)^2 = -2 \tag{7}$$

Now assume all of $r_1, r_2 \ldots r_{10}$ are real. But then the left hand side of (6) is the sum of squares of real numbers, which must not be less than 0. But $-2 < 0$, thus we have a contradiction, and at least one of $r_1 \ldots r_{10}$ must not be real. $\square$

6. Tom, Mary, and Jerry play three rounds of a game. At the conclusion of each round, the loser has to give each other player enough money to triple the holdings of that other player. It is permissible for a player to go into debt at some stage of the game (iou). The successive losers of the three rounds are Tom, then Mary, and finally Jerry. Each player ends with \$27. How much money did each person start with?

   Solution:
   Let Tom being with x dollars, Mary with y, and Jerry with z. The statement of the problem is then equivalent to the following chart:

Round	Tom	Mary	Jerry
0	x	y	z
1	$x-2y-2z$	$3y$	$3z$
2	$3x-6y-6z$	$-2x+7y-2x$	$9z$
3	$9x-18y-18z$	$-6x+21y-6z$	$-2x-2y+25z$

   Since it is possible for players to go into debt, the intermediate results do not matter. From the final result, we get three equations which we can simplify into

$$x-2y-2z=3 \tag{1}$$

$$-2x+7y-2z=9 \tag{2}$$

$$-2x-2y+25z=27 \tag{3}$$

   Subtract (3) from (2) and add twice (1) to (2) to obtain two equations in y and z:

$$9y-27z=-18 \tag{4}$$

$$3y-6z=15 \tag{5}$$

   After simplifying, subtract (5) from (4) to obtain:

$$-z=-7 \implies z=7 \implies y=19 \implies x=55 \tag{6}$$

   So Tom began with \$55, Mary with \$19, and Jerry with \$7. We can see that these values satisfy the descriptive chart, as shown:

Round	Tom	Mary	Jerry
0	55	19	7
1	3	57	21
2	9	9	63
3	27	27	27

   as desired. □

7. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}$$

Solution:
Consider the function

$$f(t) = \frac{t}{1-t^2} - \frac{3t^2\sqrt{3}}{2} = \frac{3t^4\sqrt{3} - 3t^2\sqrt{3} + 2t}{2-2t^2}$$

We can then reduce the desired inequality to

$$f(x) + f(y) + f(z) \geq 0$$

So we just need to show that $f(x) \geq 0$ if x is between 0 and 1, exclusive. The denominator of f is always nonnegative, so the problem reduces to showing that

$$3t^4\sqrt{3} + 2t - 3t^2\sqrt{3} \geq 0$$

or equivalently

$$3t^3\sqrt{3} + 2 \geq 3t\sqrt{3}$$

Finally we apply the AMGM inequality to the three numbers $3t^3\sqrt{3}, 1$, and 1 to get

$$3t^3\sqrt{3} + 2 \geq 3\sqrt[3]{3t^3\sqrt{3}} = 3t\sqrt{3}$$

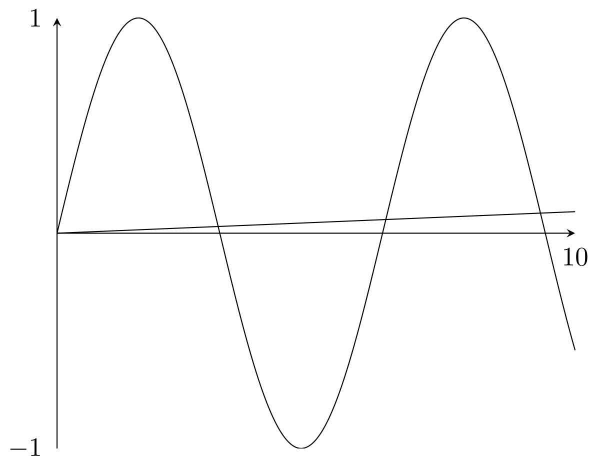
as desired. □

8. How many roots are there of the equation

$$\sin x = \frac{x}{100} \tag{1}$$

where x is measured in radians?

Solution:



First we see that both sides of (1) are odd, i.e. $-\sin(-x) = \sin(x)$ and $-(-x/100) = x/100$ so the roots will be symmetric about the y-axis. Second, there is a zero at $x = 0$. Lastly, if $|x| \geq 100$, (1) clearly has no solution.

Now let's analyze $\sin(x)$ and $x/100$ in each period of π from 0 to 100. Clearly there is no solution between $(2k-1)\pi$ and $2k\pi$, for any integer k, since $\sin(x)$ would be negative while $x/100$ is positive. But then in each remaining interval between $2k\pi$ and $(2k+1)\pi$, $\sin(x) = x/100$ exactly twice. The reason for this is as follows:

From $x = 2k\pi$ to $x = (2k+1/2)\pi$, $\sin(x)$ goes from $0 < 2k\pi/100$ to $1 > (2k+1/2)\pi/100$. Thus by the intermediate value theorem, the two must intersect.

Similarly, from $x = (2k+1/2)\pi$ to $x = (2k+1)\pi$, $\sin(x)$ goes from $1 > (2k+1/2)\pi/100$ to $0 < (2k+1)\pi/100$, so again they must intersect.

So we are finally ready to state our answer. There are $\lfloor 100/2\pi \rfloor = 15$ valid periods from 0 to 100, with 2 roots each. However, there is also an additional root at the end of the last period that is incomplete. More specifically, the period from 31π to $32\pi > 100$ is not included in our count but does contain a root that can be easily found by computation: $x \approx 96.0988$. So, we have found $2 \times 15 + 1 = 31$ roots so far. But as we noted previously, the roots are symmetric about the y-axis, so we have an additional 31 negative roots. Finally, we have the roots at $x = 0$, making a total of $2 \times 31 + 1 = 63$ roots. □