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Non-commutative quadric surfaces

S. Paul Smith and M. Van den Bergh

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1 Introduction

Many non-commutative analogues of $\mathbb{P}^3$ contain non-commutative quadric hypersurfaces. This paper studies these non-commutative quadrics, and also the consequences of their existence for the ambient non-commutative $\mathbb{P}^3$.

For example, we establish a simple criterion for recognizing when a non-commutative quadric surface is smooth: it is smooth if and only if it has two rulings. Of course, a key point is to define the terms.

The smoothness result allows us to make some further comparisons with the commutative case. For example, a generic pencil of quadrics in $\mathbb{P}^3$ has exactly four singular members and we show the same is true for the pencil of non-commutative quadrics in the Sklyanin analogue of $\mathbb{P}^3$.

Associated to a non-commutative quadric surface Q is a Grothendieck category $\mathsf{Mod}Q$ that plays the role of the quasi-coherent sheaves on it. We say that Q is smooth of dimension two if $\operatorname{Ext}^3_Q(-,-)$, but not $\operatorname{Ext}^2_Q(-,-)$, is identically zero. The problem of deciding whether a non-commutative space is smooth differs from the commutative problem because a non-commutative space can have few closed points, sometimes none at all. One cannot check smoothness by checking the homological properties of the individual points. In this sense, smoothness is not a local property as it is in the commutative case. In particular, there is no analogue of the Jacobian criterion and singular non-commutative quadrics need not have a singular point.

Non-commutative quadric surfaces are defined as degree two hypersurfaces in suitable non-commutative analogues of $\mathbb{P}^3$, the latter being a non-commutative space of the form $\operatorname{Proj} S$ where S is a not necessarily commutative connected graded ring having many of the same properties as the commutative polynomial ring in four variables (see section 2.6). Thus $Q = \operatorname{Proj} A$ where $A = S/(z)$ and $z \in S_2$ is a central regular element. (If z were a normal regular element we could replace S by a suitable Zhang twist in which z becomes central, so there is no loss of generality in assuming z is central.) Amongst other things, S is required to be a Koszul algebra and this implies that $S/(z)$ is also Koszul.

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Let S denote a 4-dimensional Sklyanin algebra [14], [15], [18], [22], [23]. Then $\operatorname{Proj} S$ contains a commutative elliptic curve E that is the zero locus of two degree two central elements in S. Linear combinations of these two central elements determine a pencil of non-commutative quadrics $Q = \operatorname{Proj} S/(z) \subset \operatorname{Proj} S$. Exactly four of these non-commutative quadrics are singular (Theorem 10.2. The base locus of the pencil is E. This is a direct analogue of the commutative case: the base locus of a generic pencil of quadrics in $\mathbb{P}^3$ is a quartic elliptic curve, and exactly four members of that pencil are singular.

The method we use to understand these non-commutative quadrics follows that of Buchweitz, Eisenbud, and Herzog in their paper [4] on maximal Cohen-Macaulay modules over quadrics, and especially the approach in the appendix of their paper. As is well known, a quadric hypersurface is smooth if and only if the even Clifford algebra determined by its defining equation is semisimple. The results in [4] establish a much deeper relation between the quadric and the Clifford algebra. We also associate to our quadric $Q = \operatorname{Proj} A$ a finite dimensional algebra C that is the analogue of an even Clifford algebra.

Let $A^!$ be the quadratic dual of A. Because A is a hypersurface ring, $\operatorname{Proj} A^!$ is an affine space. The algebra C is a coordinate ring of this space in the sense that $\operatorname{Proj} A^! \cong \mathsf{Mod} C$. We show that Q is smooth if and only if C is semisimple if and only if there are two distinct non-commutative "rulings" on Q. We show that the "lines" on Q determine two-dimensional simple C-modules; because the dimension of C is 8, it is semisimple if it has two non-isomorphic two-dimensional simple modules. The method by which we associate a C-module to a line on Q uses the fact that A is a Koszul algebra, and that the lines on Q determine graded maximal Cohen-Macaulay A-modules.

Although quantum $\mathbb{P}^2$s have been classified and are well-understood in some regards, the same is not true for quantum $\mathbb{P}^3$s. The results in this paper are a step towards gaining a similar understanding of another class of non-commutative surfaces. In sections 8 and 7 we obtain good information about the points and lines on such surfaces. Further, if the non-commutative quadric Q is smooth there is an isomorphism $K_0(Q) \cong K_0(\mathbb{P}^1 \times \mathbb{P}^1)$ of Grothendieck groups that is compatible with the Euler forms $(-,-) = \sum (-1)^i \dim \operatorname{Ext}^i_Q(-,-)$. More interestingly, the effective cones for Q and $\mathbb{P}^1 \times \mathbb{P}^1$ need not match up under this isomorphism: sometimes Q contains, in effect, a -2-curve.

All unexplained terminology for non-commutative spaces can be found in either [17] or [25].

2 Preliminaries

Throughout k denotes a field. A graded algebra A is connected if $A = k \oplus A_1 \oplus \cdots$. Most of the algebras in this paper are connected, graded k-algebras.

The Hilbert series of a graded k-vector V having finite-dimensional components is the formal series

$$H_V(t) := \sum (\dim_k V_n) t^n.$$

We write $\mathsf{Mod}C$ for the category of right modules over a ring C, and $\mathsf{mod}C$ for the full subcategory of noetherian modules.

2.1 Graded algebras

If A is a graded k-algebra, $\mathsf{GrMod}A$ denotes its category of graded right modules with degree zero module homomorphisms, and $\mathsf{grmod}A$ the full subcategory of noetherian modules.

We write $\mathsf{D}^b(A)$ for the bounded derived category of noetherian graded A-modules.

We write $\mathrm{Ext}^i_{\mathsf{Gr}A}(M,N)$ for the extension groups in $\mathsf{GrMod}A$, and define

$$\underline{\mathrm{Ext}}^*_A(M,N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_{\mathsf{Gr}A}(M,N(i)).$$

When A is right noetherian, and M a noetherian right A-module. we write $\Omega^i M$ for the i^{th} syzygy in $\mathsf{grmod}A$ of obtained from a *minimal* resolution of M. When A is connected graded, we have $\underline{\mathrm{Ext}}^i_A(\Omega^d M,k) \cong \underline{\mathrm{Ext}}^{i+d}_A(M,k)$ for all $i \geq 0$. We often write ΩM for $\Omega^1 M$.

A graded A-module M has a $\mathsf{linear\ resolution}$ if for all i the i^{th} term in its minimal projective resolution is a direct sum of copies of $A(-i)$ or, equivalently, if $\underline{\mathrm{Ext}}^i_A(M,k)_j = 0$ whenever $i+j \neq 0$. We write $\mathsf{Lin}(A)$ for the full subcategory of $\mathsf{GrMod}A$ of finitely generated graded modules having a linear resolution; it is closed under direct summands and extensions.

Suppose that M has a linear resolution. Then so do the shifts $(\Omega^n M)(n)$ of its minimal syzygies. Furthermore,

$$H_{\underline{\mathrm{Ext}}^*_A(M,k)}(t) H_A(-t) = H_M(-t). \tag{1}$$

Lemma 2.1 *If S is a graded k-algebra of finite global dimension and $H_S(t) = (1-t)^{-n}$, then the Hilbert series of any finitely generated A-module of GK-dimension one is eventually constant.*

Proof. The minimal projective resolution of a finitely generated A-module M is finite, and all terms are direct sums of shifts of A, so the Hilbert series of the module is of the form $f(t)(1-t)^{-n}$ for some $f(t) \in \mathbb{Z}[t,t^{-1}]$. The hypothesis on the GK-dimension means that we can rewrite this as $g(t)(1-t)^{-1}$ with $g(t) \in \mathbb{Z}[t,t^{-1}]$. Hence $\dim M_n = g(1)$ for $n \gg 0$. $\square$

2.2 Koszul duality

Basic information about Koszul algebras can be found in [3], [16], and [20].

Let A be a connected Koszul algebra and $A^!$ its quadratic dual.

Being Koszul says that the natural homomorphism $A^! \to \underline{\mathrm{Ext}}^*_A(k,k)$ is an isomorphism of graded k-algebras. If M is a graded A-module, the Yoneda product makes $\underline{\mathrm{Ext}}^*_A(M,k)$ a graded left $A^!$-module with degree i component $\underline{\mathrm{Ext}}^i_A(M,k)$.

A module M is stably linear if $M_{\geq n}(n)$ has a linear resolution for $n \gg 0$. We write $\mathsf{D}^b_{sl}(A)$ for the full subcategory of $\mathsf{D}^b(A)$ of complexes having stably linear homology.

By [20] there is a duality

$$K : \mathsf{D}^b_{sl}(A) \to \mathsf{D}^b_{sl}(A^!)$$

given by

$$KM = T\big(\mathrm{R}\underline{\mathrm{Hom}}_A(M, k)\big)$$

where T is the re-grading functor given by $(TV)^i_j = V^{i+j}_{-j}$ where the upper index denotes the homological (i.e., complex) degree and the lower index denotes the grading degree. The duality K restricts to a duality

$$\mathsf{Lin}(A) \to \mathsf{Lin}(A^!), \qquad M \mapsto \underline{\mathrm{Ext}}^*_A(M, k)$$

with degree i component $\underline{\mathrm{Ext}}^i_A(M, k)$. The Koszul duality functor K satisfies

$$K(M[1]) \cong (KM)[-1] \qquad \text{and} \qquad K(M(1)) \cong (KM)[-1](1). \tag{2}$$

Theorem 2.2 (Jørgensen) *[8, Thm. 3.1] Let A be a two-sided noetherian, connected, graded k-algebra that is Koszul and has a balanced dualizing complex [24], [26]. Then every finitely generated A-module is stably linear. Thus $\mathsf{D}^b_{sl}(A) = \mathsf{D}^b(A)$.*

2.3 Cohen-Macaulay rings and modules

Let A be a right and left noetherian, connected, graded k-algebra having a balanced dualizing complex $R^{\cdot}$ [24], [26].

We say that A is Cohen-Macaulay of depth d if there is an A-A-bimodule ω_A such that $R^{\cdot} \cong \omega_A[d]$. We call ω_A the dualizing module for A. By [2, Propostion 7.9], ω_A is finitely generated on each side. We say A is Gorenstein if it is Cohen-Macaulay and ω_A is an invertible bimodule. This is equivalent to the requirement that ω_A be isomorphic to $A(\ell)$ for some ℓ as both a right and as a left module.

The local cohomology functors

$$H^i_{\mathfrak{m}}(-) = \varinjlim \underline{\mathrm{Ext}}^i_A(A/A_{\geq n}, -)$$

are defined on graded right A-modules. Here $\mathfrak{m}$ denotes the maximal ideal $A_{\geq 1}$. We write $H^i_{\mathfrak{m}^\circ}$ for the local cohomology modules for left modules. The depth of an A-module M is the smallest integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$. A finitely generated module M is Cohen-Macaulay if either $M = 0$ or only one $H^i_{\mathfrak{m}}(M)$ is non-zero. For the rest of this section we assume that A is Cohen-Macaulay of depth d in the sense of the previous paragraph. Then A is a Cohen-Macaulay A-module of depth d in the sense of the present paragraph. Furthermore, there is an isomorphism

$$\omega_A \cong H^d_{\mathfrak{m}}(A)^*$$

of A-A-bimodules and, for every $M \in \mathsf{grmod} A$,

$$\underline{\mathrm{Ext}}^i_A(M, \omega_A) \cong H^{d-i}_{\mathfrak{m}}(M)^* \tag{3}$$

as graded left A-modules [26, Theorem 4.2].

2.4 The condition χ

Let A be a connected graded k-algebra.

We say A satisfies condition χ if $\underline{\mathrm{Ext}}^i_A(k, M)$ is finite dimensional for all finitely generated M and all i. By [2, Cor. 3.6], this is equivalent to $H^i_{\mathfrak{m}}(M)$ being zero in large positive degree for all i and all finitely generated M. Hence, if A is noetherian and Cohen-Macaulay, formula (3) implies that A satisfies χ on both sides [26, Theorem 4.2]. The precise relationship between the condition χ and the Cohen-Macaulay property is given by [24, Theorem 6.3].

A noetherian, connected, graded algebra A satisfying χ has finite depth, and for every $M \in \mathsf{grmod} A$ of finite projective dimension,

$$\mathrm{pdim}\, M + \mathrm{depth}\, M = \mathrm{depth}\, A.$$

As in the commutative case we call this the Auslander-Buchsbaum formula. The non-commutative version was proved by Jørgensen [7].

2.5 Non-commutative spaces

For us a non-commutative space X is a Grothendieck category $\mathsf{Mod} X$. We write $\mathsf{mod} X$ for the full subcategory of noetherian X-modules.

Suppose that A is a connected graded k-algebra.

Artin and Zhang [2] define

$$\mathrm{Proj}\, A = \mathsf{Mod}\,\mathrm{Proj}\, A := \mathsf{GrMod} A/\mathsf{Fdim} A,$$

where $\mathsf{Fdim} A$ is the full subcategory consisting of direct limits of finite dimensional modules. We write $\pi : \mathsf{GrMod} A \to \mathsf{Mod}\,\mathrm{Proj}\, A$ for the quotient functor and ω for its right adjoint. Modules in $\mathsf{Fdim} A$ are said to be torsion.

Write $X = \mathrm{Proj}\, A$ and $\mathcal{O}_X = \pi A$.

Artin and Zhang define the cohomology groups

$$H^q(X, \mathcal{F}) := \mathrm{Ext}^q_X(\mathcal{O}_X, \mathcal{F}).$$

If M is a graded A-module, there is an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \omega\pi M \to H^1_{\mathfrak{m}}(M) \to 0, \tag{4}$$

and, if $\mathcal{M} = \pi M$, then

$$H^q(X, \mathcal{M}) \cong H^{q+1}_{\mathfrak{m}}(M)_0 \tag{5}$$

for $q \geq 1$ (see [2, Prop. 7.2]).

Following [26, Defn. 2.4], we say that X is Cohen-Macaulay of dimension d if there is an X-module ω_X and isomorphisms

$$H^q(X,-) \to \operatorname{Ext}_X^{d-q}(-,\omega_X)^*$$

on $\mathsf{mod} X$ for all q.

Suppose that A is noetherian and Cohen-Macaulay of depth $d+1$. Since A satisfies χ, (5) allows us to quote [26, Thm. 2.3] which says that $X = \operatorname{Proj} A$ is Cohen-Macaulay of dimension d with $\omega_X \cong \pi(H_{\mathfrak{m}}^{d+1}(A)^*)$.

2.6 Non-commutative analogues of $\mathbb{P}^n$

The quadric surfaces of interest to us are degree two hypersurfaces in quantum $\mathbb{P}^3$s.

For the purposes of this paper we will call $\operatorname{Proj} S$ a quantum $\mathbb{P}^n$ if S is a connected graded k-algebra with the following properties:

1. S has global homological dimension $n+1$ on both sides and

$$\operatorname{Ext}^i(k,S) \cong \begin{cases} 0 & \text{if } i \neq n+1 \\ k & \text{if } i = n+1 \end{cases}$$

   for the right and left trivial modules $k = S/S_{\geq 1}$ (i.e., S is an Artin-Schelter (AS) regular algebra);

2. S is right and left noetherian;

3. $H_S(t) = (1-t)^{-n-1}$.

J.J. Zhang showed that these conditions imply that S is a Koszul algebra and has dualizing module $\omega_S \cong A(-n-1)$ [16, Thm. 5.11]. Furthermore, S satisfies χ on both sides. When $n+1 \leq 4$, S is a domain by [1].

A result of Shelton and Vancliff [13, Lemma 1.3] shows that for quantum $\mathbb{P}^3$s the hypotheses (1)-(3) are not the most efficient—one can slightly weaken them.

Write $\mathbb{P}^n_{nc} = \operatorname{Proj} S$. The hypotheses ensure that $H^{n+1}(\mathbb{P}^n_{nc},-) = 0$ and that the dimensions of $H^q(\mathbb{P}^n_{nc},\mathcal{O}(r))$ agree with those in the commutative case.

The Grothendieck group of a quantum $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}[t,t^{-1}]/(1-t)^{n+1}$ with $[\mathcal{F}(-1)] = [\mathcal{F}]t$. There is a good notion of degree for closed subspaces of $\operatorname{Proj} S$. In particular, if $z \in S$ is a homogeneous normal element, meaning that $Sz = zS$, then $\operatorname{Proj} S/(z)$ is a hypersurface of degree equal to $\deg z$. Write $A = S/(z)$. Then A is Gorenstein of depth n, and satisfies χ. In particular, $H^n(\operatorname{Proj} A,-) = 0$.

3 Maximal Cohen-Macaulay modules

Suppose that A is connected graded and Cohen-Macaulay of depth $d \geq 1$.

A noetherian A-module M is maximal Cohen-Macaulay if $\operatorname{depth} M = d$. We write $\mathsf{MCM}(A)$ for the full subcategory of $\mathsf{grmod} A$ consisting of the maximal Cohen-Macaulay modules; we consider the zero module to be maximal Cohen-Macaulay, so $\mathsf{MCM}(A)$ is an additive category.

If $i \geq d$, then every i^{th} syzygy is a maximal Cohen-Macaulay module. All the syzygies of a maximal Cohen-Macaulay module are maximal Cohen-Macaulay modules. The i^{th} syzygy of a module is not well-defined, but there is a quotient $\underline{\mathsf{MCM}}(A)$ of $\mathsf{MCM}(A)$ in which the i^{th} syzygies of a module become isomorphic.

The objects in $\underline{\mathsf{MCM}}(A)$ are the maximal Cohen-Macaulay modules, and the morphisms are

$$\operatorname{Hom}_{\underline{\mathsf{MCM}}(A)}(M, N) := \operatorname{Hom}_{\mathsf{Gr}A}(M, N)/P(M, N)$$

where $P(M, N)$ consists of the degree zero A-module maps $f : M \to N$ that factor through a projective in $\mathsf{GrMod} A$. Projective modules become isomorphic to zero in $\underline{\mathsf{MCM}}(A)$ because the identity morphism on a projective belongs to P. Hence, if M is a finitely generated module of depth n and $i + n \geq d$, the i^{th} syzygy of M is a well-defined object of $\underline{\mathsf{MCM}}(A)$ up to isomorphism.

The next two results are due to Buchweitz and are stated in his appendix to the paper [4]; see also [5].

Theorem 3.1 *Suppose that A is Gorenstein. Then $\underline{\mathsf{MCM}}(A)$ is a triangulated category with respect to the translation functor $M[-1] := \Omega M$. If M and N are maximal Cohen-Macaulay modules, then*

$$\operatorname{Hom}_{\underline{\mathsf{MCM}}(A)}(M, N[n]) \cong \operatorname{Ext}^n_{\mathsf{Gr}A}(M, N)$$

for all $n \geq 1$.

Theorem 3.2 *Let A be a Gorenstein, connected, graded, Koszul algebra over a field k, and $A^!$ its quadratic dual. The Koszul duality functor K fits into a commutative diagram*

$$\begin{array}{ccccccc} \mathsf{MCM}(A) & \longrightarrow & \mathsf{grmod} A & \longrightarrow & \mathsf{D}^b(A) & \xrightarrow{K} & \mathsf{D}^b(A^!) \\ \downarrow & & & & & & \downarrow \\ \underline{\mathsf{MCM}}(A) & & & \xrightarrow[B]{\hspace{6em}} & & & \mathsf{D}^b(\operatorname{proj} A^!) \end{array} \qquad (6)$$

in which the bottom arrow is a duality

$$\underline{\mathsf{MCM}}(A) \cong \mathsf{D}^b(\operatorname{proj} A^!), \qquad M \mapsto \mathrm{R}\underline{\operatorname{Hom}}_A(M, k).$$

The t-structure on $\underline{\mathsf{MCM}}(A)$ induced by the natural t-structure on $\mathsf{D}^b(\operatorname{proj} A^!)$ is

$$\underline{\mathsf{MCM}}(A)^{\geq p} = \{M \mid \underline{\operatorname{Ext}}^i_A(M, k)_j = 0 \text{ for } i + j > p\}$$
$$\underline{\mathsf{MCM}}(A)^{\leq p} = \{M \mid \underline{\operatorname{Ext}}^i_A(M, k)_j = 0 \text{ for } i + j < p\}$$

Its heart consists of the maximal Cohen-Macaulay modules having a linear resolution.

We will refer to the duality

$$B : \underline{\mathsf{MCM}}(A) \to \mathsf{D}^b(\operatorname{proj} A^!)$$

in Theorem 3.2 as "Buchweitz's duality".

Lemma 3.3 *Suppose that A is a connected graded, Gorenstein, Koszul algebra. Write F for the composition*

$$\mathsf{grmod} A \longrightarrow \mathsf{D}^b(A) \xrightarrow{K} \mathsf{D}^b(A^!) \longrightarrow \mathsf{D}^b(\operatorname{proj} A^!) \qquad (7)$$

If $M \in \mathsf{grmod} A$, then $F(\Omega M) \cong (FM)[1]$ and $F(\Omega M(1)) \cong (FM)(1)$.

Proof. There is an exact sequence $0 \to \Omega M \to \oplus_{i \in I} A(i) \to M \to 0$ for some multiset I, and hence a distinguished triangle $\Omega M \to \oplus_{i \in I} A(i) \to M \to$ in $\mathsf{D}^b(A)$. The image of $K(\oplus A(i))$ in $\mathsf{D}^b(\operatorname{proj} A^!)$ is zero so there is an isomorphism

$$\pi K M \xleftarrow{\sim} \pi K((\Omega M)[1]) \cong \pi K(\Omega M)[-1]$$

in $\mathsf{D}^b(\operatorname{proj} A^!)$. Hence $FM \cong F(\Omega M)[-1]$.

The other isomorphism is established in a similar way. □

Lemma 3.4 *Suppose that A is a connected graded, Gorenstein algebra. The isomorphism classes of indecomposable objects in $\underline{\mathsf{MCM}}(A)$ are in bijection with the isomorphism classes of indecomposable non-projective modules in $\mathsf{MCM}(A)$.*

Proof. Non-trivial direct summands of an object in an additive category correspond to non-trivial idempotents in its endomorphism ring.

Let M be an indecomposable non-projective in $\mathsf{MCM}(A)$. Then $E = \operatorname{End}_{\mathsf{Gr} A} M$ is a finite dimensional local ring, meaning that $E/\operatorname{rad} E$ is a division ring. Hence the endomorphism ring of M in $\underline{\mathsf{MCM}}(A)$ is also local, so M is indecomposable in $\underline{\mathsf{MCM}}(A)$.

Let M' be another indecomposable non-projective in $\mathsf{MCM}(A)$, and suppose that $f : M \to M'$ and $g : M' \to M$ become mutually inverse isomorphisms in $\underline{\mathsf{MCM}}(A)$. To show that M is isomorphic to M' in $\mathsf{MCM}(A)$, it suffices to show that fg and gf are isomorphisms in $\mathsf{MCM}(A)$. It therefore suffices to show that if $h : M \to M$ is an isomorphism in $\underline{\mathsf{MCM}}(A)$, then it is an isomorphism in $\mathsf{MCM}(A)$. But this is clear, since the isomorphisms $M \to M$ in either category are the endomorphisms that are not in the radical.

We have shown that the functor $\mathsf{MCM}(A) \to \underline{\mathsf{MCM}}(A)$ gives an injective map from the set of isomorphism classes of indecomposable non-projective Cohen-Macaulay modules to the set of isomorphism classes of indecomposable objects in $\underline{\mathsf{MCM}}(A)$. We now show this map is surjective. If M is a maximal Cohen-Macaulay module that becomes indecomposable as an object in $\underline{\mathsf{MCM}}(A)$ we may write M as a direct sum of indecomposables in $\mathsf{MCM}(A)$ and this gives a direct sum decomposition of M in $\underline{\mathsf{MCM}}(A)$ each term of which is either zero or indecomposable; hence, in $\underline{\mathsf{MCM}}(A)$, M is isomorphic to some M' where M' is an indecomposable non-projective in $\mathsf{MCM}(A)$. □

Remark. Suppose that A is Gorenstein, and let N be a maximal Cohen-Macaulay module having no non-zero projective direct summand. By applying $\underline{\mathrm{Hom}}_A(-,A)$ to $0 \to \Omega N \to P \to N \to 0$, where $P \to N \to 0$ is the start of a minimal projective resolution, one sees that ΩN also has no non-zero projective direct summand. Hence for all $d \geq 0$, $\Omega^d N$ has no non-zero projective direct summands.

Lemma 3.5 *Let A be a connected, graded, Gorenstein, Koszul algebra and*

$$B : \underline{\mathsf{MCM}}(A) \to \mathsf{D}^b(\mathrm{proj}\, A^!)$$

the equivalence in Theorem 3.2. Let S be a simple $\mathrm{Proj}\, A^!$*-module. Then*

1. *there is a unique up to isomorphism indecomposable $M \in \mathsf{MCM}(A)$ such that $BM \cong S[0]$;*
2. *if $S \cong S(d)$, then $\Omega^d M(d) \cong M$;*
3. *$\Omega^n M(n)$ has a linear resolution for all n.*

Proof. The equivalence of categories and Lemma 3.4 ensure the existence and uniqueness of M. Because $S[0]$ is in the heart of $\mathsf{D}^b(\mathrm{proj}\, A^!)$, M has a linear resolution. An induction argument using Lemma 3.3 shows that $B(\Omega^d M(d)) \cong (BM)(d) \cong S(d) \cong S$, so $M \cong \Omega^d M(d)$ in $\underline{\mathsf{MCM}}(A)$. Because M is indecomposable, it follows that M is isomorphic to a direct summand of $\Omega^d M(d)$ in $\mathsf{MCM}(A)$ and the complementary summand is projective. By the previous remark, $\Omega^d M$ has no non-zero projective direct summand, so $\Omega^d M(d) \cong M$.

Since M has a linear resolution, so does each $\Omega^n M(n)$. □

4 Smoothness

A non-commutative space X is smooth of global dimension d if d is the largest integer such that $\mathrm{Ext}^d_X(M,N) \neq 0$ for some X-modules M and N.

We now consider the question of what homological properties of a connected graded k-algebra A imply that $\mathrm{Proj}\, A$ is smooth.

The following summarizes the commutative case.

Proposition 4.1 *Let A be a graded quotient of a positively graded polynomial ring. Write $X = \mathrm{Proj}\, A$. The following are equivalent:*

1. $\mathrm{gldim}\, X \leq d$*;*
2. $\dim_k \underline{\mathrm{Ext}}^i_A(M,N) < \infty$ *for all $i \geq d+1$ and all $M, N \in \mathsf{grmod} A$;*

3. *whenever $N \to E^\bullet$ is a minimal injective resolution in* $\mathsf{GrMod}A$*,* E^i *is torsion for all* $i \geq d+1$*.*

Proof. (3) $\Leftrightarrow$ (1) If $0 \to N \to E^\bullet$ is a minimal injective resolution in $\mathsf{GrMod}A$, then $0 \to \pi N \to \pi E^\bullet$ is a minimal injective resolution in $\mathsf{Mod}X$. Thus $\operatorname{gldim} X \leq d$ if and only if $\pi E^i = 0$ for all $i \geq d+1$. Hence the equivalence of conditions (1) and (3)

(2) $\Rightarrow$ (1) Since A is commutative, $\underline{\operatorname{Ext}}^i_A(M,N)$ is an A-module. If f is a homogeneous element of A lying in $\mathfrak{m}$, then $A[f^{-1}] \otimes_A \underline{\operatorname{Ext}}^i_A(M,N) = 0$ for $i > d$, and so $\underline{\operatorname{Ext}}^i_{A[f^{-1}]} \equiv 0$ for $i > d$. Hence $\operatorname{Proj} A[f^{-1}]$ is smooth of global dimension at most d. Since X is covered by open affines of the form $\operatorname{Spec} A[f^{-1}]_0 \cong \operatorname{Proj} A[f^{-1}]$ with $f \in \mathfrak{m}$, it follows that $\operatorname{gldim} X \leq d$.

(3) $\Rightarrow$ (2) If condition (3) holds then applying $\underline{\operatorname{Hom}}_A(M,-)$ to a minimal injective resolution of N produces a complex consisting of torsion modules after the d^{th} term. Hence $\underline{\operatorname{Ext}}^i_A(M,N)$ is torsion for $i > d$. However, if M and N are noetherian, then $\underline{\operatorname{Ext}}^i_A(M,N)$ is a noetherian A-module as one sees by applying $\underline{\operatorname{Hom}}_A(-,N)$ to a minimal projective resolution of M. Hence $\underline{\operatorname{Ext}}^i_A(M,N)$ is finite dimensional for $i > d$ whenever $M, N \in \mathsf{grmod}A$. □

The proof of (3) $\Leftrightarrow$ (1) works when A is not commutative, but the other two parts of the proof fail because $\underline{\operatorname{Ext}}^i_A(M,N)$ is not an A-module when A is not commutative. Nevertheless, we will show that the implication (3) $\Rightarrow$ (2) holds if A satisfies χ.

First we need the following lemma that we learned from Kontsevich.

Lemma 4.2 *Let A be a noetherian connected graded k-algebra satisfying χ. If* $\operatorname{gldim} \operatorname{Proj} A = d < \infty$*, and* $M \in \mathsf{grmod}A$*, then there is a perfect complex* $P \in \mathsf{D}^b(\mathsf{grmod}A)$ *concentrated in homological degree* $[-d, 0]$ *and a bounded complex N together with a map in* $M_{\geq n} \oplus N \to P$ *(n large) whose cone has finite dimensional cohomology.*

Proof. Take an exact sequence $0 \to Z \to P_d \to \cdots \to P_0 \to M \to 0$ with each P_i a finitely generated free module. Applying π to this gives an element of $\operatorname{Ext}^{d+1}_X(\pi M, \pi Z)$ which must be zero, so the triangle $\pi Z[d] \to \pi P \to \pi M \to$ is split. Applying ω gives an isomorphism $\omega\pi M \oplus \omega\pi Z[d] \to \omega\pi P$. However, since χ holds, for every finitely generated module N the map $N_{\geq n} \to (\omega\pi N)_{\geq n}$ has finite dimensional cokernel (and it obviously has finite dimensional kernel). Hence, for large n there is an isomorphism $M_{\geq n} \oplus Z[d]_{\geq n} \cong P_{\geq n}$, and hence a map $M_{\geq n} \oplus Z[d]_{\geq n} \to P$ whose cone has finite dimensional cohomology. □

Proposition 4.3 *Suppose that A is a noetherian connected graded k-algebra satisfying χ. Write $X = \operatorname{Proj} A$. If* $\operatorname{gldim} X = d < \infty$*, then* $\dim_k \underline{\operatorname{Ext}}^i_A(M,N) < \infty$ *for all* $i \geq d+1$ *and all* $M, N \in \mathsf{grmod}A$*.*

Proof. By Lemma 4.2, there is a distinguished triangle $M_{\geq n} \oplus N \to P \to C \to$ such that $\oplus_q H^q(C)$ is finite dimensional. □

Notation. We write $\mathsf{MCM}_{\geq n}$ for the full subcategory of $\mathsf{MCM}(A)$ consisting of graded Cohen-Macaulay A-modules M such that $M_i = 0$ for all $i < n$. We write $\underline{\mathsf{MCM}}_{\geq n}$ for the essential image of $\mathsf{MCM}_{\geq n}$ in $\underline{\mathsf{MCM}}(A)$.

We will say that a triangulated category is semisimple if in every distinguished triangle

$$L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w}$$

at least one of u, v, and w, is zero. The condition that $w = 0$ is equivalent to the condition that u and v induce an isomorphism $M \cong L \oplus N$. It follows that if L and M are objects in a semisimple triangulated category, then $\operatorname{Hom}(L, M) \neq 0$ if and only if $\operatorname{Hom}(M, L) \neq 0$. Futhermore, the heart of every t-structure on a semisimple triangulated category is semisimple, meaning that every short exact sequence splits. The derived category of a semisimple abelian category is semisimple.

Proposition 4.4 *Let A be a connected, graded k-algebra that is Gorenstein and satisfies χ. If $\underline{\mathsf{MCM}}(A)$ is semisimple, then* $\operatorname{Proj} A$ *is smooth.*

Proof. Suppose that A has depth d, so the d^{th} syzygy of a finitely generated module is maximal Cohen-Macaulay.

Fix $K \in \mathsf{MCM}(A)$. For all integers n greater than the degrees of the minimal generators of K, $\operatorname{Hom}_{\underline{\mathsf{MCM}}}(K, -)$ vanishes on $\underline{\mathsf{MCM}}_{\geq n}$. Since $\underline{\mathsf{MCM}}(A)$ is semisimple, it follows that $\operatorname{Hom}_{\underline{\mathsf{MCM}}}(-, K)$ also vanishes on $\underline{\mathsf{MCM}}_{\geq n}$.

Now fix $M, N \in \mathsf{grmod} A$. Then

$$\begin{aligned}
\operatorname{Ext}^{d+1}_{\operatorname{Proj} A}(\pi M, \pi N) &\cong \operatorname{Ext}^{d+1}_{\mathsf{Gr} A}(M_{\geq n}, N) \qquad \text{for } n \gg 0 \\
&\cong \operatorname{Ext}^{1}_{\mathsf{Gr} A}(\Omega^d(M_{\geq n}), N) \\
&\cong \operatorname{Ext}^{d+1}_{\mathsf{Gr} A}(\Omega^d(M_{\geq n}), \Omega^d N) \qquad \text{because } A \text{ is Gorenstein} \\
&\cong \operatorname{Hom}_{\underline{\mathsf{MCM}}}(\Omega^d(M_{\geq n}), (\Omega^d N)[d+1]).
\end{aligned}$$

The previous paragraph shows that this is zero for $n \gg 0$ because $\Omega^d(M_{\geq n}) \in \mathsf{MCM}_{\geq n}$. □

Part of the argument in Proposition 4.4 can be restated in the following way.

Proposition 4.5 *Suppose that $\underline{\mathsf{MCM}}(A)$ is semisimple. If $N \in \mathsf{MCM}(A)$, there is an integer n such that $\operatorname{Ext}^1_{\mathsf{Gr} A}(M, N) = 0$ for all $M \in \mathsf{MCM}_{\geq n}$.*

Proof. Choose an integer $n >$ {the degrees of a minimal set of generators for N}. Suppose that $M \in \mathsf{MCM}_{\geq n}$. Then $\Omega M \in \mathsf{MCM}_{\geq n}$ also. Hence $\operatorname{Hom}_{\mathsf{Gr} A}(N, \Omega M)$ is zero, and so is its quotient $\operatorname{Hom}_{\underline{\mathsf{MCM}}(A)}(N, M[-1])$. But $\underline{\mathsf{MCM}}(A)$ is semisimple, so $\operatorname{Hom}_{\underline{\mathsf{MCM}}(A)}(M[-1], N) = 0$ also. Thus $\operatorname{Ext}^1_{\mathsf{Gr} A}(M, N) = 0$. □

Proposition 4.6 *Let S be a Gorenstein k-algebra of finite global dimension and $A = S/(z)$ where z is a central regular element of degree d. If $M \in \mathsf{MCM}(A)$ is not projective, then*

1. *there is a resolution* $0 \to S^s \to S^s \to M \to 0$ *of ungraded modules;*

2. $\Omega^2 M \cong M(-d)$*;*

3. $\underline{\mathrm{Ext}}_A^{i+2}(M,N) \cong \underline{\mathrm{Ext}}_A^i(M,N)(d)$ *for all* A*-modules* N *and all* $i \geq 1$.

Proof. (1) We have $\operatorname{depth}_S M = \operatorname{depth}_A M = \operatorname{depth}_A A = \operatorname{depth}_S A = \operatorname{depth} S - 1$, so $\operatorname{pdim}_S M = 1$ by the Auslander-Buchsbaum formula. Hence M has a free resolution $0 \to S^r \to S^s \to M \to 0$. But $r \leq s$ because S is noetherian and $s \leq r$ because $S^s z \subset S^r$, so $s = r$.

Although this is a resolution of M in $\mathsf{Mod}S$, the minimal resolution of M in $\mathsf{GrMod}S$ also has this form although we may have to change s and we will need to place some gradings on the two free modules.

(2) and (3) From the presentation of A we get $\operatorname{Tor}_1^S(M,A) \cong M(-d)$. Now, by applying $- \otimes_S A$ to the resolution of M_S, we obtain an exact sequence

$$0 \to M(-d) \to A^s \to A^s \to M \to 0$$

in $\mathsf{Mod}A$. We can give the two copies of A^s gradings so this becomes a sequence of graded A-modules. Thus $\Omega^2 M \cong M(-d)$. The result now follows by dimension-shifting. $\square$

5 Quadrics in quantum $\mathbb{P}^3$s

Using the notation in part (2) of the next lemma, we define

$$C(A) := A^![w^{-1}]_0. \tag{8}$$

Lemma 5.1 *Let* S *be a connected, graded, Koszul algebra of finite global dimension,* z *a central regular element of degree two, and* $A = S/(z)$. *Then*

1. A *is a Koszul algebra;*

2. *there is a central, regular element* $w \in A_2^!$ *such that* $A^!/(w) = S^!$*;*

3. *the algebra* $C(A)$ *has finite dimension equal to* $\dim_k (S^!)^{(2)}$*, the dimension of the even degree part of* S*;*

4. *the categories* $\operatorname{Proj} A^!$ *and* $\mathsf{Mod}C(A)$ *are equivalent via* $\pi N \mapsto N[w^{-1}]_0$*, where* $N \in \mathsf{GrMod}A^!$.

Proof. (1) and (2). The proof is similar to that for modding out a central regular element of degree one [10].

(3) Because S is Koszul, the hypothesis that $\operatorname{gldim} S < \infty$ implies that $S^!$ has finite dimension. It follows that $A^!_{m+2} = wA^!_m$ for large m, and hence that

$$A^![w^{-1}]_0 = A^!_0 + A^!_2 w^{-1} + \cdots = A^!_{2n} w^{-n}$$

for $n \gg 0$. In particular, $\dim_k A[w^{-1}]_0 = \dim_k A_{2n}$ for $n \gg 0$. By (1) and (2), $(1+t)H_{A^!}(t) = (1-t)^{-1}H_{S^!}(t)$ so

$$\dim_k A^!_{2n} = \dim_k S^!_0 + \dim_k S^!_2 + \cdots ,$$

for $n \gg 0$, as required.

(4) A graded $A^!$-module has finite dimension if and only if it is annihilated by a power of w, so $\mathsf{GrMod}A^!/\mathsf{Fdim}A^!$ is equivalent to $\mathsf{GrMod}A^![w^{-1}]$. Since $A^!$ is generated in degree one, $A^![w^{-1}]$ is strongly graded, and therefore $\mathsf{GrMod}A^![w^{-1}]$ is equivalent to $\mathsf{Mod}A^![w^{-1}]_0$. □

The degree shift functor (1) on $\mathsf{GrMod}A^!$ induces auto-equivalences of $\operatorname{Proj} A^!$ and $\mathsf{Mod}C(A)$ that we still denote by (1). Since w is central and homogeneous of degree two, $(2) \cong \mathrm{id}_{\mathsf{Mod}C(A)}$.

Notice that $A^!$ is noetherian because $A^!/(w)$ is.

Proposition 5.2 *Let S be a Gorenstein, connected, graded, Koszul algebra of finite global dimension, z a central regular element of degree two, and $A = S/(z)$.*

1. *There are equivalences of categories*

$$\begin{array}{ccc} \underline{\mathsf{MCM}}(A) & \xrightarrow{\;B\;} & \mathsf{D}^b(\operatorname{proj} A^!) \\ & \searrow \quad \swarrow & \\ & \mathsf{D}^b(\mathsf{mod}C(A)) & \end{array}$$

2. *If $C(A)$ is a semisimple ring, then* $\operatorname{Proj} A$ *is smooth.*

Proof. The horizontal equivalence is given by Theorem 3.2, and the southwest equivalence is given by Lemma 5.1. Part (2) follows from (1) and Proposition 4.4 because the derived category of a semisimple abelian category is semsimple, hence abelian. □

Notation and Hypotheses. We fix the following hypotheses and notation for the remainder of this section: k is an algebraically closed field, S denotes a connected, graded, Gorenstein, Koszul algebra with Hilbert series $(1-t)^{-4}$; z is a non-zero, homogeneous, central element of degree two such that $A := S/(z)$ is a domain. We write $Q := \operatorname{Proj} A$.

The hypotheses imply that $H_{S^!}(t) = (1+t)^4$, so $\operatorname{gldim} S = 4$. The previous two results apply, so the finite dimensional algebra $C(A)$ is well-defined.

It follows from Corollary 6.7 below and [1, Thm. 3.9] that S is a domain. Thus $\operatorname{Proj} S$ is a quantum $\mathbb{P}^3$ and Q is a quadric hypersurface in it. The assumption that A is a domain says that Q is "reduced and irreducible".

Proposition 5.3 *Suppose that S is a connected, graded, Gorenstein, Koszul algebra and that $H_S(t) = (1-t)^{-4}$. Let $0 \neq z \in S_2$ be a central element and suppose that $A := S/(z)$ is a domain. Then*

1. $\dim_k C(A) = 8$*;*

2. *$C(A)$ has no one-dimensional modules;*

3. *the following are equivalent:*

   (a) *$C(A)$ is semisimple;*

   (b) *$C(A)$ has two simple modules up to isomorphism;*

   (c) *$C(A) \cong M_2(k) \oplus M_2(k)$.*

Proof. (1) This does not depend on A being a domain. Because z is regular, $H_A(t) = (1+t)(1-t)^{-3}$ and $H_{A^!}(t) = (1-t)^{-1}(1+t)^3$. Thus $\dim_k A_n = 8$ for $n \gg 0$, and $\dim_k C(A) = 8$ by Lemma 5.1.

(2) First we show that if V is a subspace of $A_1^! = A_1^*$ of codimension one, then $A_1^! V = V A_1^! = A_2^!$. To see this write $V = a^\perp$ where $a \in A_1$; because A is a domain, $a \otimes A_1 \cap R = 0$, where R denotes the relations in $A_1 \otimes A_1$ defining A; hence $a^\perp \otimes A_1^* + R^\perp = A_1^* \otimes A_1^*$; thus $VA_1^! = A_2^!$. Similarly, $A_1^! V = A_2^!$.

Claim: Let T be a connected graded algebra generated in degree one, and $w \in T_d$ a central regular element of degree $d > 1$. If $T_2 = T_1 V$ for every codimension one subspace $V \subset T_1$, then $T[w^{-1}]_0$ does not have a one-dimensional module.

Proof: Suppose to the contrary that N_0 is a one-dimensional $T[w^{-1}]_0$-module. Then N_0 is the degree zero component of the $T[w^{-1}]$-module $N := T[w^{-1}] \otimes_{T[w^{-1}]_0} N_0$. Because w is a unit, $N_{di} = w^i N_0$ for all $i \in \mathbb{Z}$. In particular, $\dim_k N_{di} = 1$ for all i. Hence if $m \in N_{di-1}$, then $Vm = 0$ for some subspace $V \subset T_1$ of codimension at most one. Hence $T_2 m = 0$. It follows that

$$0 = T_2 N_{di-1} = T_3 N_{di-2} = \cdots = T_{d+1} N_{di-d}.$$

In particular, $T_{d+1} N = 0$, so $w^2 N = 0$, contradicting the fact that w is a unit in $T[w^{-1}]$. $\Diamond$

The claim applies to $T = A^!$, so (2) follows.

(3) This follows from (1) and (2) because k is algebraically closed. $\square$

Remark. Because A is a noetherian domain it has a division ring of fractions, say Q, and we may define the rank of an A-module N as $\dim_Q N \otimes_A Q$.

Notation. We define

$$\mathbb{M} := \{M \in \mathsf{MCM}(A) \mid M \text{ is indecomposable}, \ M_0 \cong k^2, \ M = M_0 A\}.$$

Proposition 5.4 *Let A and S be as in Proposition 5.3. If $M \in \mathbb{M}$, then*

1. *$M \cong \Omega^2 M(2)$;*

2. *its minimal resolution is $\cdots \to A(-2)^2 \to A(-1)^2 \to A^2 \to M \to 0$;*

3. *$\operatorname{rank} M = 1$;*

4. *$H_M(t) = 2(1-t)^{-3}$;*

5. *M is 3-critical with respect to GK-dimension.*

Furthermore, there is a bijection

$$\mathbb{M} \longleftrightarrow \{\textit{simple } C(A)\textit{-modules}\}$$
$$M \longleftrightarrow FM.$$

Proof. The hypotheses on M ensure that it is not projective.

(1) This was already established in Proposition 4.6.

(2) By (1), the minimal resolution of $\Omega^2 M$ begins $A(-2)^2 \to \Omega^2 M \to 0$. Combining this with Proposition 4.6, we see that the minimal resolution of M begins

$$\cdots \to A(-2)^2 \to A(-i) \oplus A(-j) \to A^2 \to M \to 0$$

for some i, j. However, the minimality of the resolution forces $i = j = 1$. Since $\Omega^2 M \cong M(-2)$, the full minimal resolution of M can be constructed by splicing together shifts of the exact sequence $0 \to M(-2) \to A(-1)^2 \to A^2 \to M \to 0$.

(3) Because A is a domain, its rank is one. The result now follows from the exact sequence $0 \to M(-2) \to A(-1)^2 \to A^2 \to M \to 0$.

(4) This follows from (2).

(5) Because $M(-1)$ embeds in A^2, every non-zero submodule of M has GK-dimension three; since the rank of M is one, all its non-zero submodules have rank one too. Hence every proper quotient of M has GK-dimension ≤ 2.

We now establish the bijection between $\mathbb{M}$ and the simple $C(A)$-modules. Let $M \in \mathbb{M}$. By (2), M has a linear resolution, so $FM = N[0]$ for some C-module N. But $N = \underline{\mathrm{Ext}}^*_A(M,k)[w^{-1}]_0 \cong \underline{\mathrm{Ext}}^{2i}_A(M,k)$ for $i \gg 0$, so $\dim_k N = 2$, and Proposition 5.3 now implies that N is simple.

Conversely, let N be a simple C-module. By Lemma 3.5, there is a unique indecomposable maximal Cohen-Macaulay module M such that $FM \cong N[0]$, and M has a linear resolution. By Proposition 5.3, $\dim_k N = 2$. But $N \cong \underline{\mathrm{Ext}}^*_A(M,k)[w^{-1}]_0$, so $\underline{\mathrm{Ext}}^{2i}_A(M,k) \cong k^2$ for $i \gg 0$. Hence $\Omega^{2i} M$ is generated by two elements for $i \gg 0$. But $\Omega^{2i} M \cong M(-2i)$ by Proposition 4.6, so M is generated by two elements, and these are of degree zero because M has a linear resolution. Hence $M \in \mathbb{M}$. □

Lemma 5.5 *Let A and S be as in Proposition 5.3. If $M \in \mathbb{M}$, then*

1. $\Omega M(1) \in \mathbb{M}$;
2. *there is an exact sequence* $0 \to M(-1) \to A^2 \to \Omega M(1) \to 0$;
3. *if $C(A)$ is not semisimple, then* $M \cong \Omega M(1)$;
4. *if* $M \not\cong \Omega M(1)$, *then* $\mathrm{Hom}_{\mathsf{Gr}A}(M, \Omega M(1)) = 0$.

Proof. (1) By the remark after Lemma 3.4, $\Omega M(1)$ is indecomposable. From the minimal resolution for M we see that $\Omega M(1)$ is generated in degree zero and that $\dim_k \Omega M(1)_0 = 2$.

(2) This follows from the fact that $\Omega^2 M \cong M(-2)$.

(3) If $C(A)$ is not semisimple it has only one simple module so, by (1) and the bijection in Proposition 5.4, $M \cong \Omega M(1)$.

(4) A non-zero degree-zero homomophism $\alpha : M \to \Omega M(1)$ would be injective because M and $\Omega M(1)$ are 3-critical with respect to GK-dimension, so its restriction $M_0 \to (\Omega M(1))_0$ would be an isomorphism, whence α would be surjective. This would contradict the hypothesis that $M \not\cong \Omega M(1)$. $\square$

Theorem 5.6 *The noncommutative quadric Q is smooth if and only if $C(A)$ is semisimple.*

Proof. ($\Leftarrow$) This was proved in Proposition 5.2.

($\Rightarrow$) We shall prove the contrapositive, so suppose that $C(A)$ is not semisimple. Then it has only one simple module, N say. Let $M(1) \in \mathbb{M}$ be such that $F(M(1)) \cong N[0]$. Thus M is generated in degree one.

We write $\mathcal{M}$ for πM.

By Lemma 5.5, $M \cong \Omega M(1)$, so there is an exact sequence

$$0 \longrightarrow M \longrightarrow A^2 \longrightarrow M(1) \longrightarrow 0. \tag{9}$$

This gives an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_Q^2 \longrightarrow \mathcal{M}(1) \longrightarrow 0 \tag{10}$$

in $\mathsf{Mod}Q$.

The result that $\operatorname{gldim} Q = \infty$ will be established in Step 3 below.

Step 1. $\operatorname{Ext}^1_Q(\mathcal{M}, \mathcal{M}) \neq 0$.

Proof: Because M is maximal Cohen-Macaulay

$$\operatorname{Ext}^1_{\mathsf{Gr}A}(M, M) \cong \operatorname{Hom}_{\underline{\mathsf{MCM}}(A)}(M, M[1]).$$

Applying the contravariant equivalence F this is isomorphic to

$$\operatorname{Hom}_{\mathsf{D}(C(A))}(F(M[1]), FM) \cong \operatorname{Hom}_{\mathsf{D}(C(A))}(FM, (FM)[1]).$$

But FM is a translate of the unique simple C-module N, so the last term is isomorphic to $\operatorname{Ext}^1_{C(A)}(N, N)$ which is non-zero because $C(A)$ is not semisimple. Hence $\operatorname{Ext}^1_{\mathsf{Gr}A}(M, M) \neq 0$.

Applying π to a non-split exact sequence

$$0 \to M \to D \to M \to 0 \tag{11}$$

in $\mathsf{GrMod}A$ gives an exact sequence

$$0 \to \mathcal{M} \to \mathcal{D} \to \mathcal{M} \to 0 \tag{12}$$

in $\mathsf{Mod}Q$. Write $\theta : \mathsf{MCM}(A) \to \mathsf{GrMod}A$ for the inclusion functor. There is an exact sequence of functors $0 \to H^0_{\mathfrak{m}} \to \operatorname{id}_{\mathsf{GrMod}A} \to \omega\pi \to H^1_{\mathfrak{m}} \to 0$. These two local cohomology functors vanish on $\mathsf{MCM}(A)$, so there is an isomorphism of functors $\theta \to \omega\pi\theta$. Hence, if (12) were to split via a map $g : \mathcal{M} \to \mathcal{D}$, then $\omega(g)$ would provide a splitting of (11). It follows that $\mathrm{Ext}^1_Q(\mathcal{M}, \mathcal{M}) \neq 0$.

Step 2. $\mathrm{Ext}^2_Q(\mathcal{M}(1), \mathcal{M}) \neq 0$.

Proof: Applying $\mathrm{Hom}_Q(-, \mathcal{M})$ to (10) gives an exact sequence

$$\mathrm{Ext}^1_Q(\mathcal{O}_Q, \mathcal{M})^2 \to \mathrm{Ext}^1_Q(\mathcal{M}, \mathcal{M}) \to \mathrm{Ext}^2_Q(\mathcal{M}(1), \mathcal{M}).$$

The first term is zero because depth $M = 3$ implies that

$$0 = H^2_{\mathfrak{m}}(M) \cong H^1(Q, \mathcal{M}) = \mathrm{Ext}^1_Q(\mathcal{O}_Q, \mathcal{M}).$$

The second term is non-zero by Step 1, so the third term is also non-zero, as required.

Step 3. $\mathrm{Ext}^n_Q(\mathcal{M}(n-1), \mathcal{M}) \neq 0$ for all $n \geq 2$.

Proof: We argue by induction on n. The case $n = 2$ has already been established in Step 2. Applying $\mathrm{Hom}_Q(\mathcal{M}(n), -)$ to (10) gives an exact sequence

$$\mathrm{Ext}^n_Q(\mathcal{M}(n), \mathcal{O}_Q)^2 \to \mathrm{Ext}^n_Q(\mathcal{M}(n), \mathcal{M}(1)) \to \mathrm{Ext}^{n+1}_Q(\mathcal{M}(n), \mathcal{M}) \qquad (13)$$

The first term is isomorphic to two copies of $\mathrm{Ext}^n_Q(\mathcal{M}(n-2), \mathcal{O}_Q(-2))$ which is isomorphic to $H^{2-n}(Q, \mathcal{M}(n-2))^*$ by Serre duality. This is zero for $n \geq 3$, and if $n = 2$ it is isomorphic to $\mathrm{Hom}_{\mathsf{Gr}A}(A, M)^* = M_0$ which is zero because M is generated in degree one. Since the first term of (13) is zero for all $n \geq 2$, we see from the other two terms that the induction argument goes through. $\square$

Corollary 5.7 *$C(A)$ is semisimple if and only if $M \ncong \Omega M(1)$ for all $M \in \mathbb{M}$.*

Proof. ($\Leftarrow$) If C were not semisimple it would have a unique simple module so, up to isomorphism, there would be only one module in $\mathbb{M}$; but if M is in $\mathbb{M}$ so is $\Omega M(1)$, whence $M \cong \Omega M(1)$.

($\Rightarrow$) If C is semisimple, then $\mathrm{gldim}\, Q < \infty$. But the proof of Theorem 5.6 showed that if there were an M in $\mathbb{M}$ such that $M \cong \Omega M(1)$, then $\mathrm{gldim}\, Q = \infty$. Hence there can be no such M. $\square$

Corollary 5.8 *If Q is smooth, then $\mathbb{M}$ consists of two non-isomorphic modules, say $\mathbb{M} = \{M, M'\}$, and there are exact sequences*

$$0 \to M(-1) \to A^2 \to M' \to 0$$

and

$$0 \to M'(-1) \to A^2 \to M \to 0.$$

Proof. This follows immediately from Lemma 5.5 and Corollary 5.7. $\square$

6 The Auslander property

We fix the following notation in this section: S denotes a connected, graded, Gorenstein, Koszul algebra with Hilbert series $(1-t)^{-4}$; z is a non-zero, homogeneous, central element of degree two and $A := S/(z)$.

The main result in this section, Theorem 6.6, shows that A has the Auslander property by which we mean that if $M \in \mathsf{grmod} A$ and N is a graded submodule of $\underline{\mathrm{Ext}}^j_A(M, A)$ for some j, then $\underline{\mathrm{Ext}}^i_A(N, A) = 0$ for $i < j$. By [11], this will imply that S also has the Auslander property.

The grade of $M \in \mathsf{grmod} A$ is $j(M) := \inf\{j \mid \underline{\mathrm{Ext}}^j_A(M, A) \neq 0\}$. The Auslander property is equivalent to the condition that $j(N) \geq j$ for all submodules $N \subset \underline{\mathrm{Ext}}^j_A(M, A)$. To prove that A has the Auslander property we first prove that

$$j(M) + \mathrm{GKdim}\, M = 3$$

for all $M \in \mathsf{grmod} A$.

The arguments in this section are close to those in [1, Sect. 4].

The following result is standard.

Lemma 6.1 *If R is a prime noetherian k-algebra of finite GK-dimension and $N \in \mathsf{mod} R$, the following are equivalent:*

1. $\mathrm{GKdim}\, N = \mathrm{GKdim}\, R$*;*
2. $j(N) = 0$*;*
3. $N \otimes_R Q \neq 0$*, where* $Q = \mathrm{Fract}\, R$*.*

Because A is Gorenstein its dualizing module ω_A is invertible, hence isomorphic to $A(\ell)$ for some integer ℓ as a left and as a right module. Our arguments in this section involve an examination of the convergent spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_A(\underline{\mathrm{Ext}}^{-q}_A(M, \omega_A), \omega_A) \Rightarrow \mathbb{H}^{p+q}(M) = \begin{cases} M & \text{if } p+q = 0, \\ 0 & \text{if } p+q \neq 0. \end{cases} \qquad (14)$$

We will often omit the subscript from E_2^{pq}.

Theorem 6.2 *Let A be as above and let $M \in \mathsf{grmod} A$. The E_2-page of the spectral sequence (14) looks like*

$$\begin{matrix} E^{00} & E^{10} & 0 & 0 \\ 0 & E^{1,-1} & E^{2,-1} & E^{3,-1} \\ 0 & 0 & E^{2,-2} & E^{3,-2} \\ 0 & 0 & 0 & E^{3,-3} \end{matrix} \qquad (15)$$

Proof. Since $\underline{\mathrm{Ext}}^i_A(-, A) = 0$ for $i > 3$, the non-zero terms on the E_2-page of the double-Ext spectral sequence lie in the 4×4 region depicted. Therefore the E^{20} and E^{30} terms survive to the E_∞-page. But any non-zero terms on the E_∞

page must lie on the diagonal, so $E^{20} = E^{30} = 0$. Now $\underline{\mathrm{Ext}}^3_A(M, \omega_A) \cong H^0_{\mathfrak{m}}(M)^*$ is finite dimensional so is Cohen-Macaulay of depth zero, whence $E^{p3} = 0$ for $p < 3$. This explains the zeroes in the top and bottom rows of (15).

The division ring of fractions, $Q = \mathrm{Fract}\, A$, is flat as a left and as a right A-module and $\mathrm{gldim}\, Q = 0$ so, for $i > 0$,

$$0 = \mathrm{Ext}^i_Q(M \otimes_A Q, A \otimes_A Q) \cong Q \otimes_A \underline{\mathrm{Ext}}^i_A(M, A).$$

Applying Lemma 6.1 to $N = \underline{\mathrm{Ext}}^i_A(M, A)$, we obtain $E^{0,-1} = E^{0,-2} = E^{0,-3} = 0$, giving the zeroes in the left-most column of (15).

It remains to show that $E^{1,-2} = 0$. Set $L = \underline{\mathrm{Ext}}^2_A(M, A)$; if $L = 0$ there is nothing to do, so suppose that $L \neq 0$. Since $\ker(E^{1,-2} \to E^{3,-3})$ survives to the E_∞ page, it is zero. Since $E^{3,-3}$ is finite dimensional so is $E^{1,-2} = \underline{\mathrm{Ext}}^1_A(L, A) < \infty$. If $\underline{\mathrm{Ext}}^1_A(L, A) = 0$ we are finished, so suppose otherwise.

Consider the E_2-page of the spectral sequence for L. Since $Q \otimes_A L \cong \underline{\mathrm{Ext}}^2_Q(M \otimes_A Q, A \otimes_A Q) = 0$, $\underline{\mathrm{Hom}}_A(L, A)$ is zero; hence the $q = 0$ and $q = -1$ rows look like

$$\begin{matrix} 0 & 0 & 0 & 0 \\ E^{0,-1} & E^{1,-1} & E^{2,-1} & E^{3,-1} \end{matrix}$$

Since $\underline{\mathrm{Ext}}^1_A(L, A)$ is non-zero and finite dimensional, $\underline{\mathrm{Ext}}^3_A(\underline{\mathrm{Ext}}^1_A(L, A), A) \neq 0$. But the $E^{3,-1}$ term survives to the E_∞ page, so must be zero. From this contradiction we conclude that $\underline{\mathrm{Ext}}^1_A(L, A) = 0$, as required. □

Lemma 6.3 *If $M \in \mathsf{grmod} S$ is a Cohen-Macaulay module, then*

$$\mathrm{depth}\, M + \mathrm{GKdim}\, M \equiv 0 \ (\mathsf{mod}\ 2).$$

Proof. The lemma is true for any connected graded Gorenstein algebra S of finite global dimension, n say, having Hilbert series of the form $f(t)(1-t)^{-n}$ where $f(t) \in \mathbb{Z}[t]$. The functional equation [1, (2.35)] relating the Hilbert series of a module to that of its dual becomes

$$H_{M^\vee}(t) = (-1)^d H_M(t^{-1})$$

when M is a Cohen-Macaulay module of depth d and $M^\vee = \underline{\mathrm{Ext}}^{n-d}_S(M, \omega_S)$. If $\mathrm{GKdim}\, M = r$, then $H_M(t) = g(t)(1-t)^{-r}$ for some $g(t) \in \mathbb{Z}[t, t^{-1}]$ so $H_{M^\vee}(t) = (-1)^{d+r} t^r g(t^{-1})(1-t)^{-r}$. However,

$$\lim_{t \uparrow 1} H_M(t)$$

is positive if $M \neq 0$ and the same applies to $H_{M^\vee}(t)$, so $d + r$ is even. □

Lemma 6.4 *Let A be as above and $M \in \mathsf{grmod} A$. Suppose that $j(M) = 1$ and write $M^\vee := \underline{\mathrm{Ext}}^1_A(M, A)$. Then $M^\vee$ is Cohen-Macaulay of depth two and GK-dimension two.*

Proof. Since $j(M) = 1$, the E_2-page of the spectral sequence for M looks like

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & E^{1,-1} & E^{2,-1} & E^{3,-1} \\ 0 & 0 & E^{2,-2} & E^{3,-2} \\ 0 & 0 & 0 & E^{3,-3} \end{array}.$$

Both $E^{2,-1}$ and $E^{3,-1}$ survive to the E_∞-page, so must be zero. Hence $M^\vee$ is Cohen-Macaulay of depth two and, by Lemma 6.3, $\operatorname{GKdim} M^\vee = 2$. □

Theorem 6.5 *If A is as above, then*

$$j(M) + \operatorname{GKdim} M = 3 \tag{16}$$

for all non-zero finitely generated graded A-modules M.

Proof. By Lemma 6.1, (16) holds when $M = \operatorname{GKdim} 3$ and when $j(M) = 0$. Since finite dimensional modules are precisely the Cohen-Macaulay modules of depth zero, (16) holds when $\operatorname{GKdim} M = 0$ too, so it remains to prove (16) for modules of GK-dimensions 1 and 2.

Suppose that $\operatorname{GKdim} M = 1$. By Lemma 6.1, $j(M) \geq 1$. As in the proof of Lemma 6.4, the E_2-page of the double-Ext spectral sequence for M looks like

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & E^{1,-1} & 0 & 0 \\ 0 & 0 & E^{2,-2} & E^{3,-2} \\ 0 & 0 & 0 & E^{3,-3} \end{array}$$

Hence the filtration induced on M by the spectral sequence looks like $M = F^0M = F^1M \supset F^2M \supset \cdots$, so there is a surjective map

$$M \to F^1M/F^2M = E_\infty^{1,-1} = \ker(E^{1,-1} \to E^{3,-2}).$$

This gives an exact sequence $M \to E^{1,-1} \to E^{3,-2}$. Since $\dim_k(E^{3,-2}) < \infty$, it follows that $\operatorname{GKdim}(E^{1,-1}) \leq 1$. If $E^{1,-1} \neq 0$, then $j(\underline{\operatorname{Ext}}_A^1(M, A)) = 1$ so, by Lemma 6.4, $\operatorname{GKdim} \underline{\operatorname{Ext}}_A^1(M, A)^\vee = 2$; that is, $\operatorname{GKdim}(E^{1,-1}) = 2$, which contradicts the foregoing. So we must have $E^{1,-1} = 0$. Hence $E_2^{*,-1} = 0$, so $\underline{\operatorname{Ext}}_A^1(M, A) = 0$ and $j(M) \geq 2$. However, $j(M) \neq 3$ because $\dim_k M = \infty$, so $j(M) = 2$.

Now suppose that $\operatorname{GKdim} M = 2$. By the first paragraph of the proof $j(M)$ is either 1 or 2. Suppose that $j(M) = 2$; we seek a contradiction. Let τM be the sum of all finite dimensional graded submodules of M, and consider the exact sequence $0 \to \tau M \to M \to \bar{M} \to 0$. It follows easily (cf. [1, Prop. 2.46]) that $\underline{\operatorname{Ext}}_A^i(\bar{M}, A) = 0$ for $i \neq 2$ so $\bar{M}$ is Cohen-Macaulay of depth one. But $\operatorname{GKdim} \bar{M} = \operatorname{GKdim} M = 2$ which contradicts Lemma 6.3. □

Theorem 6.6 *The algebra A satisfies the Auslander condition: if M is a finitely generated A-module and N an A-submodule of $\underline{\operatorname{Ext}}_A^j(M, A)$, then $\underline{\operatorname{Ext}}_A^i(N, A) = 0$ for $i < j$.*

Proof. Let M be a non-zero finitely generated graded A-module, and N a non-zero graded A-submodule of $\underline{\mathrm{Ext}}^i_A(M,A)$. By Theorem 6.2, the E_2 page of the spectral sequence for M looks like (15), so $j(\underline{\mathrm{Ext}}^i_A(M,A)) \geq i$. By Theorem 6.5, $\mathrm{GKdim}(\underline{\mathrm{Ext}}^i_A(M,A)) \leq 3-i$ so $\mathrm{GKdim}\, N \leq 3-i$; by Theorem 6.5 applied to N, $j(N) \geq i$. □

Corollary 6.7 *The algebra S satisfies the Auslander condition.*

Proof. This follows from [11, Thm. 3.6]. □

Because S satisfies the Auslander condition, the results in sections 1 and 2 of [12] apply. In [12, Sect. 1] a non-zero module $M \in \mathsf{grmod}S$ is said to be Cohen-Macaulay if its projective dimension is the smallest i such that $\mathrm{Ext}^i_S(M,S)$ is non-zero. Since S is Gorenstein, M is Cohen-Macaulay in the sense of [12] if and only if it is Cohen-Macaulay in the sense of the present paper.

7 Lines and rulings

We continue to assume that S and A are as in the notation just before Proposition 5.3. We continue to use the notation $Q := \mathrm{Proj}\, A$.

A graded line module for A or S is a graded module L that is cyclic and has Hilbert series

$$H_L(t) = (1-t)^{-2}.$$

We will write $\mathcal{O}_L$ for the image of L in either $\mathrm{Proj}\, A$ or $\mathrm{Proj}\, S$. The class of $\mathcal{O}_L$ in $K_0(\mathrm{Proj}\, S)$ is $(1-t)^2$. By way of comparison, if $0 \neq x \in S_1$ and $\mathcal{O}_H$ denotes the image of S/xS in $\mathrm{Proj}\, S$, the class of $\mathcal{O}_H$ is $1-t$.

Lemma 7.1 *Let $M \in \mathbb{M}$. Then*

1. $\mathrm{Hom}_{\mathsf{Gr}A}(M(-1),A) \cong k^2$;
2. *if* $0 \neq f \in \mathrm{Hom}_{\mathsf{Gr}A}(M(-1),A)$ *then f is injective, and*
3. $\mathrm{coker}\, f$ *is a line module.*

Proof. (1) There is an exact sequence

$$0 \to \underline{\mathrm{Hom}}_A(M,A) \to A^2 \to \underline{\mathrm{Hom}}_A(\Omega M, A) \to 0$$

of maximal Cohen-Macaulay *left* modules. Now $\underline{\mathrm{Hom}}_A(\Omega M, A)$ is indecomposable because M is, and is obviously generated by its degree zero component which is two-dimensional because $\mathrm{Hom}_{\mathsf{Gr}A}(M,A)=0$. Hence $\underline{\mathrm{Hom}}_A(\Omega M, A)$ belongs to $\mathbb{M}'$, the corresponding set of maximal Cohen-Macaulay *left* A-modules. By the left module version of Lemma 5.5, $\underline{\mathrm{Hom}}_A(M,A)(1)$ is also in $\mathbb{M}'$, so $\mathrm{Hom}_{\mathsf{Gr}A}(M(-1),A) \cong \underline{\mathrm{Hom}}_A(M,A)(1)_0 \cong k^2$.

(2) and (3) Because M is 3-critical and A is a domain, and hence 3-critical, every non-zero map $M(-1) \to A$ is injective. A Hilbert series computation shows that $\mathrm{coker}\, f$ is a line module. □

Proposition 7.2 *Suppose S is a connected, graded, Gorenstein, Koszul algebra with Hilbert series $(1-t)^{-4}$. Let z be a central regular element of degree two in S and set $A = S/(z)$. Let L be a line module for A. Then*

1. *L has a linear resolution as an S-module;*
2. *L has a linear resolution as an A-module, namely $\cdots \to A(-2)^2 \to A(-1)^2 \to A \to L \to 0$;*
3. *L is Cohen-Macaulay of depth two and 2-critical with respect to GK-dimension;*
4. *there is an exact sequence $0 \to M(-1) \to A \to L \to 0$ for a unique $M \in \mathbb{M}$;*
5. *if M and L are as in part (4), then $FM \cong (FL)(1)$.*

Proof. (1) By [12, Cor. 2.9], the minimal resolution of L over S is

$$0 \to S(-2) \to S(-1)^2 \to S \to L \to 0.$$

(2) If L is *any* A-module having a linear resolution over S, then L has a linear resolution over A: to see this, use the fact that $\underline{\mathrm{Ext}}^1_S(A,k) \cong k(2)$ and use the long exact sequence associated to the degenerate spectral sequence

$$\underline{\mathrm{Ext}}^p_A(L, \underline{\mathrm{Ext}}^q_S(A,k)) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_S(L,k).$$

This general fact for commutative rings is proved in [6].

(3) A line module is Cohen-Macaulay of depth two by [12, Prop. 2.8], and 2-critical by [12, Cor. 1.11].

(4) If $M(-1)$ is the kernel of a surjective map $A \to L$, then it follows from the long exact sequence for local cohomology that $M(-1)$ is Cohen-Macaulay of depth three. Every submodule of A is indecomposable because A is a domain. Hence $M(-1)$ is indecomposable. From the linear resolution of L, we see that $M(-1)$ is generated by $M(-1)_1$ and that $\dim M(-1)_1 = 2$, so $M \in \mathbb{M}$.

The uniqueness of M will follow from (5) because if M' is another element of $\mathbb{M}$ such that $M' \cong \ker(A \to L)$, then FM is isomorphic to FM' in $\mathsf{D}^b(\mathit{proj} A^!)$, whence M and M' are isomorphic in $\underline{\mathsf{MCM}}(A)$. Now apply Lemma 3.4.

(5) This follows from Lemma 3.3 because $M \cong \Omega L(1)$. □

Rulings. For each $M \in \mathbb{M}$, we define

$$\text{the ruling corresponding to } M := \{L_\phi := \operatorname{coker}\phi \mid \phi \in \mathbb{P}(\mathrm{Hom}_{\mathsf{Gr}A}(M(-1),A))\}.$$

Then

- each ruling consists of a $\mathbb{P}^1$ of line modules;
- every line module belongs to a unique ruling;
- Q has two rulings if it is smooth, and one otherwise.

The first of these facts follows from Lemma 7.1, the second from Proposition 7.2(4), and the third is a consequence of Theorem 5.6 and the fact that the cardinality of $\mathbb{M}$ equals the number of isomorphism classes of simple $C(A)$-modules.

The results in the rest of this section provide further justification for using the word ruling.

Lemma 7.3 *Let L_ϕ and L_ψ ($\phi, \psi \in \mathbb{P}^1$) be lines in the same ruling. Then*

1. *$L_\phi \cong L_\psi$ if and only if $\phi = \psi$;*
2. *$\pi L_\phi \cong \pi L_\psi$ if and only if $\phi = \psi$.*

Proof. (1) Let L_ϕ and L_ψ be in the ruling corresponding to $M \in \mathbb{M}$. Because M is indecomposable, $\operatorname{Hom}_{\mathsf{Gr}A}(M, M)$ is a local ring. It is finite dimensional and contains no non-zero nilpotents since M is GK-homogeneous. Hence $\operatorname{Hom}_{\mathsf{Gr}A}(M, M) \cong k$. Because L_ϕ is cyclic, $L_\phi \cong L_\psi$ if and only $\operatorname{Im}\phi = \operatorname{Im}\psi$. However, the images are the same if and only if $\phi = \psi\theta$ for some $\theta \in \operatorname{Hom}_{\mathsf{Gr}A}(M(-1), M(-1))$; that is, if and only if $\phi = \psi$ as elements of $\mathbb{P}^1 = \mathbb{P}(\operatorname{Hom}_{\mathsf{Gr}A}(M(-1), A)$.

(2) Because L_ψ is Cohen-Macaulay of depth two, the exact sequence (4) implies that $\omega\pi L_\psi \cong L_\psi$. Hence

$$\operatorname{Hom}_X(\pi L_\phi, \pi L_\psi) \cong \operatorname{Hom}_{\mathsf{Gr}A}(L_\phi, \omega\pi L_\psi) \cong \operatorname{Hom}_{\mathsf{Gr}A}(L_\phi, L_\psi)$$

so the result follows from (1). □

The argument in (2) and the observation that each line module belongs to a unique ruling shows that is L and L' are line modules in different rulings, then $\pi L \ncong \pi L'$.

Proposition 7.4 *Let L and L' be line modules.*

1. *If Q is smooth, then L and L' belong to different rulings if and only if there is an exact sequence*

$$0 \to L'(-1) \to A/aA \to L \to 0 \tag{17}$$

for some $0 \neq a \in A_1$.

2. *If Q is not smooth there is always an exact sequence of the form (17).*

Proof. There are exact sequences $0 \to M(-1) \to A \to L \to 0$ and $0 \to M'(-1) \to A \to L' \to 0$ in which $M, M' \in \mathbb{M}$.

(1) ($\Rightarrow$) Suppose that L and L' belong to different rulings; then $M \ncong M'$ so $M'(-1) \cong \Omega M$ by Corollary 5.7. The first term in the exact sequence

$$\operatorname{Hom}_{\mathsf{Gr}A}(A, L'(-1)) \to \operatorname{Hom}_{\mathsf{Gr}A}(M(-1), L'(-1)) \to \operatorname{Ext}^1_{\mathsf{Gr}A}(L, L'(-1)) \to 0$$

is zero, and $\mathrm{Hom}_{\mathsf{Gr}A}(M(-1), L'(-1))$ is isomorphic to

$$\begin{aligned}\mathrm{Ext}^1_{\mathsf{Gr}A}(M(-1), \Omega M(-1)) &\cong \mathrm{Hom}_{\underline{\mathsf{MCM}}}(M(-1), \Omega M(-1)[1]) \\ &= \mathrm{Hom}_{\underline{\mathsf{MCM}}}(M(-1), M(-1)) \\ &\cong k.\end{aligned}$$

Hence $\mathrm{Ext}^1_{\mathsf{Gr}A}(L, L'(-1)) \neq 0$, and there is a non-split exact sequence

$$0 \longrightarrow L'(-1) \longrightarrow V \stackrel{\alpha}{\longrightarrow} L \longrightarrow 0$$

in $\mathsf{GrMod}A$. Choose $0 \neq \phi \in \mathrm{Hom}_{\mathsf{Gr}A}(A, V)$. The composition $\alpha\phi : A \to L$ is surjective because L is cyclic, so $\dim_k \phi(A_1) \geq 2$. If $\dim_k \phi(A_1) = 2$, then $L \cong A/WA$, where $W = (\ker\phi)_1$, whence the map $V \to L$ splits, contrary to our assumption. Thus $\dim_k \phi(A_1) = 3$. Hence there is some $0 \neq a \in A_1$ and a map $\psi : A/aA \to V$ that is surjective in degrees zero and one. Let $K = \ker \alpha\psi$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A/aA & \longrightarrow & \mathrm{im}(\alpha\psi) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & L'(-1) & \longrightarrow & V & \stackrel{\alpha}{\longrightarrow} & L & \longrightarrow & 0 \end{array}$$

Since $\dim_k(\mathrm{im}\,\alpha\psi)_1 < \dim_k(A/aA)_1$, $K_1 \neq 0$, and hence the map $K \to L'(-1)$ is surjective. It follows that ψ is surjective, and hence injective because V and A/aA have the same Hilbert series. Hence we get a non-split exact sequence as claimed.

($\Leftarrow$) To show that L' is in a different ruling from L it suffices to show that $\mathrm{Ext}^1_{\mathsf{Gr}A}(L', M(-1)) = 0$.

Apply $\mathrm{Hom}_{\mathsf{Gr}A}(-, M(-1))$ to the exact sequence (17). A computation shows that $\mathrm{Ext}^1_{\mathsf{Gr}A}(A/aA(1), M(-1)) = 0$. Since $\mathrm{Ext}^2_{\mathsf{Gr}A}(A/aA(1), M(-1)) = 0$, we have $\mathrm{Ext}^1_{\mathsf{Gr}A}(L', M(-1)) \cong \mathrm{Ext}^2_{\mathsf{Gr}A}(L(1), M(-1))$. From the exact sequence $0 \to M \to A(1) \to L(1) \to 0$, we see that

$$\mathrm{Ext}^2_{\mathsf{Gr}A}(L(1), M(-1)) \cong \mathrm{Ext}^1_{\mathsf{Gr}A}(M, M(-1)) \cong \mathrm{Hom}_{\underline{\mathsf{MCM}}}(M, M(-1)[1])$$

and this is zero as we see by applying the functor F.

(2) The proof of the implication ($\Rightarrow$) works when Q is not smooth too because then $M' \cong M \cong \Omega M(1)$. $\square$

8 Points on quadrics

We continue to assume that S and A are as in the notation just before Proposition 5.3. We also suppose that Q is smooth.

A graded point module for A or S is a graded module P that is cyclic and has Hilbert series

$$H_P(t) = (1-t)^{-1}.$$

We will write $\mathcal{O}_P$ for the image of P in either $\mathrm{Proj}\,A$ or $\mathrm{Proj}\,S$. The class of $\mathcal{O}_P$ in $K_0(\mathrm{Proj}\,S)$ is $(1-t)^3$.

Lemma 8.1 *Let $M \in \mathbb{M}$. If L_ϕ is in the ruling corresponding to $\Omega M(1)$, there is an exact sequence of the form*

$$0 \longrightarrow A \longrightarrow M \longrightarrow L_\phi \longrightarrow 0. \tag{18}$$

Proof. By hypothesis, there is an exact sequence

$$0 \longrightarrow \Omega M \xrightarrow{\phi} A \xrightarrow{\overline{\phi}} L_\phi \longrightarrow 0, \tag{19}$$

There is also an exact sequence

$$0 \longrightarrow \Omega M \xrightarrow{\theta} A^2 \longrightarrow M \longrightarrow 0. \tag{20}$$

Since M is maximal Cohen-Macaulay $\operatorname{Ext}^1_{\mathsf{Gr}A}(M,A) = 0$, so the natural map

$$\operatorname{Hom}_{\mathsf{Gr}A}(A^2, A) \to \operatorname{Hom}_{\mathsf{Gr}A}(\Omega M, A) \qquad \rho \mapsto \rho \circ \theta$$

is surjective and hence an isomorphism because $\operatorname{Hom}_{\mathsf{Gr}A}(M,A) = 0$. Hence $\phi = \rho\theta$ for a unique $\rho : A^2 \to A$.

The map $\overline{\phi}\rho$ in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \xrightarrow{\theta} & A^2 & \longrightarrow & M & \longrightarrow & 0. \\ & & & & \big\downarrow{\scriptstyle \overline{\phi}\rho} & & & & \\ & & & & L_\phi & & & & \end{array}$$

is surjective because $\overline{\phi}$ and ρ are, and $\overline{\phi}\rho\theta = \overline{\phi}\phi = 0$, so there is a surjective map $\psi : M \to L_\phi$.

Now $(\ker\psi)_0 \neq 0$ because $\dim M_0 > \dim(L_\phi)_0$, so there is a non-zero map $A \to \ker\psi$. Since M is 3-critical so is $\ker\psi$, so the map $A \to \ker\psi$ is injective. But

$$\mathrm{H}_{\ker\psi}(t) = H_M(t) - H_{L_\phi}(t) = H_A(t)$$

so the map $A \to \ker\psi$ must be an isomorphism. □

Proposition 8.2 *Let $M \in \mathbb{M}$. If L_ϕ is in the ruling corresponding to $\Omega M(1)$, there is an exact sequence*

$$0 \longrightarrow L(-1) \longrightarrow L_\phi \longrightarrow P \longrightarrow 0 \tag{21}$$

in which P is a point module and L is a line module in the same ruling as L_ϕ.

Proof. By hypothesis, there is an exact sequence of the form (19).

Claim: $\dim \operatorname{Hom}_{\mathsf{Gr}A}(M(-1), L_\phi) \geq 2$. Proof: Applying $\operatorname{Hom}_{\mathsf{Gr}A}(M(-1), -)$ to (19) yields an exact sequence

$$\operatorname{Hom}_{\mathsf{Gr}A}(M(-1), A) \xrightarrow{\gamma} \operatorname{Hom}_{\mathsf{Gr}A}(M(-1), L_\phi) \xrightarrow{\delta} \operatorname{Ext}^1_{\mathsf{Gr}A}(M(-1), \Omega M).$$

If Q is smooth, then $\operatorname{Hom}_{\mathsf{Gr}A}(M(-1), \Omega M) = 0$ by Corollary 5.7, so γ is injective, and the claim follows from Lemma 7.1.

Suppose Q is not smooth. Then $M(-1) \cong \Omega M$ by Corollary 5.7; thus $\operatorname{Hom}_{\mathsf{Gr}A}(M(-1), \Omega M) \cong k$ by the argument in the proof of Lemma 7.3, and $\operatorname{Ext}^1_{\mathsf{Gr}A}(M(-1), \Omega M) \neq 0$ by the proof of Step 1 in Theorem 5.6. However, $\operatorname{Ext}^1_{\mathsf{Gr}A}(M(-1), A) = 0$ so the claim holds in this case too. $\Diamond$

The restriction of each non-zero $\psi \in \operatorname{Hom}_{\mathsf{Gr}A}(M(-1), L_\phi)$ gives a non-zero map $M(-1)_1 \to (L_\phi)_1$ between two 2-dimensional vector spaces. Now every line in the projective space $\mathbb{P}^3 = \mathbb{P}(\operatorname{Hom}_k(k^2, k^2))$ meets the quadric of singular maps, so there is a non-zero ψ such that $(\ker\psi)_1 \neq 0$. There is a non-zero map $A(-1) \to \ker\psi$; this map is injective because $A(-1)$ and $\ker\psi$ are 3-critical; the cokernel of the composition $\alpha : A(-1) \to \ker\psi \to M(-1)$ is cyclic because $\dim M_0 = 1 + \dim A(-1)_1$ and M is generated by M_0; the Hilbert series of $\operatorname{coker}\alpha$ is

$$H_{M(-1)}(t) - H_{A(-1)}(t) = t(1-t)^{-2}$$

so $\operatorname{coker}\alpha$ is a shifted line module, say $L(-1)$.

Because $\psi\alpha = 0$, it follows from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(-1) & \xrightarrow{\alpha} & M(-1) & \xrightarrow{\overline{\alpha}} & L(-1) & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle\psi} & & & & \\ & & & & L_\phi & & & & \end{array}$$

that $\psi = \beta\overline{\alpha}$ for some $\beta : L(-1) \to L_\phi$. Because $L(-1)$ and L_ϕ are 2-critical β is injective, and $\operatorname{coker}\beta$ is cyclic with Hilbert series $(1-t)^{-2} - t(1-t)^{-2}$. Hence $\operatorname{coker}\beta$ is a point module.

It remains only to show that L is in the same ruling as L_ϕ. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M(-1) & \longrightarrow & A(-1)^2 & \xrightarrow{\theta} & M(-1) & \longrightarrow & 0. \\ & & & & & & \big\downarrow{\scriptstyle\overline{\alpha}} & & \\ & & & & & & L(-1) & & \end{array}$$

Since L is cyclic, the restriction of $\overline{\alpha}\theta$ to one of the copies of $A(-1)$ is surjective so, after a Hilbert series computation, we see that the kernel of $\overline{\alpha}\theta$ must be isomorphic to $\Omega M(-1)$. From the exact sequence $0 \to \Omega M(-1) \to A(-1) \to L(-1) \to 0$ we see that L is in the same ruling as L_ϕ. $\square$

Lemma 8.3 *Given a line module L, there is an exact sequence of the form*

$$0 \longrightarrow L(-1) \longrightarrow L_\phi \longrightarrow P \longrightarrow 0 \qquad (22)$$

in which P is a point module and L_ϕ is a line module in the same ruling as L.

Proof. This follows from duality and Proposition 8.2 for left modules.

Since L is Cohen-Macaulay of depth two, $\underline{\mathrm{Ext}}^1_A(L, A)$ is a *left* line module. By Proposition 8.2 for left modules, there is an exact sequence

$$0 \to L'(-1) \to \underline{\mathrm{Ext}}^1_A(L, A) \to P' \to 0$$

where L' and P' are a left line and point module, respectively. Since P' is Cohen-Macaulay of depth one, applying $\underline{\mathrm{Hom}}_A(-, A)$ to this exact sequence gives an exact sequence

$$0 \to \underline{\mathrm{Ext}}^1_A(\underline{\mathrm{Ext}}^1_A((L, A), A) \to \underline{\mathrm{Ext}}^1_A(L', A)(1) \to \underline{\mathrm{Ext}}^1_A(P', A) \to 0.$$

Twisting this by (-1) gives the desired exact sequence (22). □

9 The Grothendieck group of a smooth quadric

We continue to assume that S and A are as in the notation just before Proposition 5.3. We also assume that Q is smooth. We will write C for the algebra $C(A)$ that is isomorphic to $M_2(k) \oplus M_2(k)$.

We write $K_0(\mathsf{A})$ for the Grothendieck group of an abelian category A and define $K_0(Q) := K_0(\mathsf{mod}Q)$. We will show that there is an isomorphism $K_0(Q) \cong K_0(\mathbb{P}^1 \times \mathbb{P}^1)$ of abelian groups that is compatible with the Euler forms. However, the discussion after Theorem 10.2 shows that the effective cones need not coincide under this isomorphism.

The localization sequence for K-theory gives the exact rows in the diagram

$$\begin{array}{ccccccc} K_0(\mathsf{fdim}A^!) & \xrightarrow{\alpha} & K_0(\mathsf{grmod}A^!) & \xrightarrow{\beta} & K_0(\mathsf{mod}C) & \longrightarrow & 0 \\ & & \phi \big\downarrow \cong & & & & \\ K_0(\mathsf{fdim}A) & \longrightarrow & K_0(\mathsf{grmod}A) & \longrightarrow & K_0(Q) & \longrightarrow & 0. \end{array} \tag{23}$$

The isomorphism ϕ is induced by the Koszul duality functor $K : \mathsf{D}^b(\mathsf{grmod}A) \to \mathsf{D}^b(\mathsf{grmod}A^!)$ and the fact that $K_0(\mathsf{A}) \cong K_0(\mathsf{D}^b(\mathsf{A}))$. Thus $\phi([N]) = [KN]$ is an isomorphism of abelian groups.

We use the degree shift functors (± 1) on the categories $\mathsf{fdim}A^!$, $\mathsf{grmod}A^!$, $\mathsf{fdim}A$, $\mathsf{grmod}A$, and $\mathsf{mod}Q$, to make their Grothendieck groups into $\mathbb{Z}[t, t^{-1}]$-modules via

$$t.[M] = [M(-1)].$$

Exact functors between these categories that "commute" with the shift functors induce $\mathbb{Z}[t, t^{-1}]$-module homomorphisms between their Grothendieck groups.

Let M and M' be the two indecomposable maximal Cohen-Macaulay modules such that $\mathbb{M} = \{M(1), M'(1)\}$.

We will use the notation

$$a = [A], m = [M], m' = [M'], \ell := a - m, \ell' := a - m'$$

for these elements of $K_0(\mathsf{grmod}A)$. We will use the same notation for the images of a, m, m', ℓ and ℓ' in $K_0(Q)$ and will always take care to indicate which Grothendieck group we are working in.

The line modules L and L' in the rulings determined by M and M' respectively occur in exact sequences of the form $0 \to M \to A \to L \to 0$ and $0 \to M' \to A \to L' \to 0$, so all the line modules in a single ruling give the same class in $K_0(\mathsf{grmod}A)$, namely $[L] = \ell = a - m$ and $\ell' := [L'] = a - m'$ respectively.

Proposition 9.1 *The Grothendieck group of Q is free of rank 4 with basis $\{a, m, m', at\}$ and the action of $\mathbb{Z}[t, t^{-1}]$ on it is given by*

$$\begin{aligned} mt &= 2at - m' \\ m't &= 2at - m, \\ at^2 &= a(1+4t) - 2(m+m'). \end{aligned}$$

As an R-module,

$$K_0(Q) \cong \frac{Ra \oplus R\ell}{\big(\ell(1-t)^2,\ a(1-t^2) - 2\ell(1-t)\big)}. \tag{24}$$

Proof. Write $R = \mathbb{Z}[t, t^{-1}]$ and $C = A^![w^{-1}]_0$.

By Dévissage, $K_0(\mathsf{fdim}A^!) \cong K_0(\mathsf{grmod}k)$. Taking Hilbert series gives a map $K_0(\mathsf{grmod}A^!) \to \mathbb{Z}[[t]][t^{-1}]$; since $K_0(\mathsf{grmod}k) \cong \mathbb{Z}[t, t^{-1}]$ it follows that the map α in (23) is injective. Since Q is smooth, $K_0(\mathsf{mod}C) = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2] \cong \mathbb{Z}^2$ where S_1 and S_2 are the two simple C-modules; hence the top row of (23) splits, and

$$K_0(\mathsf{grmod}A^!) = R[k] \oplus \mathbb{Z}[\tilde{S}_1] \oplus \mathbb{Z}[\tilde{S}_2]$$

where $\tilde{S}_1$ and $\tilde{S}_2$ are the liftings of S_1 and S_2 via the functor $- \otimes_C A^![w^{-1}]_{\geq 0}$. Transferring this to A via Koszul duality, we see that

$$K_0(\mathsf{grmod}A) = Ra \oplus \mathbb{Z}m \oplus \mathbb{Z}m'.$$

From the exact sequences in Corollary 5.8, we obtain relations

$$2at = mt + m' = m't + m \tag{25}$$

in $K_0(\mathsf{grmod}A)$. Hence there is a surjective map

$$\frac{Ra \oplus Rm}{(2at - m - (2at - mt)t)} \to K_0(\mathsf{grmod}A) \tag{26}$$

of $\mathbb{Z}[t, t^{-1}]$-modules. However, it follows from (25) that there is also a surjective map

$$K_0(\mathsf{grmod}A) = Ra \oplus \mathbb{Z}m \oplus \mathbb{Z}m' \to \frac{Ra \oplus Rm}{(2at - m - (2at - mt)t)}$$

so we conclude that (26) is an isomorphism.

Since $\ell = a - m$, we also have

$$K_0(\mathsf{grmod}A) \cong \frac{Ra \oplus R\ell}{(a(1-t)^2 - \ell(1-t^2))}.$$

Now $K_0(\mathsf{fdim}A) \cong K_0(\mathsf{grmod}k) \cong \mathbb{Z}[t, t^{-1}]$ with basis $[k] \leftrightarrow 1$, so

$$K_0(Q) \cong K_0(\mathsf{grmod}A)/([k]),$$

where $([k])$ denotes the $\mathbb{Z}[t, t^{-1}]$-submodule generated by $[k]$.

We now compute $[k]$. From the Hilbert series for $A^!$, we see that the truncated minimal resolution of k looks like

$$0 \to N \to A(-2)^7 \to A(-1)^4 \to A \to k \to 0.$$

It is clear that N is a maximal Cohen-Macaulay module and that $N(3)$ has a linear resolution.

Let $F : \mathsf{grmod}A \to \mathsf{D}^b(\operatorname{proj} A^!)$ be the functor in Lemma 3.3. Since $N \cong \Omega^3 k$, that lemma shows that

$$F(N(3)) \cong (Fk)(3) \cong A^!(3).$$

The equivalence $\operatorname{proj} A^! \to \mathsf{mod}C$ sends $A^!$ to C. The degree twist (1) on $\operatorname{proj} A^!$ induces an auto-equivalence of $\mathsf{mod}C$, but every auto-equivalence of $\mathsf{mod}C$ sends ${}_CC$ to ${}_CC$. Thus, if G is the composition

$$\mathsf{grmod}A \xrightarrow{\ F\ } \mathsf{D}^b(\operatorname{proj} A^!) \xrightarrow{\ \sim\ } \mathsf{D}^b(\mathsf{mod}C),$$

then

$$G(N(3)) \cong C(3) \cong C.$$

The functor G sends the two maximal Cohen-Macaulay modules $M(1)$ and $M'(1)$ to the two simple left C-modules, so we see that

$$G(N(3)) \cong G(M(1)^{\oplus 2} \oplus M'(1)^{\oplus 2}).$$

It now follows from Buchweitz's duality (Theorem 3.2) that

$$N \cong M(-2)^{\oplus 2} \oplus M'(-2)^{\oplus 2}.$$

The truncated resolution of k and equation (25) therefore give

$$[k] = (1 - 4t + 7t^2)a - (2m + 2m')t^2 = (1 + 4t - t^2)a - 2(m + m').$$

Hence, in $K_0(Q)$, $at^2 = a(1 + 4t) - 2(m + m')$. It follows that $K_0(Q)$ has basis $\{a, m, m', at\}$, as claimed. □

Remark. By Proposition 8.2, A has some graded point modules and we therefore introduce the notation

$$p := \ell(1-t) \qquad \text{and} \qquad p' := \ell'(1-t)$$

for the classes of these point modules in $K_0(Q)$. It follows from (25) that $(m - m')(1 - t) = 0$, whence

$$p = \ell(1-t) = (a - m)(1-t) = (a - m')(1-t) = \ell'(1-t) = p'.$$

Proposition 9.2 *The sets* $\{a, m, m', p\}$ *and* $\{a, \ell, \ell', p\}$ *provide* $\mathbb{Z}$*-bases for* $K_0(Q)$. *The* $\mathbb{Z}[t, t^{-1}]$*-action is given by*

$$a(1-t) = \ell + \ell' t = \ell' + \ell t, \quad \ell(1-t) = \ell'(1-t) = p, \quad p(1-t) = 0.$$

Proof. Recall that $\ell = a - m$ and $\ell' = a - m'$, so $\{a, \ell, \ell', at\}$ is a basis for $K_0(Q)$. Furthermore

$$p = \ell(1-t) = (a-m)(1-t) = a - at - m + (2at - m') = a + at - m - m',$$

and it follows from this that the two claimed bases are indeed bases for $K_0(Q)$. The action of t is already implicit, if not explicit, in the calculations made in the proof of Proposition 9.1. □

The annihilator of $K_0(Q)$ as a $\mathbb{Z}[t, t^{-1}]$-module is $(1-t)^3$. The submodule of $K_0(Q)$ annihilated by $(1-t)$ is $\mathbb{Z}p \oplus \mathbb{Z}(\ell - \ell')$.

Taking Hilbert series gives a $\mathbb{Z}[t, t^{-1}]$-module homomorphism $K_0(\mathsf{grmod} A) \to \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$, $[N] \mapsto H_N(t)$. Likewise there is a homomorphism $q : K_0(\mathsf{grmod} A) \to \mathbb{Z}[t, t^{-1}]$ defined by

$$q[N] = H_N(t)(1-t)^3.$$

Because $q[k] = (1-t)^3$, there is an induced $\mathbb{Z}[t, t^{-1}]$-module homomorphism

$$\bar{q} : K_0(Q) \to \mathbb{Z}[t, t^{-1}]/(1-t)^3.$$

One has

$$\bar{q}(a) = 1 + t, \quad \bar{q}(\ell) = \bar{q}(\ell') = 1 - t, \quad \bar{q}(p) = (1-t)^2.$$

Suppose that $N \in \mathsf{grmod} A$ has GK-dimension one. Then $H_N(t) = f(t)(1-t)^{-1}$ for some $f(t) \in \mathbb{Z}[t, t^{-1}]$, so $\bar{q}[\pi N]$ belongs to the ideal of $\mathbb{Z}[t, t^{-1}]/(1-t)^3$ generated by $(1-t)^2$. It follows that

$$[\pi N] \in \mathbb{Z}p \oplus \mathbb{Z}(\ell - \ell').$$

The Euler form. The Euler form on $K_0(Q)$ is denoted by $(-,-)$ and is defined by

$$([M],[N]) = \sum_{i=0}^{2} (-1)^i \dim_k \operatorname{Ext}^i_Q(M, N).$$

Proposition 9.3 *The Euler form on* $K_0(Q)$ *is given by*

$$(a,a) = (a,\ell) = (a,\ell') = (a,p) = (p,a) = 1; \quad (\ell,a) = (\ell',a) = -1,$$

and

$$(\ell,\ell) = (\ell',\ell') = (\ell,p) = (\ell',p) = (p,\ell) = (p,\ell') = (p,p) = 0$$

and

$$(\ell,\ell') = (\ell',\ell) = -1.$$

Proof. Let P be a graded point module occuring in an exact sequence of the form $0 \to L_\phi(-1) \to L_\psi \to P \to 0$ where L_ψ and L_ϕ are line modules in the same ruling. From the Cohen-Macaulayness of A, M, M', P we see that

$$(a,a) = 1, \quad (a,m) = (a,m') = 0, \quad (a,p) = 1,$$

whence

$$(a,a) = (a,\ell) = (a,\ell') = (a,p) = 1.$$

Serre duality on Q takes the form $\operatorname{Ext}^i_Q(\mathcal{F},\mathcal{G}) \cong \operatorname{Ext}^{2-i}_Q(\mathcal{G},\mathcal{F}(-2))^*$ for $\mathcal{F},\mathcal{G} \in \mathsf{mod}Q$. Hence $(x,y) = (y, xt^2)$ for all $x, y \in K_0(Q)$. Also, $(xt, yt) = (x,y)$.

We have $(\ell, a) = (a, \ell t^2) = (a, \ell - 2p) = -1$, and similarly, $(\ell', a) = -1$. Also, $(p,a) = (a, pt^2) = (a,p) = 1$. In summary,

$$(\ell,a) = (\ell',a) = -1, \quad (p,a) = 1.$$

Now we show that $(m,m) = 1$. The first step is to show that $\operatorname{Ext}^1_Q(\mathcal{M},\mathcal{M}) = 0$. If $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{M} \to 0$ is exact, then applying ω gives an exact sequence $0 \to M \to F \to M \to 0$ because $R^1\omega M = H^2_{\mathfrak{m}}(M) = 0$ and $\omega\pi M \cong M$. But $\operatorname{Ext}^1_{\mathsf{Gr}A}(M,M) \cong \operatorname{Hom}_{\underline{\mathsf{MCM}}}(M, M[1]) = 0$, where the last equality follows by applying the functor F, so the sequence in $\mathsf{GrMod}A$ splits; but the original sequence in $\mathsf{Mod}Q$ is obtained by applying ω to this split sequence, so it splits too. Hence $\operatorname{Ext}^1_Q(\mathcal{M},\mathcal{M}) = 0$. Now, $\operatorname{Ext}^2_Q(\mathcal{M},\mathcal{M}) \cong \operatorname{Hom}_Q(\mathcal{M},\mathcal{M}(-2))^* \cong \operatorname{Hom}_{\mathsf{Gr}A}(M, M(-2))^* = 0$ because $M(-2)_1 = 0$. Finally, $\operatorname{Hom}_Q(\mathcal{M},\mathcal{M}) \cong \operatorname{Hom}_{\mathsf{Gr}A}(M,M) = k$, so $(m,m) = 1$.

Using this gives

$$(\ell,\ell) = (a-m, a-m) = (a, a-m) - (m,a) + (m,m) = 1 - (a-\ell, a) + 1 = 0,$$

and similarly, $(\ell',\ell') = 0$. Using Serre duality, we obtain

$$0 = (\ell,\ell) = (\ell, \ell t^2) = (\ell, \ell - 2p) = -2(\ell,p)$$

which gives $(\ell,p) = 0$. Therefore

$$(\ell,\ell') = (\ell, \ell' + \ell - p) = (\ell, \ell' + \ell t) = (\ell, a(1-t)) = -1 - (\ell, at)$$

and hence

$$(\ell,\ell') = -1 - (at, \ell t^2) = -1 - (a, \ell t) = -1 - (a, \ell - p) = -1.$$

Similarly, $(\ell',\ell) = -1$.

Finally,

$$(p,p) = (\ell(1-t), p) = -(\ell t, p) = -(\ell, pt^{-1}) = -(\ell,p) = 0,$$

and

$$(p,\ell) = (\ell, pt^2) = (\ell,p) = 0.$$

This completes the proof. $\square$

Proposition 9.3 is exactly as in the commutative case—of course, our proof applies to that case too.

The interpretation of $(\ell, \ell') = -1$ is that a line in one ruling meets a line in the other ruling with multiplicity one. In the commutative case this means that two such lines span a hyperplane. Proposition 7.4(1) is the appropriate analogue of this. It is therefore sensible to introduce the notation

$$h := [\mathcal{O}_Q] - [\mathcal{O}_Q(-1)] = \ell + \ell' t = \ell' + \ell t.$$

We now define an intersection pairing on $K_0(Q)$ by

$$b.c := -(b, c).$$

Proposition 9.4

$$\begin{aligned} \ell.\ell = \ell.p = h.p = p.h = p.\ell = \ell'.\ell' = 0 \\ \ell.\ell' = \ell.h = \ell'.h = h.\ell' = h.\ell = 1 \\ \ell.\ell' = \ell'.\ell = 1 \end{aligned}$$

Proof. The calculations are as follows:

$$\begin{aligned} (h, \ell) &= (\ell + \ell' t, \ell) = (\ell' t, \ell) = (\ell' - p, \ell) = -1 \\ (h, p) &= (\ell + \ell' t, p) = (\ell, p) + (\ell', p t^{-1}) = 0 \\ (\ell, h) &= (h, \ell t^2) = (h, \ell - 2p) = -1 \\ (p, h) &= (h, p t^2) = (h, p) = 0. \end{aligned}$$

□

One other computation of interest is $(m, m') = 0$.

Proposition 9.5 *Suppose Q is smooth. Let L and L' be non-isomorphic line modules for A, and $\mathcal{O}_L$ and $\mathcal{O}'_L$ their images in* $\operatorname{Proj} Q$. *The following are equivalent:*

1. $\operatorname{Ext}^1_Q(\mathcal{O}_L, \mathcal{O}_{L'}) = 0$;
2. $\operatorname{Ext}^i_Q(\mathcal{O}_L, \mathcal{O}_{L'}) = 0$ *for all* i;
3. *L and L' belong to the same ruling.*

Proof. Because L' is Cohen-Macaulay of depth 2, $\omega\pi L' \cong L'$. Hence

$$\operatorname{Hom}_Q(\mathcal{O}_L, \mathcal{O}_{L'}) \cong \operatorname{Hom}_{\mathsf{Gr}A}(L, \omega\pi L') \cong \operatorname{Hom}_{\mathsf{Gr}A}(L, L') = 0.$$

By Serre duality $\operatorname{Ext}^2_Q(\mathcal{O}_L, \mathcal{O}_{L'}) \cong \operatorname{Hom}_Q(\mathcal{O}_{L'}, \mathcal{O}_L(-2))$; because L is Cohen-Macaulay of depth 2, this is isomorphic to $\operatorname{Hom}_{\mathsf{Gr}A}(L', L(-2))$ which is zero because $L(-2)_0 = 0$. Hence (1) $\Leftrightarrow$ (2).

By Proposition 9.3, L and L' belong to the same ruling if and only if $([\mathcal{O}_L], [\mathcal{O}_{L'}]) = 0$. This, together with the observations in the previous paragraph, shows that (3) is equivalent to (1) and (2). □

If Q is not smooth then $\operatorname{Ext}^1_Q(\mathcal{O}_L, \mathcal{O}'_L) \neq 0$ and, if $L \not\cong L'$, then $\operatorname{Ext}^2_Q(\mathcal{O}_L, \mathcal{O}_{L'})$ and $\operatorname{Hom}_Q(\mathcal{O}_L, \mathcal{O}_{L'})$ are both zero.

10 The Sklyanin quadrics

Throughout this section S denotes a four-dimensional Sklyanin algebra. We recall some results from the papers [12], [15], [16], [18], [22], and [23].

The data used to define S is a triple $(E, \mathcal{L}, \tau)$ consisting of an elliptic curve E, a degree four line bundle $\mathcal{L}$ on it, and a translation automorphism τ of E, and S is a particular quotient of the tensor algebra on $H^0(E, \mathcal{L})$. In particular, S is a connected graded k-algebra, generated in degree one. Because $S_1 = H^0(E, \mathcal{L})$ we can, and will, consider E as a fixed quartic curve in $\mathbb{P}(S_1^*)$. We fix an origin 0 for E such that four points of E are coplanar if and only if their sum is zero. We therefore identify τ with a point on E so the translation automorphism becomes $p \mapsto p + \tau$.

The Hilbert series of S is $(1-t)^{-4}$. Like the polynomial ring, S is Gorenstein, and its dualizing module is $\omega_S \cong S(-4)$ as a one-sided S-module. Furthermore, S is a noetherian domain and a Koszul algebra. Thus $\operatorname{Proj} S$ is a quantum $\mathbb{P}^3$ in the sense of section 2.6.

The center of S contains two linearly independent homogeneous elements, say Ω_1 and Ω_2, of degree two. These give rise to a pencil of quotients $A = S/(\Omega)$, Ω a non-zero linear combination of Ω_1 and Ω_2, and hence a pencil of non-commutative quadric hypersurfaces $\operatorname{Proj} A \subset \operatorname{Proj} S$. Every such A is a domain, and is Gorenstein with dualizing module $\omega_A \cong A(-2)$ as a one-sided A-module.

The ring $S/(\Omega_1, \Omega_2)$ is a twisted homogeneous coordinate ring of E, so $\operatorname{Proj} S/(\Omega_1, \Omega_2)$ gives E as a closed subspace of $\operatorname{Proj} S$. It is the base locus of the pencil of non-commutative quadrics.

A generic pencil of quadrics in $\mathbb{P}^3$ has exactly four singular members. The smooth quadrics have two rulings on them, and the singular ones have only one ruling. The Sklyanin quadrics behave in a similar way, as we now explain.

The following rule sets up a bijection between the line modules for S and the secant lines to E in $\mathbb{P}(S_1^*)$: if $p, q \in E$, and $W \subset S_1$ is the subspace of linear forms vanishing on the secant line $\overline{pq}$ through p and q, then S/SW is a line module that we denote by $L(\overline{pq})$ [12].

The pencil of commutative quadrics in $\mathbb{P}(S_1^*)$ containing E may be labelled as Y_z, $z \in E/\pm \cong \mathbb{P}^1$, in such a way that Y_z is the union of all the secant lines $\overline{pq}$ such that $p + q = z$. It follows that $Y_z = Y_{-z}$ and the four singular quadrics are Y_ω, $\omega \in E_2$, the 2-torsion subgroup of E. When $z \notin E_2$, the two rulings on Y_z are given by $\{\overline{pq} \mid p + q = z\}$ and $\{\overline{pq} \mid p + q = -z\}$.

If $z \in E$, then there is a non-zero linear combination $\Omega(z)$ of Ω_1 and Ω_2 such that

$$\Omega(z).L(\overline{pq}) = 0 \Longleftrightarrow p + q = z \text{ or } p + q = -z - 2\tau$$

(see [12, Sect. 6]). We label the non-commutative quadrics in $\operatorname{Proj} S$ as

$$Q_z := \operatorname{Proj} S/(\Omega(z)), \qquad z \in E.$$

Thus $Q_z = Q_{-z-2\tau}$. If $z \notin E_2 + \tau$, we say there are two families of line modules for A giving "lines" on Q_z, namely $\{L(\overline{pq}) \mid p + q = z\}$ and $\{L(\overline{pq}) \mid p + q = -z - 2\tau\}$.

The degree two divisors $(p)+(q)$ such that $p+q=z$ are parametrized by the fiber over z of the addition map $S^2E \to E$. These fibers are isomorphic to $\mathbb{P}^1$, which is why we say these lines form a family.

The next result shows that these "families" coincide with the "rulings" defined in section 7.

Proposition 10.1 *Let L and L' be line modules for A. Then L and L' belong to the same ruling if and only if they belong to the same family.*

Proof. Let $Q_z = \operatorname{Proj} A$. Suppose that $L = L(\overline{pq})$ and $L' = L(\overline{p'q'})$ where $p+q, p'+q' \in \{z, -z-2\tau\}$.

($\Leftarrow$) Suppose that $p+q = p'+q'$. There are points $r, s \in E$ such that p, q, r, and s span a secant plane, say that given by $a=0$ for $0 \neq a \in A_1$, and p', q', r, and s also span a secant plane, say that given by $b=0$ for $0 \neq b \in A_1$.

Set $L'' = L(\overline{r-\tau, s-\tau})$. By the argument in the proof of [19, Lemma 4.5], there are exact sequences

$$0 \to L''(-1) \to A/Aa \to L \to 0$$

and

$$0 \to L''(-1) \to A/Ab \to L' \to 0.$$

By Proposition 7.4, L and L'' belong to different rulings, and so do L' and L''; hence L and L' belong to the same ruling.

($\Rightarrow$) Suppose that $p+q \neq p'+q'$. In this case $p+q+(p'+\tau)+(q'+\tau)=0$, so $p, q, p'+\tau, q'+\tau$ span a secant plane. By [19, Lemma 4.5], there is an exact sequence of the form $0 \to L'(-1) \to A/xA \to L \to 0$, so L and L' belong to the same ruling by Proposition 7.4. $\square$

Theorem 10.2 *The Sklyanin quadric Q_z is smooth if and only if $z+\tau \notin E_2$. The four singular quadrics are $Q_{\omega-\tau}$, $\omega \in E_2$.*

Proof. If $z+\tau \notin E_2$, then $Q_z = Q_{-z-2\tau}$ has two families of line modules, namely $L(\overline{pq})$ such that $p+q=z$ and $p+q=-z-2\tau$, whereas if $z+\tau \in E_2$, there is only one family of line modules for Q_z, namely $L(\overline{pq})$ such that $p+q = z = -z-2\tau$. Now by Theorem 10.1, there are two rulings on Q_z if and only if $z+\tau \notin E_2$, so the result follows from Theorem 5.6. $\square$

Although the pencil of Sklyanin quadrics behaves like a generic pencil of quadrics in $\mathbb{P}^3$ there is a significant difference. The singular locus of a singular commutative quadric Q in a generic pencil in $\mathbb{P}^3$ is a point, and that point lies on all the lines on Q. However, the results in [19] (see also [15, Sect. 10]) show there is no analogous result for the Sklyanin quadrics. For simplicity, we will explain this only when τ has infinite order.

When τ has infinite order the closed points in $\operatorname{Proj} S$ consist of those on E and a discrete family that may be labelled as

$$\{p_{\omega+i\tau} \mid \omega \in E_2, i \in \mathbb{N}\}$$

in such a way that (a) $p_{\omega+i\tau}$ lies on the non-commutative secant line $\overline{pq}$ if and only if $p+q=\omega+i\tau$, and (b) if $\mathcal{F}_{\omega+i\tau}$ is the simple module corresponding to $p_{\omega+i\tau}$, then $\dim_k H^0(\operatorname{Proj} S, \mathcal{F}_{\omega+i\tau}) = i+1$. Thus, all the lines in one of the two rulings on the smooth quadrics $Q_{\omega+i\tau} = Q_{\omega-(i+2)\tau}$, $i \in \mathbb{N}$, pass through a common point. The lines on a singular quadric $Q_{\omega-\tau}$ do not pass through a common point.

Let $i \in \mathbb{N}$ and $\omega \in E_2$. By [19, Sect. 4], if $p+q=\omega+i\tau$ there is an exact sequence

$$0 \to \mathcal{O}_{\overline{p-(i+1)\tau, q-(i+1)\tau}}(-i) \to \mathcal{O}_{\overline{pq}} \to \mathcal{F}_{\omega+i\tau} \to 0 \tag{27}$$

of $Q_{\omega+i\tau}$-modules; because $(p-(i+1)\tau)+(q-(i+1)\tau) \neq p+q$, the two lines in (27) belong to different rulings; it also follows from (27) that the class of $\mathcal{F}_{\omega+i\tau}$ in $K_0(Q_{\omega+i\tau})$ is

$$[\mathcal{F}_{\omega+i\tau}] = \ell - \ell' t^{i+1} = \ell - \ell' + (i+1)p.$$

This shows that the positive cone of $K_0(Q_{\omega+i\tau})$ is not the same as that of $K_0(\mathbb{P}^1 \times \mathbb{P}^1)$. A computation in $K_0(Q_{\omega+i\tau})$ using Proposition 9.3 gives

$$(\mathcal{F}_{\omega+i\tau}, \mathcal{F}_{\omega+i\tau}) = 2,$$

so $p_{\omega+i\tau}$ behaves like a curve with self-intersection -2.

Similar behavior is exhibited by the primitive quotient rings of the enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ (cf. [9], [21] and [23]). More precisely, the homogenized enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ is the coordinate ring of a quantum $\mathbb{P}^3$ that contains a pencil of non-commutative quadrics and those non-commutative quadrics behave like the Sklyanin quadrics. In particular, the finite-dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ provide points on certain of these quadrics that also behave like -2-curves—they have self-intersection -2.

The quadrics in a generic pencil in $\mathbb{P}^3$ can be viewed as the fibers of a family $X \to \mathbb{P}^1$. The total space $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ is smooth. It seems likely that the analogous non-commutative 3-fold $X_{nc} \subset \operatorname{Proj} S \times \mathbb{P}^1$ is also smooth, but we do not know how to tackle this problem.

Our methods may be applied to the pencil of non-commutative quadrics in the non-commutative $\mathbb{P}^3$ that is associated to the enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$. This pencil of non-commutative quadrics is analogous to the commutative pencil of quadrics that is generated by a double plane $w^2=0$ and $x^2+y^2+z^2=0$. The non-commutative pencil contains a "double plane" and one more singular non-commutative quadric that corresponds to the unique primitive quotient of $U(\mathfrak{sl}(2,\mathbb{C}))$ having infinite global dimension. This particular quotient of $U(\mathfrak{sl}(2,\mathbb{C}))$ is a simple ring so has no finite dimensional simple module; this is analogous to the fact that the singular Sklyanin quadrics are not the ones having a point that causes infinite global dimension. The homological properties of the quotients of $U(\mathfrak{sl}_2)$ are described in [21].

Let Q be a smooth non-commutative quadric surface. It would be interesting to show that there is a map $Q \to \mathbb{P}^1$ in the sense of [17, Defn. 2.3], to define and study the fibers of such a map, and to show that Q is the disjoint union of these fibers in a suitable sense. It would also be interesting to examine quadric hypersurfaces in non-commutative analogues of $\mathbb{P}^n$ for $n > 3$.

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