MAPS BETWEEN NON-COMMUTATIVE SPACES

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Abstract. Let J be a graded ideal in a not necessarily commutative graded k-algebra $A = A_0 \oplus A_1 \oplus \cdots$ in which $\dim_k A_i < \infty$ for all i. We show that the map $A \to A/J$ induces a closed immersion $i : \operatorname{Proj}_{nc} A/J \to \operatorname{Proj}_{nc} A$ between the non-commutative projective spaces with homogeneous coordinate rings A and A/J. We also examine two other kinds of maps between non-commutative spaces. First, a homomorphism $\phi : A \to B$ between not necessarily commutative $\mathbb{N}$-graded rings, induces an affine map $\operatorname{Proj}_{nc} B \supset U \to \operatorname{Proj}_{nc} A$ from a non-empty open subspace $U \subset \operatorname{Proj}_{nc} B$. Second, if A is a right noetherian connected graded algebra (not necessarily generated in degree one), and $A^{(n)}$ is a Veronese subalgebra of A, there is a map $\operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$; we identify open subspaces on which this map is an isomorphism. Applying these general results when A is (a quotient of) a weighted polynomial ring produces a non-commutative resolution of (a closed subscheme of) a weighted projective space.

1. Introduction

Following Rosenberg [8] and Van den Bergh [13], a non-commutative space X is a Grothendieck category $\mathsf{Mod}X$. A map $g : Y \to X$ between two spaces is an adjoint pair of functors (g^*, g_*) with $g_* : \mathsf{Mod}Y \to \mathsf{Mod}X$ and g^* left adjoint to g_*. The map g is affine [8, page 278] if g_* is faithful and has a right adjoint. For example, a ring homomorphism $\varphi : R \to S$ induces an affine map $g : Y \to X$ between the affine spaces defined by $\mathsf{Mod}Y := \mathsf{Mod}S$ and $\mathsf{Mod}X := \mathsf{Mod}R$.

We say that $g : Y \to X$ is a closed immersion if it is affine and the essential image of $\mathsf{Mod}Y$ in $\mathsf{Mod}X$ under g_* is closed under submodules and quotients.

This paper concerns maps between non-commutative projective spaces of the form $\operatorname{Proj}_{nc} A$. We recall the definition.

Let k be a field. An $\mathbb{N}$-graded k-algebra A is locally finite if $\dim_k A_i < \infty$ for all i. The non-commutative projective space X with homogeneous coordinate ring A is defined by

$$\mathsf{Mod}X := \mathsf{GrMod}A/\mathsf{Fdim}A$$

(see Section 2), and

$$\operatorname{Proj}_{nc} A := (\mathsf{Mod}X, \mathcal{O}_X),$$

where $\mathcal{O}_X$ is the image of A in $\mathsf{Mod}X$. Thus $\operatorname{Proj}_{nc} A$ is an enriched quasi-scheme in the languauge of [13]. Let Y be another non-commutative projective space with homogeneous coordinate ring B. A map $f : \operatorname{Proj}_{nc} B \to \operatorname{Proj}_{nc} A$ is a map $f : Y \to X$ such that $f^*\mathcal{O}_X \cong \mathcal{O}_Y$.

When A is a commutative $\mathbb{N}$-graded k-algebra we write $\operatorname{Proj} A$ for the usual projective scheme. We will always view a quasi-separated, quasi-compact scheme

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X as a non-commutative space by associating to it the enriched space $(\mathsf{Qcoh}X, \mathcal{O}_X)$. The rule $X \mapsto (\mathsf{Qcoh}X, \mathcal{O}_X)$ is a faithful functor.

The main results in this paper are Theorems 3.2, 3.3, 4.1, and Proposition 4.8.

Theorem 3.2 shows that a surjective homomorphism $A \to A/J$ of graded rings induces a closed immersion $i : \operatorname{Proj}_{nc} A/J \to \operatorname{Proj}_{nc} A$. The functors i^* and i_* are the obvious ones. It seems to be a folklore result that i^* is left adjoint to i_* but we could not find a proof in the literature so we provide one here; we also show that i_* is faithful, that its essential image is closed under subquotients, and that it has a right adjoint (this is what we mean by a "closed immersion"). Several people have been aware for some time that this is the appropriate intuitive picture but, as far as I know, no formal definition has been given and so no explicit proof has been given.

If A is a graded subalgebra of B, the commutative results suggests there is should be a closed subspace Z of $Y = \operatorname{Proj}_{nc} B$ and an affine map $g : Y\backslash Z \to \operatorname{Proj}_{nc} A$. Theorem 3.3 establishes such a result under reasonable hypotheses on A and B. In fact, that result is set in a more general context, namely a homomorphism $\phi : A \to B$ of graded rings. Corollary 3.4 then says that if $\phi : A \to B$ and B is a finitely presented left A-module, then there is an affine map $g : \operatorname{Proj}_{nc} B \to \operatorname{Proj}_{nc} A$. This is a (special case of a) non-commutative analogue of the commutative result that a finite morphism is affine.

If A is a quotient of a commutative polynomial ring, and $A^{(n)}$ is the graded subring with components $(A^{(n)})_i = A_{ni}$, then there is an isomorphism of schemes $\operatorname{Proj} A \cong \operatorname{Proj} A^{(n)}$. Verevkin [12] proved that $\operatorname{Proj}_{nc} A \cong \operatorname{Proj}_{nc} A^{(n)}$ when A is no longer commutative, but is connected and generated in degree one. Theorem 4.1 shows that when A is not required to be generated in degree one, there is still a map $\operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$, and Proposition 4.8 describes open subspaces on which this map is an isomorphism.

The results here are modelled on the commutative case, and none are a surprise. In large part the point of this paper is to make the appropriate definitions so that results from commutative algebraic geometry carry over verbatim to the non-commutative setting. Thus we formalize and make precise some of the terminology and intuition in papers like [2] and [7].

In Example 4.9 we show how our results apply to a quotient of a weighted polynomial ring to obtain a birational isomorphism $\operatorname{Proj}_{nc} A \to X = \operatorname{Proj} A$ where X is a commutative subscheme of a weighted projective space. It can happen that X is singular whereas $\operatorname{Proj}_{nc} A$ is smooth. Thus we can view $\operatorname{Proj}_{nc} A \to \operatorname{Proj} A$ as a non-commutative resolution of singularities.

We freely use basic notions and terminology for non-commutative spaces from the papers [9], [10], and [13].

Acknowledgements. This work was stimulated by Darin Stephenson's paper [11]. I would like to thank him for explaining his results, and also suggesting that Proposition 4.10 should be true. The final paragraph of this paper applies our results to Stephenson's algebras. I thank James Zhang for bringing [3] to my attention.

2. Definitions and preliminaries

Throughout this paper we assume that A is a locally finite $\mathbb{N}$-graded algebra over a field k. Thus $A = A_0 \oplus A_1 \oplus \cdots$, and $\dim_k A_i < \infty$ for all i. The augmentation ideal

$\mathfrak{m}$ of A is $A_1 \oplus A_2 \oplus \cdots$. If A_0 is finite dimensional and A is right noetherian, then it follows that $\dim_k A_i < \infty$ for all i because $A_{\geq i}/A_{\geq i+1}$ is a noetherian $A/\mathfrak{m}$-module. We write $\mathsf{GrMod}A$ for the category of $\mathbb{Z}$-graded right A-modules and define

$$\mathsf{Tails}A := \mathsf{GrMod}A/\mathsf{Fdim}A,$$

where $\mathsf{Fdim}A$ is the full subcategory consisting of direct limits of finite dimensional A-modules. Equivalently, $\mathsf{Fdim}A$ consists of those modules every element of which is annihilated by a suitably large power of $\mathfrak{m}$. We write π for the quotient functor $\mathsf{GrMod}A \to \mathsf{Tails}A$ and ω for its right adjoint.

The projective space with homogeneous coordinate ring A is the space X defined by $\mathsf{Mod}X := \mathsf{Tails}A$. We write $\operatorname{Proj}_{nc} A = (\mathsf{Mod}X, \mathcal{O}_X)$ where $\mathcal{O}_X$ denotes the image of A in $\mathsf{Tails}A$.

A closed subspace Z of a space X is a full subcategory $\mathsf{Mod}Z$ of $\mathsf{Mod}X$ that is closed under submodules and quotient modules in $\mathsf{Mod}X$ and such that the inclusion functor $i_* : \mathsf{Mod}Z \to \mathsf{Mod}X$ has both a left adjoint i^* and a right adjoint $i^!$. A map $\alpha : Y \to X$ of non-commutative spaces is a closed immersion if it is an isomorphism from Y onto a closed subspace of X.

The complement $X \backslash Z$ to a closed subspace Z is defined by

$$\mathsf{Mod}X\backslash Z := \mathsf{Mod}X/\mathsf{T},$$

the quotient category of $\mathsf{Mod}X$ by the localizing subcategory T consisting of those X-modules M that are the direct limit of modules N with the property that N has a finite filtration $N = N_n \supset N_{n-1} \supset \cdots N_1 \supset N_0 = 0$ such that each N_i/N_{i-1} is in $\mathsf{Mod}Z$. Because T is a localizing category, there is an exact quotient functor $j^* : \mathsf{Mod}X \to \mathsf{Mod}X\backslash Z$, and its right adjoint $j_* : \mathsf{Mod}X\backslash Z \to \mathsf{Mod}X$. The pair (j^*, j_*) defines a map $j : X\backslash Z \to X$. We call it an open immersion.

We sometimes write $\mathsf{Mod}_Z X$ for the category T and call it the category of X-modules supported on Z.

Following Rosenberg [8, pg. 278], a map $f : Y \to X$ is affine if f_* is faithful and has both a left adjoint f^* and a right adjoint $f^!$. If f_* is faithful the counit $\mathrm{id}_Y \to f^! f_*$ is monic and the unit $f^* f_* \to \mathrm{id}_Y$ is epic.

Watt's Theorem for graded modules. Let A and B be $\mathbb{Z}$-graded k-algebras. We recall the analogue of Watt's Theorem proved by Del Rio [3, Proposition 3] that describes the k-linear functors $\mathsf{GrMod}A \to \mathsf{GrMod}B$ that have a right adjoint.

A bigraded A-B-bimodule is an A-B-bimodule

$$M = \bigoplus_{(p,q)\in\mathbb{Z}^2} {}_pM_q$$

such that $A_i . {}_pM_q . B_j \subset {}_{i+p}M_{q+j}$ for all $i, j, p, q \in \mathbb{Z}$. Write $\otimes$ for $\otimes_k$. If L is a graded right A-module we define

$$L \mathbin{\bar{\otimes}_A} M = \bigoplus_{q\in\mathbb{Z}} (L \mathbin{\bar{\otimes}_A} M)_q$$

where $(L \mathbin{\bar{\otimes}_A} M)_q$ is the image of $\bigoplus_p (L_{-p} \otimes {}_pM_q)$ under the canonical map $L \otimes M \to L \otimes_A M$. This gives $L \mathbin{\bar{\otimes}_A} M$ the structure of a graded right B-module; it is a B-module direct summand of the usual tensor product $L \otimes_A M$. If N is a graded right B-module we define $\underline{\underline{\mathrm{Hom}}}_B(M, N)$ to consist of those B-module homomorphisms that vanish on all except a finite number of ${}_pM_*$ and send each ${}_pM_q$ to N_q. This is made into a graded right A-module by declaring that

$$\underline{\underline{\mathrm{Hom}}}_B(M,N)_p = \mathrm{Hom}_{\mathsf{Gr}B}({}_{-p}M_*, N).$$

Then

$$\mathrm{Hom}_{\mathsf{Gr}B}(L\,\bar{\otimes}_A\,M, N) \cong \mathrm{Hom}_{\mathsf{Gr}A}(L, \underline{\underline{\mathrm{Hom}}}_B(M,N)). \tag{2-1}$$

Theorem 2.1 (Del Rio). [3] *Let A and B be graded k-algebras, and $F : \mathsf{GrMod}A \to \mathsf{GrMod}B$ a k-linear functor having a right adjoint. Then $F \cong -\bar{\otimes}_A M$ where M is the bigraded A-B-bimodule*

$$M = \bigoplus_{p\in\mathbb{Z}} F(A(p))$$

with homogeneous components ${}_pM_q = F(A(p))_q$.

If F also commutes with the twists by degree, then F is given by tensoring with a graded A-B-bimodule, say $V = \oplus_n V_n$. The corresponding M in this case is $M = \oplus V(p)$ with ${}_pM_q = V(p)_q$.

The left A-action on M is given by declaring that $x \in A_i$ acts on ${}_pM_*$ as $F(\lambda_x)$, where $\lambda_x : A(p) \to A(p+i)$ denotes left multiplication by x.

3. MAPS INDUCED BY GRADED RING HOMOMORPHISMS

Throughout this section we assume that A and B are locally finite $\mathbb{N}$-graded algebras over a field k.

We consider the problem of when a homomorphism $\phi : A \to B$ of graded rings induces a map $g : \mathrm{Proj}_{nc} B \to \mathrm{Proj}_{nc} A$ and, if it does, how the properties of g are determined by the properties of ϕ.

Associated to ϕ is an adjoint triple $(f^*, f_*, f^!)$ of functors between the categories of graded modules. Explicitly, $f^* = -\otimes_A B$, $f_* = -\otimes_B B_A$ is the restriction map, and $f^! = \oplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Gr}B}(B(-p), -)$. We wish to establish conditions on ϕ which imply that these functors factor through the quotient categories in the following diagrams:

$$\begin{tikzcd}
\mathsf{GrMod}B \arrow[r, "f_*"] \arrow[d, "\pi'"'] & \mathsf{GrMod}A \arrow[d, "\pi"] & \mathsf{GrMod}B \arrow[d, "\pi'"'] & \mathsf{GrMod}A \arrow[l, "{f^*, f^!}"'] \arrow[d, "\pi"] \\
\mathsf{Tails}B & \mathsf{Tails}A & \mathsf{Tails}B & \mathsf{Tails}A
\end{tikzcd}$$

Lemma 3.1. *Let A and B be Grothendieck categories with localizing subcategories $\mathsf{S} \subset \mathsf{A}$ and $\mathsf{T} \subset \mathsf{B}$. Let $\pi : \mathsf{A} \to \mathsf{A}/\mathsf{S}$ and $\pi' : \mathsf{B} \to \mathsf{B}/\mathsf{T}$ be the quotient functors, and let ω and ω' be their right adjoints. Consider the following diagram of functors:*

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathsf{B} \arrow[d, "\pi'"] \\
\mathsf{A}/\mathsf{S} & \mathsf{B}/\mathsf{T}.
\end{tikzcd}$$

1. *If $F(\mathsf{S}) \subset \mathsf{T}$, then there is a unique functor $G : \mathsf{A}/\mathsf{S} \to \mathsf{B}/\mathsf{T}$ such that $\pi' F = G\pi$.*
2. *If $H : \mathsf{B} \to \mathsf{A}$ is a right adjoint to F, then $\pi H \omega'$ is a right adjoint to G.*
3. *If H is a right adjoint to F and G' is a right adjoint to G, then $H(\mathsf{T}) \subset \mathsf{S}$ if and only if $G'\pi' \cong \pi H$.*

Proof. (1) The existence and uniqueness of a functor G such that $\pi' F = G\pi$ is due to Gabriel [6, Coroll. 2, p. 368].

(2) To show that G has a right adjoint it suffices to show that it is right exact and commutes with direct sums. If $\mathcal{M}_\lambda$ is a collection of objects in A/S, then each is of the form $\mathcal{M}_\lambda = \pi M_\lambda$ for some object M_λ in A. Both π' and F commute with direct sums because they have right adjoints, so $G\pi$ commutes with direct sums; π also commutes with direct sums. Therefore

$$G(\oplus \mathcal{M}_\lambda) = G(\oplus \pi M_\lambda) \cong G\pi(\oplus M_\lambda) \cong \oplus G\pi M_\lambda = \oplus G\mathcal{M}_\lambda.$$

Thus G commutes with direct sums.

To see that G is right exact consider an exact sequence

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0 \tag{3-1}$$

in A/S. By Gabriel [6, Coroll. 1, p. 368], (3-1) is obtained by applying π to an exact sequence $0 \to L \to M \to N \to 0$ in A. Both π' and F are right exact because they have right adjoints, so $\pi' FL \to \pi' FM \to \pi' FN \to 0$ is exact. In other words $G\mathcal{L} \to G\mathcal{M} \to G\mathcal{N} \to 0$ is exact.

Hence G has a right adjoint, say G'. It follows that $\omega G'$ is a right adjoint to $G\pi$. But $G\pi = \pi' F$ has $H\omega'$ as a right adjoint so $\omega G' \cong H\omega'$. Since $\pi\omega \cong \mathrm{id}_{\mathsf{A}/\mathsf{S}}$, $G' \cong \pi H \omega'$. Since a right adjoint is only determined up to natural equivalence $\pi H \omega'$ is a right adjoint to G.

(3) If $H(\mathsf{T}) \subset \mathsf{S}$, then πH vanishes on T so, by Gabriel [6, Corollaire 2, page 368], there is a functor $V : \mathsf{B}/\mathsf{T} \to \mathsf{A}/\mathsf{S}$ such that $V\pi' = \pi H$. Thus $V \cong \pi H \omega'$, and this is isomorphic to G' by (2). Hence $G'\pi' \cong \pi H$. Conversely, if $G'\pi' \cong \pi H$, then $\pi H(\mathsf{T}) = 0$, so $H(\mathsf{T}) \subset \mathsf{S}$. □

Warning. The functor H in part (2) of Lemma 3.1 need not have the property that $H(\mathsf{T})$ is contained in S. An explicit example of this is provided by taking $\mathsf{B} = \mathsf{A}$, $F = H = \mathrm{id}_{\mathsf{A}}$, $\mathsf{S} = 0$, and $\mathsf{T} = \mathsf{B}$.

Theorem 3.2. *Let J be a graded ideal in an $\mathbb{N}$-graded k-algebra A. Then the homomorphism $A \to A/J$ induces a closed immersion $i : \operatorname{Proj}_{nc} A/J \to \operatorname{Proj}_{nc} A$.*

Proof. Write $X = \operatorname{Proj}_{nc} A$ and $Z = \operatorname{Proj}_{nc} A/J$. Write $\mathfrak{m} = A_1 \oplus A_2 \oplus \cdots$. Thus $\mathsf{Mod}X = \mathsf{GrMod}A/\mathsf{Fdim}A$. We write $\pi : \mathsf{GrMod}A \to \mathsf{Mod}X$ for the quotient functor and ω for a right adjoint to it. Similarly, $\pi' : \mathsf{GrMod}A/J \to \mathsf{Mod}Z$ is the quotient functor, and ω' is a right adjoint to it. See [12] and [1, Section 2] for more information about this.

Let $f_* : \mathsf{GrMod}A/J \to \mathsf{GrMod}A$ be the inclusion functor. It has a left adjoint $f^* = - \otimes_A A/J$, and a right adjoint $f^!$ that sends a graded A-module to the largest submodule of it that is annihilated by J.

By [6, Coroll. 2, p. 368] (=part(1) of Lemma 3.1), there is a unique functor i_* such that

$$\begin{tikzcd} \mathsf{GrMod}A/J \arrow[r, "f_*"] \arrow[d, "\pi'"'] & \mathsf{GrMod}A \arrow[d, "\pi"] \\ \mathsf{Mod}Z \arrow[r, "i_*"'] & \mathsf{Mod}X \end{tikzcd}$$

commutes, and i_* is exact because f_* is [6, Coroll. 3, p. 369]. Thus $i_*\pi' = \pi f_*$.

By Lemma 3.1(2), i_* has a right adjoint, namely $i^! := \pi' f^! \omega$. It is clear that $f^!$ sends $\mathsf{Fdim}A$ to $\mathsf{Fdim}A/J$, so $\pi' f^! \cong i^! \pi$ by Lemma 3.1(3).

It is clear that f^* sends $\mathsf{Fdim}A$ to $\mathsf{Fdim}A/J$, so by [6, Coroll. 2, p. 368], there is a functor $i^* : \mathsf{Mod}X \to \mathsf{Mod}Z$ such that $\pi' f^* = i^* \pi$. Since f_* is right adjoint to f^*, it follows from Lemma 3.1(2) that $\pi f_* \omega'$ is a right adjoint to i^*. But $\pi f_* \omega' = i_* \pi' \omega' \cong i_*$. Hence i^* is left adjoint to i_*.

We now show that i_* is faithful. Since i_* has a left and a right adjoint it is exact, so it suffices to show that if $i_* \mathcal{M} = 0$, then $\mathcal{M} = 0$. Suppose that $i_* \pi' M = 0$ for some $M \in \mathsf{GrMod}A/J$. Then $\pi f_* M = 0$, and we conclude that M is in $\mathsf{Fdim}A$, and hence in $\mathsf{Fdim}A/J$; therefore $\pi' M = 0$. Hence i_* is faithful.

We will show that i_* is full after establishing the following fact.

Claim: $\omega \pi f_* \cong f_* \omega' \pi'$. Proof: Let $M \in \mathsf{GrMod}A/J$, let τM denote the largest submodule of M that is in $\mathsf{Fdim}A/J$ (equivalently, in $\mathsf{Fdim}A$), and set $\bar{M} = M/\tau M$. Then $\pi' M = \pi' \bar{M}$ and $\pi f_* M = \pi f_* \bar{M}$, so the two functors take the same value on M if and only if they take the same value on $\bar{M}$. Hence we can, and will, assume that $M = \bar{M}$; i.e., $\tau M = 0$.

We must show that $\omega \pi M = \omega' \pi' M$. By definition $\omega \pi M$ is the largest essential extension $0 \to M \to \omega \pi M \to T \to 0$ such that $T \in \mathsf{Fdim}A$. The definition of $\omega' \pi' M$ is analogous (although T is then required to belong to $\mathsf{Fdim}A/J$), so it suffices to prove that $\omega \pi M$ is in $\mathsf{GrMod}A/J$. The surjective map $\omega \pi M \otimes_A J \to (\omega \pi M)J$ is such that the composition $M \otimes_A J \to \omega \pi M \otimes_A J \to (\omega \pi M)J$ is zero, so there is a surjective map $T \otimes_A J \to (\omega \pi M)J$. However, $T \otimes_A J$ belongs to $\mathsf{Fdim}A$ because T does, so $(\omega \pi M)J \in \mathsf{Fdim}A$. This implies that $M \cap (\omega \pi M)J \in \mathsf{Fdim}A$. But $\tau M = 0$, so $M \cap (\omega \pi M)J = 0$, and it follows that $(\omega \pi M)J = 0$ because M is essential in $\omega \pi M$. In other words, $\omega \pi M \in \mathsf{GrMod}A/J$. This completes the proof of the Claim. □

We have $f^* f_* \cong \mathrm{id}_{\mathsf{GrMod}A/J}$ and $\pi' \omega' \cong \mathrm{id}_{\mathsf{Mod}Z}$, so

$$i^* i_* \cong (\pi' f^* \omega)(\pi f_* \omega') \cong \pi' f^* f_* \omega' \pi' \omega' \cong \mathrm{id}_{\mathsf{Mod}Z} .$$

It follows from this that i_* is full.

To see that i_* is a closed immersion, it remains to check that $i_*(\mathsf{Mod}Z)$ is closed under submodules and quotients in $\mathsf{Mod}X$. Let $\mathcal{M} \in \mathsf{Mod}Z$ and suppose that $0 \to \mathcal{L} \to i_* \mathcal{M} \to \mathcal{N} \to 0$ is an exact sequence in $\mathsf{Mod}X$. There is an exact sequence $0 \to \omega \mathcal{L} \to \omega i_* \mathcal{M} \to \omega \mathcal{N} \to R^1 \omega \mathcal{L}$ in $\mathsf{GrMod}A$. Let N denote the image of $\omega i_* \mathcal{M}$ in $\omega \mathcal{N}$. Then $\mathcal{L} \cong \pi \omega \mathcal{L}$ and $\mathcal{N} \cong \pi N$ because π is exact and $R^1 \omega \mathcal{L} \in \mathsf{Fdim}A$. Now $\mathcal{M} = \pi' M$ for some $M \in \mathsf{GrMod}A/J$, so $\omega i_* \mathcal{M} = \omega i_* \pi' M = \omega \pi f_* M \cong f_* \omega' \pi' M$ from which we conclude that $\omega i_* \mathcal{M}$ is annihilated by J. Therefore $\omega \mathcal{L}$ is also annihilated by J, so is of the form $f_* L$ for some $L \in \mathsf{GrMod}A/J$; hence $\mathcal{L} \cong \pi \omega \mathcal{L} \cong \pi f_* L \cong i_* \pi' L \in i_*(\mathsf{Mod}Z)$. Since N is a quotient of $\omega i_* \mathcal{M}$ it is also annihilated by J, and a similar argument shows that $\mathcal{N} \in i_*(\mathsf{Mod}Z)$.

Since $f^* A = A/J$, $i^* \mathcal{O}_X = \mathcal{O}_Z$. □

We retain the notation of the theorem.

Because i_* is fully faithful, we often view $\mathsf{Mod}Z$ as a full subcategory of $\mathsf{Mod}X$ and speak of $Z = \operatorname{Proj}_{nc} A/J$ as a closed subspace of $X = \operatorname{Proj}_{nc} A$ and call it the zero locus of J.

It is *not* the case that every closed subspace of $\operatorname{Proj}_{nc} A$ is the zero locus of a two-sided ideal in A. For example, if $A = k_q[x,y]$ is the ring defined by the relation $yx = qxy$ where $0 \neq q \in k$, then $\operatorname{Proj}_{nc} A \cong \mathbb{P}^1$, but the closed points of $\mathbb{P}^1$ are not cut out by two-sided ideals when $q \neq 1$: for example, $(\alpha x + \beta y)A$ is not a two-sided ideal when $q \neq 1$ and $\alpha\beta \neq 0$. This is essentially due to the fact that the auto-equivalence $\mathcal{M} \to \mathcal{M}(1)$ of $\mathsf{Mod}X$ induced by the degree shift on A does not generally send Z-modules to Z-modules.

A more difficult question is whether every closed subspace of $\operatorname{Proj}_{nc} A$ is the zero locus of a two-sided ideal in *some* homogeneous coordinate ring of $\operatorname{Proj}_{nc} A$. We do not know the answer to this question.

Theorem 3.3. *Suppose that $\phi : A \to B$ is a map of locally finite $\mathbb{N}$-graded k-algebras. Write $X = \operatorname{Proj}_{nc} A$ and $Y = \operatorname{Proj}_{nc} B$. Let $\mathfrak{m}$ be the augmentation ideal of A, and let I be the largest two-sided ideal of B contained in $\phi(\mathfrak{m})B$. Let $Z \subset Y$ be the zero locus of I. If $B\phi(\mathfrak{m})^n \subset \phi(\mathfrak{m})B$ for some integer n, then ϕ induces an affine map*

$$g : Y \backslash Z \to X.$$

Proof. The category of modules over $Y\backslash Z$ is $\mathsf{Mod}Y/\mathsf{Mod}_Z Y$. This is equivalent to the quotient category $\mathsf{GrMod}B/\mathsf{T}$ where T consists of those modules M with the property that every element of M is killed by some power of I. Let $\pi' : \mathsf{GrMod}B \to \mathsf{GrMod}B/\mathsf{T}$ be the quotient functor. We have functors $(f^*, f_*, f^!)$ between the graded module categories and a diagram

$$\begin{tikzcd}
\mathsf{GrMod}B \arrow[r, "f_*"] \arrow[d, "\pi'"'] & \mathsf{GrMod}A \arrow[d, "\pi"] \\
\mathsf{Mod}\, Y\backslash Z & \mathsf{Mod}X.
\end{tikzcd}$$

To check that f^* sends $\mathsf{Fdim}A$ to T it suffices to check that $f^*(A/\mathfrak{m}) \in \mathsf{T}$ because f^* commutes with direct limits and with the degree twist (1). However, $f^*(A/\mathfrak{m}) = B/\phi(\mathfrak{m})B$ is in T because $I \subset \phi(\mathfrak{m})B$. Hence there is a unique functor $g^* : \mathsf{Tails}A \to (\mathsf{GrMod}B)/\mathsf{T}$ satisfying $g^*\pi = \pi' f^*$.

To check that f_* sends T to $\mathsf{Fdim}A$ it suffices to check that $f_*(B/I)$ is in $\mathsf{Fdim}A$. However, $(B/I).\mathfrak{m}^n = B\phi(\mathfrak{m})^n + I/I$; the hypothesis that $B\phi(\mathfrak{m})^n \subset \phi(\mathfrak{m})B$ ensures that $B\phi(\mathfrak{m})^n \subset I$, so $(B/I).\mathfrak{m}^n = 0$. Hence there is an exact functor $g_* : \mathsf{GrMod}B/\mathsf{T} \to \mathsf{Tails}A$ such that $g_*\pi' = \pi f_*$.

By Lemma 3.1(2), g_* has a right adjoint $g^! = \pi' f^! \omega$.

To show that g_* is faithful we must show that if M is a graded B-module such that $g_*\pi' M = 0$, then $M \in \mathsf{T}$. Since $\pi f_* M = 0$, as an A-module M is a direct limit $\varinjlim M_\lambda$ where each M_λ is a finite dimensional A-module. There is an epimorphism

$$\varinjlim (M_\lambda \otimes_A B) \cong (\varinjlim M_\lambda) \otimes_A B \cong M \otimes_A B \to M$$

of graded B-modules. Since $M_\lambda \otimes_A B$ equals $f^* M_\lambda$, it is in T; but T is closed under direct limits and quotients, so M is in T. Thus g_* is faithful. □

The following consequence of the theorem slightly extends a result of Van den Bergh [13, Proposition 3.9.11].

Corollary 3.4. *Let $\phi : A \to B$ be a homomorphism of graded rings such that B becomes a finitely presented graded left A-module. Then ϕ induces an affine map $g : \operatorname{Proj}_{nc} B \to \operatorname{Proj}_{nc} A$.*

Proof. If we apply $A/\mathfrak{m} \otimes_A -$ to a finite presentation of B as a left A-module, we see that $B/\phi(\mathfrak{m})B$ has finite dimension. Thus, as a right A-module $B/\phi(\mathfrak{m})B$ is annihilated by $\mathfrak{m}^n$ for some $n \gg 0$. Equivalently, $B\phi(\mathfrak{m})^n \subset \phi(\mathfrak{m})B$. Thus the hypotheses of the theorem are satisfied. It remains to show that Z is empty.

Let I denote the right annihilator in B of $B/\phi(\mathfrak{m})B$. We have already observed that $\phi(\mathfrak{m})^n \subset I$. Since $A/\mathfrak{m}^n$ is finite dimensional, so is $A/\mathfrak{m}^n \otimes_A B \cong B/\phi(\mathfrak{m})^n B$. Thus B/I is finite dimensional. Hence the zero locus of I in $\operatorname{Proj}_{nc} B$ is empty. □

Remark. If, in Theorem 3.3, B_A is finitely presented, then we have the useful technical fact that $g^!\pi = \pi' f^!$. This follows from Lemma 3.1(3) once we show that $f^!$ sends $\mathsf{Fdim}A$ to $\mathsf{Fdim}B$. Let $M = \varinjlim M_\lambda$ be a direct limit of finite dimensional A-modules. If B is a finitely presented right A-module, then $\operatorname{Hom}_{\mathsf{Gr}A}(B, -)$ commutes with direct limits, so $\operatorname{Hom}_{\mathsf{Gr}A}(N, \varinjlim M_\lambda) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}A}(B, M_\lambda)$; this is a direct limit of finite dimensional B-modules because B_A is a finitely generated. Hence $f^!(\mathsf{Fdim}A) \subset \mathsf{Fdim}B$.

4. The Veronese mapping

Throughout this section A is a locally finite $\mathbb{N}$-graded k-algebra and n is a positive integer. The n^{th} Veronese subalgebra $A^{(n)}$ is defined by

$$A_i^{(n)} := A_{ni}.$$

It is a classical result in algebraic geometry that if A is a finitely generated commutative connected graded k-algebra generated in degree one, then $\operatorname{Proj} A \cong \operatorname{Proj} A^{(n)}$. This isomorphism is implemented by the Veronese embedding.

Verevkin proved a non-commutative version of this result when A is noetherian and generated in degree one [12, Theorem 4.4].

Theorem 4.1 and Proposition 4.8 show what happens when A need not be commutative and need not be generated in degree one.

Theorem 4.1. *Let A be a right noetherian locally finite $\mathbb{N}$-graded k-algebra. Fix a positive integer n. There is a map $g : \operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$. Furthermore, g_* has a right adjoint.*

We will use the notation $X := \operatorname{Proj}_{nc} A^{(n)}$ and $X' := \operatorname{Proj}_{nc} A$.

We need two preliminary results before proving the theorem. First we explain how the functors defining the map $g : \operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$ in the theorem are induced by functors between the categories $\mathsf{GrMod}A$ and $\mathsf{GrMod}A^{(n)}$.

If L is a graded A-module we define the graded $A^{(n)}$-module $L^{(n)}$ by

$$L_i^{(n)} := L_{ni}.$$

The rule $L \mapsto L^{(n)}$ extends to give an exact functor

$$f_* : \mathsf{GrMod}A \to \mathsf{GrMod}A^{(n)}.$$

The functor f_* is not faithful when $n \geq 2$ because $f_*((A/\mathfrak{m})(1)) = 0$.

The next result shows that f_* has a left adjoint f^* and a right adjoint $f^!$.

Proposition 4.2. *Let A be a locally finite $\mathbb{N}$-graded k-algebra. Fix a positive integer n. Let W and W' be the spaces with module categories*

$$\mathsf{Mod}W = \mathsf{GrMod}A^{(n)}$$

and

$$\mathsf{Mod}W' = \mathsf{GrMod}A.$$

Then there is a map $f : W' \to W$ with direct image functor given by $f_*L = L^{(n)}$.

Proof. It is clear that f_* is an exact functor commuting with direct sums. By the graded version of Watt's Theorem, $f_* \cong -\bar{\otimes}_A M$ where

$$M = \bigoplus_{p \in \mathbb{Z}} A(p)^{(n)}$$

with components ${}_pM_q = (A(p)^{(n)})_q = A(p)_{nq}$. The right action of $A^{(n)}$ is given by right multiplication, and each $A(p)^{(n)}$ is a right $A^{(n)}$-submodule. The left action of A is given by left multiplication whereby $a \in A_i$ acts by sending $A(p)_{nq}$ to $A(p+i)_{nq}$.

Define $f^* : \mathsf{GrMod}A^{(n)} \to \mathsf{GrMod}A$ by $f^*N = N \otimes_{A^{(n)}} A$ with the usual right action of A, and grading given by

$$(N \otimes_{A^{(n)}} A)_s = \sum_{ni+j=s} N_i \otimes A_j.$$

It is not hard to show that f^* is a left adjoint to f_*. Therefore $f^* \cong -\bar{\otimes}_{A^{(n)}} Q$ where

$$Q = \bigoplus_{p \in \mathbb{Z}} f^*(A^{(n)}(p)) \cong \bigoplus_{p \in \mathbb{Z}} A(np);$$

multiplication $A^{(n)}(p) \otimes_{A^{(n)}} A \to A(np)$ gives an isomorphism of graded right A-modules. Thus ${}_pQ_* \cong A(np)$ with its usual grading. One can verify directly that $f_* \cong \underline{\underline{\mathrm{Hom}}}_A(Q, -)$.

The right adjoint to f_* is the functor $f^! = \underline{\underline{\mathrm{Hom}}}_{A^{(n)}}(M, -)$. If N is a graded right $A^{(n)}$-module

$$(f^!N)_i = \mathrm{Hom}_{\mathsf{Gr}A^{(n)}}({}_{-i}M_*, N) = \mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(-i), N).$$

If N is a graded $A^{(n)}$-module, then $f_*f^*(N) = N$ so f_*f^* is naturally equivalent to id_X. □

Let $\pi' : \mathsf{GrMod}A \to \mathsf{Tails}A$ and $\pi : \mathsf{GrMod}A^{(n)} \to \mathsf{Tails}A^{(n)}$ be the quotient functors. To prove Theorem 4.1, we must find functors g^*, g_*, and $g^!$ making the following diagrams commute:

$$\begin{tikzcd} \mathsf{GrMod}A \arrow[r, "f_*"] \arrow[d, "\pi'"'] & \mathsf{GrMod}A^{(n)} \arrow[d, "\pi"] \\ \mathsf{Tails}A \arrow[r, "g_*"'] & \mathsf{Tails}A^{(n)}. \end{tikzcd} \qquad \begin{tikzcd} \mathsf{GrMod}A \arrow[d, "\pi'"'] & \mathsf{GrMod}A^{(n)} \arrow[l, "{f^*, f^!}"'] \arrow[d, "\pi"] \\ \mathsf{Tails}A & \mathsf{Tails}A^{(n)}. \arrow[l, "{g^*, g^!}"] \end{tikzcd}$$

Since f_* sends $\mathsf{Fdim}A$ to $\mathsf{Fdim}A^{(n)}$ there is a functor $g_* : \mathsf{Tails}A \to \mathsf{Tails}A^{(n)}$ such that $g_*\pi' = \pi f_*$.

To ensure that f^* and $f^!$ induce functors between the quotient categories we must impose a noetherian hypothesis. Although there is no noetherian hypothesis in Proposition 4.2, in Theorem 4.1 it is assumed that A is right noetherian. This hypothesis ensures that f^* sends $\mathsf{Fdim}A^{(n)}$ to $\mathsf{Fdim}A$.

Lemma 4.3. *Let A be a locally finite $\mathbb{N}$-graded k-algebra.*

1. *f_* sends noetherian A-modules to noetherian $A^{(n)}$-modules;*
2. *if A is right noetherian so is $A^{(n)}$, and A is a finitely generated right $A^{(n)}$-module.*
3. *if A is left noetherian, then f^* sends $\mathsf{Fdim}A^{(n)}$ to $\mathsf{Fdim}A$;*

4. *if A is left and right noetherian, then there is a functor $g^* : \mathsf{Tails}A^{(n)} \to \mathsf{Tails}A$ such that $g^*\pi = \pi' f^*$;*
5. *if A is left noetherian, then $f^!$ sends $\mathsf{Fdim}A^{(n)}$ to $\mathsf{Fdim}A$;*
6. *if A is left and right noetherian, then there is a functor $g^! : \mathsf{Tails}A^{(n)} \to \mathsf{Tails}A$ such that $g^!\pi = \pi' f^!$;*

Proof. (1) Let M be a right noetherian graded A-module. If N is a submodule of $M^{(n)}$ then $N = NA \cap M^{(n)}$. Hence any proper ascending chain of submodules in $M^{(n)}$ would give a proper ascending chain of submodules of M by multiplying by A. Since M contains no such chain, neither does $M^{(n)}$.

(2) Applying (1) to $M = A$ shows that $A^{(n)}$ is right and left noetherian. If $N = A \oplus A(1) \oplus \cdots \oplus A(n-1)$, then $N^{(n)} \cong A$ as a left and as a right $A^{(n)}$-module, so applying (1) to N gives the rest of the result.

(3) If N is a finite dimensional right $A^{(n)}$-module, then $N \otimes_{A^{(n)}} A$ is a finite dimensional A-module because A is a finitely generated *left* $A^{(n)}$-module.

(4) This follows at once from (3).

(5) We must show that if N is a finite dimensional graded right $A^{(n)}$-module, then $\underline{\underline{\mathrm{Hom}}}_{A^{(n)}}(M, N)$ is finite dimensional, where

$$M = \bigoplus_{p \in \mathbb{Z}} A(p)^{(n)}.$$

The degree $-p$ component of $\underline{\underline{\mathrm{Hom}}}_{A^{(n)}}(M, N)$ is $\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N)$, and it suffices to show that this is zero for almost all p and is always finite dimensional. By the noetherian hypothesis, $A(p)^{(n)}$ is a finitely generated $A^{(n)}$-module, so $\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N)$ has finite dimension. We must show that $\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N)$ is zero if $|p|$ is sufficiently large.

Fix p. Now

$$A(p + nj)^{(n)} \cong A(p)^{(n)}(j)$$

so

$$\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p+nj)^{(n)}, N) \cong \mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N(-j)).$$

Since $A(p)^{(n)}$ is finitely generated and N is finite dimensional, when $|j|$ is sufficiently large $\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N(-j))$ is zero.

It therefore follows that $\mathrm{Hom}_{\mathsf{Gr}A^{(n)}}(A(p)^{(n)}, N)$ is zero for $|p|$ sufficiently large. This completes the proof of (5), and (6) follows from this. □

Proof of Theorem 4.1. By Lemma 4.3 there are functors $g^*, g_*, g^!$ between the categories $\mathsf{Mod}\,\mathrm{Proj}_{nc}\, A^{(n)} = \mathsf{Tails}A^{(n)}$ and $\mathsf{Mod}\,\mathrm{Proj}_{nc}\, A = \mathsf{Tails}A$ satisfying

$$g^*\pi = \pi' f^*, \quad g_*\pi' = \pi f_*, \quad g^!\pi = \pi' f^!.$$

Applying Lemma 3.1 to f^* we see that g^* has $\pi f_* \omega'$ as a right adjoint. But this is naturally equivalent to g_*, so g_* is a right adjoint to g^*. Similarly, $g^!$ is a right adjoint to g_*. Since $f^* A^{(n)} = A$, $g_* \mathcal{O}_{\mathrm{Proj}_{nc} A^{(n)}} = \mathcal{O}_{\mathrm{Proj}_{nc} A}$. This completes the proof of Theorem 4.1. □

In the next result $\mathrm{Proj}\, A$ is the usual commutative scheme viewed as a noncommutative space with module category $\mathsf{Qcoh}(\mathrm{Proj}\, A)$.

Corollary 4.4. *If A is a finitely generated commutative connected graded k-algebra, there is a map $g : \mathrm{Proj}_{nc}\, A \to \mathrm{Proj}\, A$. Furthermore, g_* has a right adjoint $g^!$.*

Proof. For some sufficiently large n, $A^{(n)}$ is generated in degree one so

$$\mathsf{Tails}A^{(n)} \cong \mathsf{Tails}A \cong \mathsf{Qcoh}\,\mathrm{Proj}\,A.$$

Therefore Theorem 4.1 gives the result. $\square$

Remarks. 1. Since g_* has both a left and a right adjoint it is exact, and hence its right adjoint $g^!$ preserves injectives. There is therefore a convergent spectral sequence

$$\mathrm{Ext}^p_{\mathrm{Proj}_{nc}A}(M, R^q g^! N) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{Proj}_{nc}A^{(n)}}(g_*M, N)$$

for M and N modules over $\mathrm{Proj}_{nc}A$ and $\mathrm{Proj}_{nc}A^{(n)}$ respectively.

2. If J is a two-sided ideal of A, then the natural map $A \to A/J$ induces an isomorphism $A^{(n)}/J^{(n)} \to (A/J)^{(n)}$, so there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}_{nc}A/J \arrow[r, "i"] \arrow[d] & \mathrm{Proj}_{nc}A \arrow[d, "g"] \\
\mathrm{Proj}_{nc}A^{(n)}/J^{(n)} \arrow[r] & \mathrm{Proj}_{nc}A^{(n)}
\end{tikzcd}$$

where the horizontal maps are the natural closed immersions.

Proposition 4.5. *Let A be a right noetherian, locally finite, $\mathbb{N}$-graded k-algebra. Suppose that A is prime and* $\mathrm{Fract}_{gr}A$ *contains a copy of $A(n)$ for all $n \in \mathbb{Z}$. Then*

1. $\mathrm{Proj}_{nc}A$ *and* $\mathrm{Proj}_{nc}A^{(n)}$ *are integral spaces in the sense of* [10], *and*
2. $g : \mathrm{Proj}_{nc}A \to \mathrm{Proj}_{nc}A^{(n)}$ *is a birational isomorphism in the sense that it induces an isomorphism between the function fields.*

Proof. That $X = \mathrm{Proj}_{nc}A$ is an integral space is proved in [10, Th. 4.5]. It is also shown there that the function field of X is

$$\mathrm{Fract}_{gr}A := \{ab^{-1} \mid a, b \in A \text{ homogeneous of the same degree and } b \text{ is regular}\}.$$

It is clear that $\mathrm{Fract}_{gr}A^{(n)} \subset \mathrm{Fract}_{gr}A$, and the reverse inclusion follows from the observation that $ab^{-1} = ab^{n-1}b^{-n}$. $\square$

Remarks. 1. If A is prime noetherian and has a regular element of degree d for all $d \gg 0$, then $\mathrm{Fract}_{gr}A$ contains a copy of $A(n)$ for all $n \in \mathbb{Z}$, so the previous result applies.

2. If z is a normal regular element, then the complement to the zero locus of z in $\mathrm{Proj}_{nc}A$ is $\mathrm{Proj}_{nc}A[z^{-1}]$; the category of modules over this is $\mathsf{GrMod}A[z^{-1}]$; if d is the smallest positive integer such that $A[z^{-1}]$ has a unit of degree d, this is an affine space with coordinate ring

$$\begin{pmatrix} R_0 & R_1 & \cdots & R_{d-1} \\ R_{-1} & R_0 & \cdots & R_{d-2} \\ \vdots & & & \vdots \\ R_{-d+1} & R_{-d+2} & \cdots & R_0 \end{pmatrix}$$

where $R = A[z^{-1}]$.

3. In the previous result, if s and t are homogeneous regular elements of relatively prime degrees in A, and st is normal, then $A[(st)^{-1}]$ has a unit of degree one, so the open complement to the zero locus of st in $\mathrm{Proj}_{nc}A$ is the affine space with coordinate ring $A[(st)^{-1}]_0$. This ring is equal to $A^{(n)}[(st)^{-n}]_0$, so the open complement is is isomorphic to the open complement to the zero locus of $(st)^n$ in $\operatorname{Proj}_{nc} A^{(n)}$.

4. If $\operatorname{Fract}_{gr} A$ fails to contain a copy of $A(n)$ for all n, the map $\operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$ need not be a birational isomorphism. For example, take $A = k[x]$ with $\deg x = 2$.

Example 4.6. *If A is not generated in degree one, then g_* need not be faithful.*

Let $A = k[x,z]$ be the polynomial ring with $\deg x = 1$ and $\deg z = n \geq 2$. The image under π of $M = A/(x)$ is a simple module $\mathcal{O}_p$ in $\operatorname{Proj}_{nc} A$. We have $\mathcal{O}_p(1) \neq 0$, but $(M(1))^{(n)} = 0$, so $g_*(\mathcal{O}_p(1)) = 0$.

One might anticipate that $g : \operatorname{Proj}_{nc} A \to \operatorname{Proj}_{nc} A^{(n)}$ is an isomorphism on suitable open subspaces: in the previous example, g restricts to an isomorphism from the complement to the zero locus of x in $\operatorname{Proj}_{nc} A$ to the complement to the zero locus of x in $\operatorname{Proj}_{nc} A^{(n)}$. We prove a general result of this type in Proposition 4.8. First we need a lemma.

For each integer r, define

$$A^{(n)+r} := \sum_{j \in \mathbb{Z}} A_{nj+r}.$$

Obviously, $A^{(n)+r} A^{(n)+s} \subset A^{(n)+r+s}$ so each $A^{(n)+r}$ is an $A^{(n)}$-$A^{(n)}$-bimodule, and these bimodules depend only on $r (\operatorname{mod} n)$. Define

$$I_r := A^{(n)+r} A = \sum_{j \in \mathbb{Z}} A_{nj+r} A$$

and

$$I := \bigcap_{r \in \mathbb{Z}} I_r = I_1 \cap I_2 \cap \cdots \cap I_n.$$

Although I_r is in general only a right ideal of A, $I_r^{(n)}$ is a two-sided ideal of $A^{(n)}$. Since $A_q I_r \subset I_{q+r}$, I is a two-sided ideal of A.

Notice that $A^{(n)+r} A^{(n)-r} = I_r^{(n)}$.

Lemma 4.7. *With the above notation, $I^{2n} \subset I^{(n)} A$.*

Proof. From the containment

$$I^2 \;\subset\; I_r I = A^{(n)+r} I \;\subset\; A^{(n)+r} I_{n-r} = A^{(n)+r} A^{(n)+n-r} A = I_r^{(n)} A,$$

it follows that

$$I^{2n} \;\subset\; I_1^{(n)} I^{2n-2} \;\subset\; I_1^{(n)} I_2^{(n)} I^{2n-4} \;\subset \cdots \subset\; I_1^{(n)} \cdots I_n^{(n)} A.$$

But this last term is contained in

$$\big(I_1^{(n)} \cap \cdots \cap I_n^{(n)}\big) A = I^{(n)} A,$$

which completes the proof. □

Proposition 4.8. *With the hypotheses of Theorem 4.1, the map g restricts to an isomorphism $g : \operatorname{Proj}_{nc} A \backslash Z' \to \operatorname{Proj}_{nc} A^{(n)} \backslash Z$ where Z' and Z are the zero loci of I and $I^{(n)}$ respectively.*

Proof. Write $X' = \operatorname{Proj}_{nc} A$, $X = \operatorname{Proj}_{nc} A^{(n)}$, $U' = X' \backslash Z'$ and $U = X \backslash Z$. Write $\alpha : U \to X$ and $\beta : U' \to X'$ for the inclusions. We will use Lemma 3.1 to show there is an isomorphism $h : U' \to U$ such that the diagram

$$\begin{tikzcd} U' \arrow[r, "\beta"] \arrow[d, "h"'] & X' = \operatorname{Proj}_{nc} A \arrow[d, "g"] \\ U \arrow[r, "\alpha"'] & X = \operatorname{Proj}_{nc} A^{(n)} \end{tikzcd}$$

commutes.

Let T be the localizing subcategory of $\mathsf{GrMod}A$ consisting of those modules L such that every element of L is killed by a suitably large power of I. Let S be the localizing subcategory of $\mathsf{GrMod}A^{(n)}$ consisting of those modules N such that every element of N is killed by a suitably large power of $I^{(n)}$. These two localizing subcategories contain all the finite dimensional modules. The spaces U and U' are defined by

$$\mathsf{Mod}U := (\mathsf{GrMod}A^{(n)})/\mathsf{S} \qquad \text{and} \qquad \mathsf{Mod}U' := (\mathsf{GrMod}A)/\mathsf{T}.$$

Let f_* and f^* be the functors defined in Proposition 4.2. We will show that $f_*(\mathsf{T}) \subset \mathsf{S}$ and $f^*(\mathsf{S}) \subset \mathsf{T}$. The first of these inclusions is obvious: if every element of an A-module L is annihilated by a power of I, then every element of $L^{(n)}$ is annihilated by a power of $I^{(n)}$. To show that $f^*(\mathsf{S}) \subset \mathsf{T}$ it suffices to show that $f^*(A^{(n)}/I^{(n)})$ belongs to T. But $f^*(A^{(n)}/I^{(n)}) \cong A/I^{(n)}A$, and by Lemma 4.7, $A/I^{(n)}A$ is annihilated by I^{2n} so belongs to T.

We now use Lemma 3.1 in the context of the following diagram

$$\begin{tikzcd} \mathsf{GrMod}A^{(n)} \arrow[r, "f^*"] \arrow[d, "\alpha^*\pi"'] & \mathsf{GrMod}A \arrow[d, "\beta^*\pi'"] \\ \mathsf{Mod}U & \mathsf{Mod}U'. \end{tikzcd}$$

Because $f^*(\mathsf{S}) \subset \mathsf{T}$, there exists a functor $h^* : \mathsf{Mod}U \to \mathsf{Mod}U'$ such that $h^*\alpha^*\pi = \beta^*\pi' f^*$. Because f_* is right adjoint to f^*, $h_* := \alpha^*\pi f_*\omega'\beta_*$ is right adjoint to h^*. Thus, h^* and h_* define a map $h : U' \to U$. Since $g_*\pi' = \pi f_*$, a computation gives $\alpha_* h_* \cong g_*\beta_*$. Therefore $\alpha h = g\beta$.

It remains to show that h is an isomorphism.

The unit $\mathrm{id}_{\mathsf{GrMod}A^{(n)}} \to f_* f^*$ is an isomorphism because the natural map $L \to (L \otimes_{A^{(n)}} A)^{(n)}$ is an isomorphism for all $L \in \mathsf{GrMod}A^{(n)}$. Because $f_*(\mathsf{T}) \subset \mathsf{S}$, part (3) of Lemma 3.1 gives $h_*\beta^*\pi' \cong \alpha^*\pi f_*$. Therefore,

$$h_* h^* \cong h_* h^* \alpha^* \pi \omega \alpha_* = h_* \beta^* \pi' f^* \omega \alpha_* \cong \alpha^* \pi f_* f^* \omega \alpha_* \cong \mathrm{id}_U .$$

To show that the natural transformation $h^* h_* \to \mathrm{id}_{U'}$ is an isomorphism, we first consider the natural transformation $f^* f_* \to \mathrm{id}_{\mathsf{GrMod}A}$. For an A-module M this is the multiplication map

$$f^* f_* M = M^{(n)} \otimes_{A^{(n)}} A \to M.$$

We claim that the kernel and cokernel of this map belong to T.

Suppose that $\sum m_i \otimes a_i \in M^{(n)} \otimes_{A^{(n)}} A$ is in the kernel. Then $\sum_i m_i a_i = 0$. By taking homogeneous components we can reduce to the case where each a_i belongs to $A_{n-r}+A_{2n-r}+\cdots$ for some $r\in\{1,\dots,n\}$. Then, if $b\in A_{nj+r}$ for some j, then

$$\Big(\sum_i m_i\otimes a_i\Big)b=\sum_i m_i\otimes a_ib=\sum_i m_ia_ib\otimes 1=0.$$

Thus $\big(\sum_i m_i\otimes a_i\big)I_r=0$. Hence the kernel is annihilated by I, so belongs to T. The cokernel of $f^*f_*M\to M$ is $M/M^{(n)}A$. If $r\in\{1,\dots,n\}$, then $M_{nj-r}I_r\subset M^{(n)}A$. Hence I annihilates the cokernel.

Since the kernel and cokernel of $f^*f_*\to\mathrm{id}$ belong to T, there is an isomorphism $\beta^*\pi'f^*f_*\to\beta^*\pi'$. Hence

$$h^*h_*=h^*\alpha^*\pi f_*\omega'\beta_*\cong\beta^*\pi'f^*f_*\omega'\beta_*\cong\mathrm{id}_{U'}.$$

□

In Example 4.6, $I_1=(x)$ and $I_2=A$, so $I=(x)$, whence Z' is the zero locus of x. This explains why we need to remove the zero locus of x to get the isomorphism.

If A is generated in degree one, then $I_r=A_{\ge r}$ for $r\in\{0,1,\dots,n-1\}$, so $A/I_r\in\mathsf{Fdim}A$, whence $A/I\in\mathsf{Fdim}A$. It follows that Z and Z' are empty, and therefore $U=X$ and $U'=X'$. We therefore recover Verevkin's result $X\cong X'$ when A is generated in degree one over A_0.

Example 4.9. *Let A be a weighted polynomial ring. That is, $A=k[x_0,\dots,x_n]$ where $\deg x_i=q_i\ge 1$. Write $Q=(q_0,\dots,q_n)$. Then $\mathbb{P}^n_Q:=\operatorname{Proj}A$ is called a* weighted projective space. *It is isomorphic to the quotient variety $\mathbb{P}^n/\mu_Q$, where $\mu_Q=\mu_{q_0}\times\cdots\times\mu_{q_n}$. There is a large integer d such that $A^{(d)}$ is generated in degree one. Hence*

$$\mathbb{P}^n_Q=\operatorname{Proj}A=\operatorname{Proj}A^{(d)}\cong\operatorname{Proj}_{nc}A^{(d)}.$$

By Theorem 4.1, there is a map

$$g:\operatorname{Proj}_{nc}A\to\operatorname{Proj}_{nc}A^{(d)}\cong\mathbb{P}^n_Q.$$

This is an isomorphism on an open subspace by Proposition 4.8. Since A has global homological dimension $n+1$, $\operatorname{Proj}_{nc}A$ has global homological dimension n. We therefore think of $\operatorname{Proj}_{nc}A$ as a smooth space of dimension n and the map g as a non-commutative resolution of $\mathbb{P}^n_Q$. Let $X\subset\mathbb{P}^n_Q$ be the closed subscheme cut out by an ideal J in A. Then there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Proj}_{nc}A/J \arrow[r, "i"] \arrow[d, "f"'] & \operatorname{Proj}_{nc}A \arrow[d, "g"] \\
X \arrow[r] & \mathbb{P}^n_Q
\end{tikzcd}$$

in which f is a birational isomorphism and i is a closed immersion. It can happen that $\operatorname{Proj}_{nc}A/J$ is smooth even when X is singular. Thus $\operatorname{Proj}_{nc}A$ is a "non-commutative resolution" of X. An interesting case to examine in some detail is that where X is an orbifold of a Calabi-Yau three-fold.

If $A=k[x]$ with $\deg x=2$, and $n=2$, then $\operatorname{Proj}_{nc}A\cong\operatorname{Spec}k\times k$ and $\operatorname{Proj}A\cong\operatorname{Spec}k$. Furthermore, $Z'=X'$ and $Z=X$. This is a special case of the next result, the truth of which was suggested by Darin Stephenson.

Proposition 4.10. *Let A be a locally finite $\mathbb{N}$-graded k-algebra such that $A_i = 0$ whenever $i \not\equiv 0 (\mathrm{mod}\ n)$. Then $\operatorname{Proj}_{nc} A$ is isomorphic to the disjoint union of n copies of $\operatorname{Proj}_{nc} A^{(n)}$.*

Proof. Let $p_r : \mathsf{GrMod}A \to \mathsf{GrMod}A$ be the functor defined by

$$p_r(M) = \bigoplus_{i\in\mathbb{Z}} M_{r+in}$$

on objects, and $p_r(\theta) = \theta|_{p_r(M)}$ whenever $\theta \in \operatorname{Hom}_{\mathsf{Gr}A}(M, N)$. The hypothesis on A ensures that each $p_r(M)$ is a graded A-submodule of M, so p_r is indeed a functor from $\mathsf{GrMod}A$ to itself. It is clear that $\mathrm{id}_{\mathsf{GrMod}A} = p_0 \oplus \cdots \oplus p_{n-1}$, where this direct sum is taken in the abelian category of k-linear functors from $\mathsf{GrMod}A$ to itself; essentially, this is the observation that $M = p_0(M) \oplus \cdots \oplus p_{n-1}(M)$, and that any map $\theta : M \to N$ of graded A-modules respects this decomposition. Furthermore, each p_r is idempotent and the p_rs are mutually orthogonal. It follows from this that there is a decomposition of $\mathsf{GrMod}A$ as a product of categories, each component being the full subcategory on which p_r is the identity.

It is clear that the shift functor (1) cyclicly permutes these subcategories, so they are all equivalent to one another and (n) is an auto-equivalence of each component. However, any one of these categories together with its autoequivalence (n) is equivalent to $\mathsf{GrMod}A^{(n)}$ with its auto-equivalence (1). Thus $\mathsf{GrMod}A$ is equivalent to the product of n copies of $\mathsf{GrMod}A^{(n)}$.

Finally, this decomposition descends to the Tails categories. □

5. AN ORE EXTENSION AND AN EXAMPLE

The morphism

$$\begin{aligned} p : \mathbb{P}^n \backslash \{(0, \dots, 0, 1)\} &\to \mathbb{P}^{n-1} \\ (\alpha_0, \dots, \alpha_n) &\mapsto (\alpha_0, \dots, \alpha_{n-1}) \end{aligned}$$

is called the projection with center $(0, \dots, 0, 1)$. This section examines a non-commutative analogue of this basic operation.

Consider a connected graded k-algebra R and a connected graded Ore extension

$$S = R[t; \sigma, \delta]$$

with respect to a graded automorphism σ and a graded σ-derivation δ of degree $n \geq 1$. Thus $S = \oplus_{n=0}^{\infty} Rt^n$ and $tr = r^\sigma t + \delta(r)$ for all $r \in R$. Since $\delta(R_i) \subset R_{i+n}$ for all i, by setting $\deg t = n$, S becomes a connected graded algebra.

One expects that the inclusion map $R \to S$ induces a map $\operatorname{Proj}_{nc} S \to \operatorname{Proj}_{nc} R$. Indeed, the projection map above can be obtained as a special case of this.

Let $\mathfrak{m}$ denote the augmentation ideal of R. Since $\delta(\mathfrak{m}) \subset \mathfrak{m}$, $\mathfrak{m}S$ is a two-sided ideal of S. Furthermore, $S/\mathfrak{m}S \cong k[t]$ as graded rings.

Proposition 5.1. *With the above notation, let Z denote the zero locus of $\mathfrak{m}S$ in Y.*

1. *$Z \cong \operatorname{Spec} k^{\times n}$.*
2. *There is an affine map $g : \operatorname{Proj}_{nc} S \backslash Z \to \operatorname{Proj}_{nc} R$.*

Proof. (2) The existence of g is a special case of Theorem 3.3. That proposition applies because $S\mathfrak{m} \subset \mathfrak{m}S$.

(1) The quotient ring $S/\mathfrak{m}S$ is isomorphic to the polynomial ring $k[t]$ with $\deg t = n$, so this follows from Proposition 4.10. □

We think of $\operatorname{Proj}_{nc} S$ as a "cone over $\operatorname{Proj}_{nc} R$ with vertex Z". It would be interesting to describe the "fibers" of the map g.

When $\deg t > 1$, the Ore extension $S = R[t; \sigma, \delta]$ is not generated by its elements of degree one. This sometimes causes technical problems; however, if R is generated in degree one, then the n^{th} Veronese $S^{(n)}$ is generated in degree one. We can then combine Theorems 3.3 and 4.1 to analyze the space with homogeneous coordinate ring S as follows.

Proposition 5.2. *The inclusion of the n-Veronese subalgebras of S and $S/\mathfrak{m}S$ gives a commutative diagram of rings and an induced commutative diagram of spaces as in the following diagram:*

$$\begin{tikzcd}
k[t]^{(n)} \arrow[r] & k[t] = S/S\mathfrak{m} \\
S^{(n)} \arrow[r] \arrow[u] & S \arrow[u] \\
R^{(n)} \arrow[r] \arrow[u] & R \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Spec} k \cong v \arrow[d] & Z' \cong \operatorname{Spec} k^{\times n} \arrow[l] \arrow[d] \\
\operatorname{Proj}_{nc} S^{(n)} & \operatorname{Proj}_{nc} S \arrow[l, "g"'] \\
\operatorname{Proj}_{nc} S^{(n)} \backslash \{v\} \arrow[u] \arrow[d, "\alpha"'] & \operatorname{Proj}_{nc} S \backslash Z' \arrow[l] \arrow[u] \arrow[d, "\beta"] \\
\operatorname{Proj}_{nc} R^{(n)} & \operatorname{Proj}_{nc} R \arrow[l]
\end{tikzcd}$$

An application. In [11], a family of three-dimensional Artin-Schelter regular algebras A is constructed and studied. Although the algebraic properties of A are quite well understood, our understanding of the corresponding geometric object $\operatorname{Proj}_{nc} A$ is rudimentary. The algebras are of the form $A = R[t; \sigma, \delta]$ with $\deg t = 2$ and R a two-dimensional Artin-Schelter regular algebra generated in degree one. It is well-known that R and its Veronese subalgebras are (not necessarily commutative) homogeneous coordinate rings of $\mathbb{P}^1$. By Proposition 5.2, there is a commutative diagram of spaces and maps

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d] & Z' \cong \operatorname{Spec} k^{\times n} \arrow[l] \arrow[d] \\
\operatorname{Proj}_{nc} A^{(n)} & \operatorname{Proj}_{nc} A \arrow[l, "g"'] \\
\operatorname{Proj}_{nc} A^{(n)} \backslash \{v\} \arrow[u] \arrow[d, "\alpha"'] & \operatorname{Proj}_{nc} A \backslash Z' \arrow[l, "g"'] \arrow[u] \arrow[d, "\beta"] \\
\mathbb{P}^1 & \mathbb{P}^1. \arrow[l, "\cong"']
\end{tikzcd} \tag{5-1}$$

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