The Mordell Inequality and Hurwitz Lattice Packings

Stephanie L. Vance

University of Washington

March 27, 2008

Lattices in Quaternionic Vector Spaces

- A generalization of lattices in $\mathbb{R}^n$ to different vector spaces.
- Vector spaces over the Hamilton quaternions

$$\mathbb{H} = \{a+bi+cj+dk : a, b, c, d \in \mathbb{R},\ i^2 = j^2 = -1, ij = -ji = k\}$$

- Which ring should play the role of the integers?
- A natural choice would be the integer quaternions

$$\{a + bi + cj + dk : a, b, c, d \in \mathbb{Z}\}.$$

The Ring of Hurwitz Integers in $\mathbb{H}$

Another option is the Hurwitz integers:

$$\mathcal{H} = \left\{ a + bi + cj + dk \in \mathbb{H} : \; a, b, c, d \in \mathbb{Z} \; \text{ or } \; a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}.$$

- $\mathcal{H}$ is almost a Euclidean domain (non-commutative)
- $\mathcal{H}$ is the unique maximal order in the $\mathbb{Q}$-algebra of rational quaternions
- $\mathcal{H}$ is similar to the root lattice D_4

Definition

A Hurwitz lattice in a f.d. $\mathbb{H}$-vector space is a lattice that has the structure of a left Hurwitz module.

Comment: Hurwitz lattices are Free $\mathcal{H}$-modules.

Examples:

- The root lattices D_4 and E_8
- The Barnes-Wall lattice Λ_{16}
- The Leech lattice Λ_{24}

Non-Example: The Coxeter-Todd lattice K_{12}

A Hermitian and Euclidean Structure

Let E denote an n-dimensional quaternionic vector space.

- For a fixed basis $\{b_1, \ldots, b_n\}$ of E define a Hermitian product on E,

$$h\left(\sum_i \alpha_i b_i, \sum_i \beta_i b_i\right) = \sum_i \alpha_i \overline{\beta}_i; \quad \text{with} \quad \alpha_i, \beta_i \in \mathbb{H}.$$

- Define an inner product $\langle \cdot, \cdot \rangle$ on E by,

$$\langle x, y \rangle = \mathrm{Trd}_{\mathbb{H}/\mathbb{R}}(h(x, y)) = h(x, y) + \overline{h(x, y)}, \text{ with } x, y \in E.$$

Hurwitz Lattice Invariants

Let Λ be an $4n$-dimensional Hurwitz lattice

- The (squared) norm of Λ:

$$\mathrm{N}(\Lambda) = \min\{\langle x, x\rangle : x \in \Lambda\}$$

- The determinant of Λ:

$$\det(\Lambda) = \det\left(\langle b_i, b_j\rangle\right)_{1\leq i,j\leq 4n}, \text{ with } \{b_1, \ldots, b_{4n}\} \text{ a basis of } \Lambda$$

- The density of Λ:

$$\frac{\mathrm{N}(\Lambda)^{4n/2} V_n}{\sqrt{\det(\Lambda)}}$$

Lattice projections and Relative Hurwitz Lattices

Proposition: Let Λ be a Hurwitz lattice in E and let F be a $\mathbb{H}$-vector subspace of E.

1. $\Lambda \cap F$ is a full-dimensional Hurwitz lattice in F if and only if $\pi_{F^\perp}(\Lambda)$ is full-dimensional Hurwitz lattice in $F^\perp$.
2. If $\Lambda \cap F$ is a full-dimensional Hurwitz lattice in F, then $\det(\Lambda) = \det(\Lambda \cap F)\det(\pi_{F^\perp}(\Lambda))$.

These statements are proved for ordinary lattices in f.d. Euclidean spaces by J. Martinet. Observing that both $\Lambda \cap F$ and $\pi_{F^\perp}(\Lambda)$ are Hurwitz modules, the proposition readily follows from his proof.

Hurwitz Lattices and Duality

Dual lattices

- The $\mathbb{Z}$-dual with respect to an inner product:

$$\Lambda^* = \{x \in E \ : \ \langle x, \Lambda \rangle \subseteq \mathbb{Z}\}.$$

- The $\mathcal{H}$-dual with respect to a Hermitian product

$$\Lambda^\# = \{x \in E : h(x, \Lambda) \subseteq \mathcal{H}\}.$$

- The dual lattices satisfy:
  - $(\Lambda^*)^* = \Lambda$ and $(\Lambda^\#)^\# = \Lambda$
  - $\Lambda^\# \subseteq \Lambda^*$
  - $\Lambda^* = \left(\frac{1+i}{2}\right) \Lambda^\#$

More Lattice Determinants and Dual Lattices

Let Λ be a Hurwitz lattice in E.

- $\det(\Lambda)\det(\Lambda^*) = 1$
- $\det(\Lambda)\det(\Lambda^\#) = 2^{4n}$

Let F be a subspace of E with $\Lambda \cap F$ a full-dimensional lattice.

- $(\Lambda \cap F)^\# = \pi_F(\Lambda^\#)$
- $\Lambda^\# \cap F^\perp = (\pi_{F^\perp}(\Lambda))^\#$

Hurwitz Lattice Sphere Packings

Problem:

Determine the optimal density of Hurwitz lattice packings in each quaternionic dimension.

The Hermite Invariant

- The Hermite invariant of a *n*-dimensional Lattice:

$$\gamma(\Lambda) = \frac{\mathrm{N}(\Lambda)}{\det(\Lambda)^{1/n}}$$

- Hermite's Constant for dimension *n*:

$$\gamma_n = \sup\{\gamma(\Lambda) : \Lambda \text{ is an } n\text{-dimensional lattice } \}$$

- Mordell's Inequality:
  For $n \geq 3$, we have the inequality

$$\gamma_n \leq (\gamma_{n-1})^{\frac{n-1}{n-2}}.$$

Mordell's Inequality for Hurwitz Lattices

The Hurwitz Hermite constant for dimension $4n$:

$$\gamma(\mathcal{H}, 4n) = \max\{\gamma(\Lambda) : \Lambda \text{ is an } 4n\text{-dimensional Hurwitz lattice}\}.$$

Theorem

For each $n \geq 3$,

$$\gamma(\mathcal{H}, 4n) \leq \gamma(\mathcal{H}, 4(n-1))^{\frac{n-1}{n-2}} \left(\frac{1}{\sqrt{2}}\right)^{1/(n-2)}.$$

$\gamma(\mathcal{H}, 16)$ and the Barnes-Wall lattice Λ_{16}

The Hermite Hurwitz constant for dimensions ≤ 24

n	1	2	3	4	5	6	7
$\gamma(\mathcal{H}, 4n)$	$\sqrt{2}$	2	$2^{7/6}{}^{*}$	$2\sqrt{2}^{**}$	$2^{17/10}{}^{**}$	4	$2^{41/14}{}^{**}$
lattice	D_4	E_8	Λ_{12}	Λ_{16}	Λ_{20}	Λ_{24}	Λ_{28}

*Computation due to Sigrist and Schürmann

**Conjectured bound

Assuming the Mordell bound for Hurwitz lattices holds,

$$\begin{aligned}\gamma(\mathcal{H}, 16) &\leq (2^{7/6})^{3/2}\left(\frac{1}{\sqrt{2}}\right)^{1/2} \\ &= 2\sqrt{2}\end{aligned}$$

This would imply that $\gamma(\mathcal{H}, 16) = \gamma(\Lambda_{16})$.

Proving the Theorem

The technique used to prove the inequality follows that given for the original version of Mordell's theorem in J. Martinet's *Perfect Lattices in Euclidean Spaces*.

Proof: Let Λ be a Hurwitz lattice in an n-dimensional $\mathbb{H}$-vector space E. Let x be a minimal vector in $\Lambda^{\#}$ and let F denote the $(n-1)$-dimensional subspace $E \cap (\mathbb{H}x)^{\perp}$. The relative Hurwitz lattice $\Lambda \cap F$ in F is a full-dimensional Hurwitz lattice because $\Lambda^{\#} \cap F^{\perp} = \mathcal{H}x$ is a full-dimensional lattice in $F^{\perp} = \mathbb{H}x$.

Computing $\det(\Lambda \cap F)$

$$\begin{aligned}
\det(\Lambda \cap F) &= \det(\Lambda)\det(\pi_{F^\perp}(\Lambda))^{-1} \\
&= \det(\Lambda)\,\det((\Lambda^\# \cap F^\perp)^\#)^{-1} \\
&= \det(\Lambda)\,\det(\Lambda^\# \cap F^\perp)(2)^{-4}
\end{aligned}$$

The lattice $\Lambda^\# \cap F = \mathcal{H}x$ has the $\mathbb{Z}$-basis $\{x, ix, jx, \left(\frac{1+i+j+k}{2}\right)x\}$. The determinant of the corresponding gram matrix is

$$\begin{aligned}
\mathrm{N}(\Lambda^\#)^4(1/4) &= \gamma(\Lambda^\#)^4 \det(\Lambda^\#)^{1/n}(1/4) \\
&= \gamma(\Lambda^\#)^4 \left(\det(\Lambda)^{-1}(1/2)^{-4n}\right)^{1/n}(1/4) \\
&= \gamma(\Lambda^\#)^4 \det(\Lambda)^{-1/n}4.
\end{aligned}$$

Proof (continued)

We can now write the determinant of $\Lambda \cap F$ as

$$\begin{aligned} \det(\Lambda \cap F) &= \det(\Lambda)\det(\Lambda^{\#} \cap F^{\perp})(2)^{-4} \\ &= \det(\Lambda)\left(\gamma(\Lambda^{\#})^4 \det(\Lambda)^{-1/n}4\right)(2)^{-4} \\ &= \det(\Lambda)^{\frac{n-1}{n}}\gamma(\Lambda^{\#})^4(1/4) \end{aligned}$$

we get the identity

$$\gamma(\Lambda \cap F) = (1/4)^{-1/(4(n-1))}\frac{\gamma(\Lambda)}{\gamma(\Lambda^{\#})^{1/(n-1)}}\frac{\mathrm{N}(\Lambda \cap F)}{\mathrm{N}(\Lambda)}.$$

The factor $\mathrm{N}(\Lambda \cap F)$ is bounded from below by $\mathrm{N}(\Lambda)$, yielding the sequence of inequalities

$$(\sqrt{2})^{1/(n-1)}\frac{\gamma(\Lambda)}{\gamma(\Lambda^{\#})^{1/(n-1)}} \leq \gamma(\Lambda \cap F) \leq \gamma(\mathcal{H}, 4(n-1)).$$

Proof (continued)

We now have the inequality,

$$(\sqrt{2})^{1/(n-1)}\frac{\gamma(\Lambda)}{\gamma(\Lambda^{\#})^{1/(n-1)}} \leq \gamma(\mathcal{H}, 4(n-1)). \tag{1}$$

If we switch the roles of $\Lambda^{\#}$ and $\Lambda = (\Lambda^{\#})^{\#}$ in the previous steps we obtain a second inequality,

$$(\sqrt{2})^{1/(n-1)}\frac{\gamma(\Lambda^{\#})}{\gamma(\Lambda)^{1/(n-1)}} \leq \gamma(\mathcal{H}, 4(n-1)). \tag{2}$$

Then using these two inequalities we get

$$\left(\sqrt{2}\right)^{n/(n-1)}\left(\frac{\gamma(\Lambda)}{\gamma(\Lambda^{\#})^{1/(n-1)}}\right)^{n-1}\left(\frac{\gamma(\Lambda^{\#})}{\gamma(\Lambda)^{1/(n-1)}}\right) \leq \gamma(\mathcal{H}, 4(n-1))^{n}. \tag{3}$$

Proof (continued)

The $\mathrm{N}(\Lambda^{\#})$ terms cancel out such that

$$\left(\sqrt{2}\right)^{n/(n-1)} \frac{\gamma(\Lambda)^{n-1}}{\gamma(\Lambda)^{1/(n-1)}} \leq \gamma(\mathcal{H}, r(n-1))^{n},$$

which simplifies to

$$\left(\sqrt{2}\right)^{n/(n-1)} \gamma(\Lambda)^{n(n-2)/(n-1)} \leq \gamma(\mathcal{H}, 4(n-1))^{n}.$$

These inequalities hold for all Hurwitz lattices in E. Therefore,

$$\left(\sqrt{2}\right)^{n/(n-1)} \gamma(\mathcal{H}, 4n)^{n(n-2)/(n-1)} \leq \gamma(\mathcal{H}, 4(n-1))^{n}$$

with which we obtain the desired inequality

$$\gamma(\mathcal{H}, 4n) \leq \gamma(\mathcal{H}, 4(n-1))^{\frac{n-1}{n-2}} \left(\frac{1}{\sqrt{2}}\right)^{1/(n-2)}.$$

Q.E.D.

Remarks

1. When the inequality is tight...
2. We can iterate the inequality to skip more than one dimension (not very effective though)
3. A more general version can be proved for totally definite quaternion algebras and imaginary quadratic number fields with class number 1.
4. Eisenstein Latices and the Coxeter-Todd lattice K_{12}

What's Next?

- Compute the corresponding Hermite constant for low dimensions for more types of lattices for which the general version of the Mordell inequality may be applied
- Attempt to generalize the theorem so that the class number 1 part of the hypothesis may be omitted under certain conditions
- Consider other number fields with degree greater than 2