Summer Institute for Mathematics at the University of Washington

2008 Problems

1. What is the probability that two points on a circle of radius r are more than r units apart?

2. Let $a > 0, b > 0, c > 0$. Prove that
$$\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

3. How many times does the digit '1' appear in $9 + 99 + 999 + \cdots + 999\ldots99$, where the last term has 2008 9's.

4. Assume a, b, c, d are distinct integers and that $(x-a)(x-b)(x-c)(x-d) = 9$ has an integral solution r. Prove that $a + b + c + d = 4r$.

5. If two of the solutions $r \neq s$ of $x^3 + bx^2 + cx + d = 0$ satisfy $r = -s$, prove that $bc = d$.

6. Let A, B, C be the angles of a triangle. Prove that
$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

7. The king has several children. The probability that two children, selected at random, will be of the same sex is $1/2$. The probability that the two children will both be girls is the same as the probability that one child selected at random will be a boy. The king has at least four children. How many children does he have?

8. Let t_i be the lengths of the line segments that bisect angle A_i of a triangle. The segments go from A_i to the opposite side. Let a_i be the lengths of the sides opposite angle A_i. Prove the following inequalities:
$$\sum_{i=1}^{3} 1/t_i > \sum_{i=1}^{3} 1/a_i$$
$$\sum_{i=1}^{3} t_i < \sum_{i=1}^{3} a_i.$$