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Curves on Quasi-Schemes

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Abstract. This paper concerns curves on noncommutative schemes, hereafter called quasi-schemes. A quasi-scheme X is identified with the category $\mathsf{Mod}X$ of quasi-coherent sheaves on it. Let X be a quasi-scheme having a regularly embedded hypersurface Y. Let C be a curve on X which is in 'good position' with respect to Y (see Definition 5.1) – this definition includes a requirement that X be far from commutative in a certain sense. Then C is isomorphic to $\mathbb{V}_n^1$, where n is the number of points of intersection of C with Y. Here $\mathbb{V}_n^1$, or rather $\mathsf{Mod}\mathbb{V}_n^1$, is the quotient category $\mathsf{GrMod}k[x_1, \ldots, x_n]/\{\mathrm{Kdim} \leqslant n-2\}$ of $\mathbb{Z}^n$-graded modules over the commutative polynomial ring, modulo the subcategory of modules having Krull dimension $\leqslant n-2$. This is a hereditary category which behaves rather like $\mathsf{Mod}\mathbb{P}^1$, the category of quasi-coherent sheaves on $\mathbb{P}^1$.

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0. Introduction

There are several motivations for this paper, one of which is suggested by the title.

Noncommutative algebraic geometry has not yet developed a conceptual framework which allows one to say, for example, that a curve in one space (quasi-scheme) is isomorphic to a curve in another space. This gap is particularly apparent when one considers line modules. One wishes to think of line modules as lines, and then to decide whether the lines in one quantum projective space are isomorphic to those in another. There are at least two reasons this has not been done: first, it is unclear how to define a morphism between schemes; second a line module is a single module whereas a space (quasi-scheme) is an Abelian category (Definition 1.1). This paper is concerned with the second issue. We can avoid the first issue since our concern is to show that certain pairs of quasi-schemes are isomorphic, and it is clear that isomorphism should be defined in terms of equivalence of categories. The following simple question illustrates the problems surrounding the second issue. Let R be a ring with a right ideal I. How should we associate to R/I a full Abelian subcategory of $\mathsf{Mod}R$, which is a substitute for $\mathsf{Mod}R/I$ which only exists when I is a two-sided ideal? If one has a sound recipe for producing such a category, one can then compare R/I with S/J, where J is a right ideal in another ring S, by comparing the categories '$\mathsf{Mod}R/I$' and '$\mathsf{Mod}S/J$'.

We address this problem under some hypotheses which, although rather special, cover several important examples of current interest. The situation we consider is that of a quasi-scheme X having a sub-quasischeme Y sitting as a regularly embedded hypersurface in X (see Definitions 2.1 and 4.1). Thus we have a Grothendieck category $\mathsf{Mod}X$, which we think of as quasi-coherent sheaves on X, and a full subcategory $\mathsf{Mod}Y$ which we think of as the quasi-coherent sheaves on the hypersurface Y. In several important applications Y is commutative, by which we mean that $\mathsf{Mod}Y$ is equivalent to the category of quasi-coherent sheaves on a commutative scheme. We define a certain type of X-module called a *curve module*, and associate to it a subcategory $\mathsf{Mod}C$ of $\mathsf{Mod}X$. $\mathsf{Mod}C$ should be thought of as the quasi-coherent sheaves on the curve of which C is the structure sheaf or coordinate ring. If C is in general position with respect to Y (Definition 5.1), then we are able to give a detailed description of $\mathsf{Mod}C$. It is similar to the category of coherent sheaves on $\mathbb{P}^1$; for example, it is hereditary, meaning that $\mathrm{Ext}^2(-,-) \equiv 0$; every Noetherian object in it is a direct sum of line bundles and uniserial modules of finite length; its Picard group is $\mathbb{Z}^n$, where n is the number of points where C meets Y. This description is used to show that $\mathsf{Mod}C$ is isomorphic to a certain quotient category of the graded modules over the n-dimensional commutative polynomial ring graded by $\mathbb{Z}^n$. Thus, $\mathsf{Mod}C$ is a very special category, and it follows that many different quasi-schemes may contain isomorphic curves.

The notion of general position referred to in the previous paragraph contains within it a requirement that X be far from the commutative case. Thus our results say nothing about a quasi-scheme which arises from a ring which is a finite module over its center.

Section 7 shows that the results apply to the enveloping algebra of the two-dimensional non-Abelian Lie algebra; the quantum affine plane $k_q[x, y]$ with relation $yx = qxy$; the quantum $\mathbb{P}^2$'s of Artin, Tate, and Van den Bergh; the quantum quadrics in the Sklyanin $\mathbb{P}^3$; the quantum $\mathbb{P}^3$'s investigated by Vancliff [23] arising from 2×2 quantum matrices; and more. Each of these quasi-schemes has a (homogeneous) coordinate ring containing a normal, regular, nonunit and the 'zero locus' of this element is a regularly embedded hypersurface. Thus, each of these spaces contains curves isomorphic to curves lying in the other spaces.

The following simple example illustrates our results. Let $A = k_q[x, y]$ with defining relation $yx = qxy$, and $0 \neq q \in k$ is not a root of 1. Let C be the right A-module $A/(y + f)A$ where $f \in k[x]$ has degree $n - 1$. Write f as a product of linear factors, say $f = a\prod_{i=1}^{n-1}(x - r_i)$. We say C is in general position if the elements $q^l r_i$ are different for different (l, i). If we think of C as a curve in quantum affine 2-space, then the general position condition says that C meets the x- and y-axes transversally; recall that the points in this quantum plane are the points on the x- and y-axes. Given a critical module M with respect to some dimension function, we define the subcategory $\mathsf{C}(M)$ 'generated by M' to be the smallest full subcategory of $\mathsf{Mod}A$ satisfying the following conditions:

(1) if N is critical and M and N have the same injective hull, then $N \in \mathsf{C}(M)$;
(2) if $L \in \mathsf{C}(M)$, then submodules and quotient modules of L are in $\mathsf{C}(M)$;
(3) $\mathsf{C}(M)$ is closed under direct sums.

We prove that if $C = A/(y+f)A$ is in general position, then there is an equivalence of categories

$$\mathsf{C}(C) \cong \mathbb{V}_n^1 := \mathsf{GrMod}_{\mathbb{Z}^n} k[x_1, \dots, x_n]/(\mathrm{Kdim} \leqslant n-2),$$

where $k[x_1, \dots, x_n]$ is the commutative polynomial ring graded by $\mathbb{Z}^n$ in the obvious way. It is easy to check that C has exactly n simple quotient modules corresponding to the n simple quotients of $C =: \mathcal{O}_{\mathbb{V}_n^1}$.

If M is a module over a Noetherian ring R satisfying a polynomial identity, then $\mathsf{C}(M) = \mathsf{Mod}(R/\mathrm{Ann}\, M)$ because $R/\mathrm{Ann}\, M$ embeds in a finite direct sum of copies of M.

The applications of this paper to line modules in quantum projective spaces explain to some extent the fact that the lattice of submodules which are again line modules is often isomorphic to $\mathbb{N}^n$ ($n = 3$ for the quantum $\mathbb{P}^2$'s in [3], $n = 2$ for the four-dimensional Sklyanin algebra, etc.).

Although the word curve appears throughout this paper, we do *not* have a definition of a noncommutative curve with which we are happy. We do know many examples which should be included in any reasonable definition, but the boundary of the definition will not become clear until a greater variety of examples has been examined.

1. Quasi-Schemes

In this section k denotes a commutative ring. Later on k will denote an algebraically closed field. Modules will always be right modules, and the category of right modules over a ring R is denoted $\mathsf{Mod}R$.

We will adopt the language and framework set out by Van den Bergh in [27].

To avoid set theoretic problems we work in a fixed universe. Sets belonging to the universe are called *small*, all Hom-sets in a category are required to be small. A *small category* is one in which the objects form a small set. All index sets for limits or colimits are required to be small. We will use the phrase 'direct limit' only for a colimit indexed by a directed set.

Recall that a *Grothendieck category* is an Abelian category which is co-complete (i.e., has all colimits), has exact direct limits, and has a set of generators. If in addition it has a set of Noetherian generators it is said to be *locally Noetherian*. A Grothendieck category has enough injectives [11, Théorème 1.10.1], and even has injective envelopes [8]. If T is a localizing subcategory of a Grothendieck category

C, then the quotient category C/T is again a Grothendieck category [16, Corollary 4.6.2]. A Grothendieck category is complete.

DEFINITION 1.1 ([17, 27]). A *quasi-scheme* X is a Grothendieck category, $\mathsf{Mod}X$. Objects in $\mathsf{Mod}X$ are called *X-modules*. We write $\mathsf{mod}X$ for the full subcategory of Noetherian modules. We say X is *commutative* if $\mathsf{Mod}X$ is equivalent to the category of quasi-coherent sheaves on a scheme in the sense of Grothendieck.

Sometimes it is useful to specify a distinguished X-module called a *structure sheaf*, and usually denoted by $\mathcal{O}_X$. It is also natural to insist that $\mathcal{O}_X$ be large in some suitable sense. For example, one might ask that as s runs through all auto-equivalences of $\mathsf{Mod}X$, the objects $s\mathcal{O}_X$ form a generating set for $\mathsf{mod}X$. The structure sheaf $\mathcal{O}_C$ of the curves C which we study in this paper are large in this sense.

Notice that $X = \mathsf{Mod}X$; we tend to use the letter X alone when we wish to think of X as a geometric object.

The next definition isolates an important subclass of Grothendieck categories; one property of these is that they have all limits.

DEFINITION 1.2. Let C be a k-linear Abelian category. We define the category

$$\mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}k) := \text{left exact contravariant } k\text{-linear functors } \mathsf{C} \to \mathsf{Mod}k.$$

The morphisms are the natural transformations. We view C as a full subcategory of $\mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}k)$ through the Yoneda embedding $U \mapsto \mathrm{Hom}_{\mathsf{C}}(-, U)$.

PROPOSITION 1.3 ([9, Chapitre II]). *Let* C *be a small Abelian category, and write* $\overline{C} = \mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}\mathbb{Z})$. *Then* $\overline{C}$ *has the following properties*:

(1) $\overline{C}$ *is a Grothendieck category*;
(2) *the inclusion* $\mathsf{C} \to \overline{C}$ *is full, faithful, and exact*;
(3) *every object in* $\overline{C}$ *is a direct limit of objects in* C;
(4) *if* C *is Noetherian, then*
   (a) *an object in* $\overline{C}$ *is injective if and only if it is an exact functor*;
   (b) *every Noetherian object in* $\overline{C}$ *belongs to* C;
   (c) *every object in* $\overline{C}$ *is the direct limit of its Noetherian subobjects*.

The hypothesis that C is small ensures that the family of generators $\mathrm{Hom}_{\mathsf{C}}(-, U)$ for $\mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}\mathbb{Z})$ is a small set. There is a version of the previous result for k-linear Abelian categories.

The two prototypical examples of quasi-schemes are the usual commutative schemes, and $\mathsf{Mod}R$, where R is a ring which need not be commutative. The next example gives some important examples.

EXAMPLE 1.4. Let A be a k-algebra graded by a group G. The category of G-graded A-modules is denoted $\mathsf{GrMod}A$, or $\mathsf{GrMod}_G A$. Since $\mathsf{GrMod}_G A$ is a

Grothendieck category it is a quasi-scheme. Let Tors be a localizing subcategory of $\mathsf{GrMod}_G A$. By [16, 4.6.2], the quotient $\mathsf{GrMod}_G A/\mathsf{Tors}$ is a Grothendieck category, hence a quasi-scheme.

If A is right Noetherian, the full subcategory of modules whose finitely generated submodules have Krull dimension $\leqslant l$ is a localizing subcategory, so modding out such a subcategory gives another quasi-scheme. This is our basic example of a quasi-scheme.

If the group G is trivial, and $\ell = -1$, this quotient category is $\mathsf{Mod}A$, which we will sometimes call Spec A. If $G = \mathbb{Z}$, and $\ell = 0$, we sometimes call this quotient category $\mathsf{Tails}A$.

The notion of Krull dimension can be defined in any Abelian category C. First the objects of Krull dimension $\leqslant 0$ are the direct limits of Artinian modules. These form a Serre subcategory, say C_0, so the quotient category C/C_0 exists; the objects in C which are not in C_0, but whose image is in $(\mathsf{C}/\mathsf{C}_0)_0$ are said to have Krull dimension $\leqslant 1$. One keeps going in the obvious way. A Noetherian *1-critical* object in an Abelian category is a non-Artinian object, all of whose proper quotients are Artinian.

DEFINITION 1.5. A full subcategory E of an Abelian category D is *closed* if it is closed under subquotients, and the inclusion functor has a right adjoint. A closed subcategory $\mathsf{E} \subset \mathsf{D}$ is *biclosed* [27] if it is closed, and the inclusion functor has a left adjoint. We write $i_*\colon \mathsf{E} \to \mathsf{D}$ for the inclusion, and write $i^!$ and i^* for the right and left adjoints respectively.

If E is closed in D, then i_* is exact.

LEMMA 1.6. *Let D be a k-linear Abelian category, and let E be a full subcategory closed under subquotients and direct sums.*

(1) *If D is Noetherian, or locally Noetherian, then E is closed.*
(2) *If D is Noetherian, then $\mathsf{Lex}(\mathsf{E}^{\mathrm{op}}, \mathsf{Mod}k)$ is closed in $\mathsf{Lex}(\mathsf{D}^{\mathrm{op}}, \mathsf{Mod}k)$.*
(3) *If E is closed in D, and D has enough injectives, so does E.*
(4) *If D is Grothendieck, then E is closed* [27, Proposition 3.4.3].
(5) *If D is Grothendieck, so is E.*
(6) *If D is locally Noetherian, so is E.*

Proof. (1) An object M in D has a largest subobject belonging to E. To see this, if M has subobjects U_i belonging to E, then $\bigoplus_{\text{finite}} U_i$ belongs to E, so its image in M belongs to E; if M is Noetherian, this image stabilizes as the finite set increases, so all the U_i are contained in a single subobject of M belonging to E; if D is locally Noetherian, then the direct sum of *all* the U_i's is in E, and the image of that in M is the largest subobject of M belonging to E.

The adjoint functor assigns to an object of D its largest subobject belonging to E, and assigns to a morphism its restriction to these subobjects; if $f\colon M \to N$ is a morphism in D, and $U \subset M$ belongs to E, so does $f(U)$.

(2) First we show that $\bar{\mathsf{E}}$ embeds in $\bar{\mathsf{D}}$. Let E' be the full subcategory of $\bar{\mathsf{D}}$ consisting of those objects all of whose Noetherian subobjects belong to E. Then E' satisfies Ab5, so is locally Noetherian. But every locally Noetherian category is the completion of its full subcategory of Noetherian objects [9, Théorème 1, p. 356], so E' is equivalent to $\bar{\mathsf{E}}$. Thus, $\bar{\mathsf{E}}$ may be viewed as a full subcategory of $\bar{\mathsf{D}}$. It is straightforward to show that it is a closed subcategory.

(3) A right adjoint to an exact functor sends injectives to injectives, so if M is in E with injective envelope I in D, the adjoint functor sends I to an injective in E containing M.

(5) By [27, Proposition 3.4.3.1], E has direct limits. Since the inclusion $i_*\colon \mathsf{E} \to \mathsf{D}$ is a left adjoint, it commutes with direct limits; so E has exact direct limits. Since $i^!$ is right exact, it sends a set of generators in D to a set of generators in E. Hence, E is Grothendieck. □

A locally Noetherian category is cocomplete by definition, and is also complete [9, p. 358]. Therefore a closed subcategory of a locally Noetherian category has small products; but the product in the subcategory need not equal the product in the larger category. Thus the inclusion functor need not preserve products; it will preserve products if and only if the subcategory is biclosed.

2. Subschemes

DEFINITION 2.1. Let X be a quasi-scheme. A *(bi)closed subscheme* $Y \subset X$ is a (bi)closed subcategory $\mathsf{Mod}Y \subset \mathsf{Mod}X$. If we write $i\colon Y \to X$ to indicate this, we write $i_*\colon \mathsf{Mod}Y \to \mathsf{Mod}X$ for the inclusion functor. We call i_* the *direct image functor*, its left adjoint i^* the *inverse image functor*, and its right adjoint $i^!$ the *support functor*. We call $i^!M$ the part of M supported on Y.

Remark 2.2. Let Y be a biclosed subscheme of X. Then the direct image functor i_* is exact, and

(1) i^* is right exact because it is a left adjoint (similarly, $i^!$ is left exact);
(2) because i_* is a full embedding it follows, say from the Yoneda embedding, that the adjunctions $i^*i_* \to \mathrm{id}_Y$ and $\mathrm{id}_Y \to i^!i_*$ are natural equivalences;
(3) because $\mathsf{Mod}Y$ is closed under subquotients, if M is an X-module, the adjunction morphisms $M \to i_*i^*M$ and $i_*i^!M \to M$ are epic and monic, respectively;
(4) if N is an simple Y-module, then it is a simple X-module (because $\mathsf{Mod}Y$ is closed under submodules and quotients in $\mathsf{Mod}X$);
(5) if Y is a closed subscheme of X, then the Krull dimension of a Y-module is independent of whether it is viewed as an object in $\mathsf{Mod}X$ or $\mathsf{Mod}Y$.

If $\varphi\colon R \to S$ is a ring homomorphism, there are functors

- $f^*\colon \mathsf{Mod}R \to \mathsf{Mod}S$ defined by $f^* = S \otimes_R -$,

- the restriction functor $f_*\colon \mathsf{Mod}S \to \mathsf{Mod}R$, namely $f_* = {}_RS \otimes_S - = \mathrm{Hom}_S(S_R, -)$, and
- $f^!\colon \mathsf{Mod}R \to \mathsf{Mod}S$ defined by $f^! = \mathrm{Hom}_R(S_S, -)$.

Both (f^*, f_*) and $(f_*, f^!)$ are adjoint pairs. If φ is surjective, then $f_*\colon \mathsf{Mod}S \to \mathsf{Mod}R$ is a biclosed subscheme.

The following result is due to Rosenberg [18, Example 3.1] and [19, Proposition 6.4.1, p. 127].

PROPOSITION 2.3. *Let R be a ring, and C a biclosed subcategory of $\mathsf{Mod}R$. Write $i_*\colon \mathsf{C} \to \mathsf{Mod}R$ for the inclusion. Define $S = \mathrm{Hom}_{\mathsf{C}}(i^*R, i^*R)$. Then the functor $\mathrm{Hom}_{\mathsf{C}}(i^*R, -)\colon \mathsf{C} \to \mathsf{Mod}S$ is an equivalence of categories, the canonical ring homomorphism $\varphi\colon R \to S$ induced by i^* is surjective, and $\varphi_* \cong i_*$, and $\varphi^* \cong i^*$.*

Proof. Since i^* is a left adjoint it is right exact, so sends progenerators to progenerators, whence i^*R is a progenerator in C. Thus C is a module category, and the rest of the proof is straightforward. $\square$

PROPOSITION 2.4. *Let $i\colon Y \to X$ be a closed subscheme. Let $P \in \mathsf{Mod}Y$ and $Q \in \mathsf{Mod}X$. There is a Grothendieck spectral sequence*

$$E_2^{pq} = \mathrm{Ext}_Y^p(P, R^q i^! Q) \Rightarrow_p \mathrm{Ext}_X^n(i_*P, Q).$$

Proof. Let $F = \mathrm{Hom}_Y(P, -)$. A right adjoint to an exact functor preserves injectives so, if E is injective in $\mathsf{Mod}X$, $i^!E$ is injective in $\mathsf{Mod}Y$. Thus $i^!$ is right acyclic for F. Hence there is a third quadrant Grothendieck spectral sequence

$$(R^pF)(R^qi^!)(Q) \Rightarrow R^n(F \circ i^!)(Q).$$

But $F \circ i^! = \mathrm{Hom}_Y(P, -) \circ i^! \cong \mathrm{Hom}_X(i_*P, -)$, thus giving the result. $\square$

3. Some Examples

NOTATION. We will consider objects graded by the group $\mathbb{Z}^n$. We fix a basis

$$\varepsilon_1 = (1, 0, \dots, 0), \dots, \varepsilon_n = (0, \dots, 0, 1).$$

We also define $\varepsilon := \varepsilon_1 + \cdots + \varepsilon_n$. If $i \in \mathbb{Z}^n$ we write $i = (i_1, \dots, i_n)$. We define a partial ordering $i \geqslant j$ if $i_s \geqslant j_s$ for all s. We define $|i| = i_1 + \cdots + i_n$.

EXAMPLE 3.1. Let $S = k[X_1, \dots, X_n]$ be the commutative polynomial ring graded by $\mathbb{Z}^n$ with $\deg X_s = \varepsilon_s$. Then $\mathbb{V}_n^\ell := \mathsf{GrMod}S/\{\mathrm{Kdim} \leqslant n - \ell - 1\}$ is a quasi-scheme with Krull dimension ℓ. Let $\pi\colon \mathsf{GrMod}S \to \mathbb{V}_n^\ell$ be the quotient functor. We view $\mathbb{V}_n^1$ as a noncommutative curve, and $\mathbb{V}_n^2$ as a noncommutative surface. The 'line bundles' on $\mathbb{V}_n^\ell$ are $\{\mathcal{L}(i) := \pi S(i) \mid i \in \mathbb{Z}^n\}$, where $S(i)$ is the shift of S, namely $S(i)_j = S_{i+j}$. There is a monomorphism $\mathcal{L}(i) \to \mathcal{L}(j)$

whenever $i \leqslant j$. The 'points' of $\mathbb{V}_n^1$ are the objects $p_s = \pi(S/(X_s))$ for $s = 1, \dots, n$, and their shifts.

The examples $\mathbb{V}_n^1$ play a central role in this paper: we will show that many noncommutative surfaces contain curves which are isomorphic to $\mathbb{V}_n^1$. The results suggest that when the surface is far from the commutative case (for example, it does not have a homogeneous coordinate ring which satisfies a polynomial identity) most curves on the surface are isomorphic to $\mathbb{V}_n^1$ for some n. In particular, even when a noncommutative ring is not graded its module category may contain subcategories which are equivalent to (a quotient category of) the category of graded modules over an algebra graded by an Abelian group of large rank.

Section 8 gives more information about $\mathbb{V}_n^1$, emphasizing its similarity to $\mathbb{P}^1$.

DEFINITION 3.2. Let X be a quasi-scheme. The *cohomological dimension* of an X-module M is $\operatorname{cd} M := \max\{d \mid \operatorname{Ext}_X^d(M, N) \neq 0$ for some $N\}$. The *global dimension* of X is $\operatorname{gldim} X := \max\{\operatorname{cd} M \mid M$ is an X-module$\}$. If X has a structure sheaf $\mathcal{O}_X$, the *cohomological dimension* of X is $\operatorname{cd} X := \operatorname{cd} \mathcal{O}_X$.

DEFINITION 3.3. An *exact dimension function* on X is a function $\partial\colon \mathsf{Mod} X - \{0\} \to \mathbb{Z}$ such that whenever $0 \to L \to M \to N \to 0$ is exact, $\partial M = \max\{\partial L, \partial N\}$. A module M is ∂-*pure* if $\partial L = \partial M$ for all nonzero submodules L of M. An injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$ of an X-module is *pure* if $\partial I^n = \partial M - n$ for $0 \leqslant n \leqslant \partial M$ and $\partial I^n = 0$ for $n > \partial M$.

Krull dimension, Kdim, in the sense of Gabriel and Rentschler is exact and finitely partitive and Gelfand–Kirillov dimension, GKdim, is often exact and finitely partitive.

PROPOSITION 3.4. *Let X be a Noetherian quasi-scheme with structure sheaf $\mathcal{O}_X$. Suppose that*

- *$\mathcal{S} = \{s_j\}$ is a set of auto-equivalences of X such that $\{s_j \mathcal{O}_X\}$ generates* $\mathsf{Mod} X$,
- *the minimal injective resolution of $\mathcal{O}_X$ is pure with respect to an exact dimension function ∂,*
- *∂ is invariant with respect to all s_j,*
- *every indecomposable injective is pure with respect to ∂.*

Suppose that $0 \leqslant \ell \leqslant \operatorname{gldim} X < \infty$. *Let* T *be the full subcategory of* $\mathsf{Mod} X$ *consisting of all modules M such that $\partial M \leqslant \partial \mathcal{O}_X - \ell - 1$. Let $\pi\colon$* $\mathsf{Mod} X \to \mathsf{Mod} X/\mathsf{T}$ *be the quotient functor, and define* $\mathsf{Mod} Z = \mathsf{Mod} X/\mathsf{T}$. *Then Z is a quasi-scheme with* $\operatorname{gldim} Z = \ell$. *Furthermore, if ∂ denotes Krull dimension, then* $\operatorname{Kdim} Z = \ell$.

Remark 3.5. For many rings R, if $\mathsf{Mod} X$ is a suitable quotient category of $\mathsf{Mod} R$ and $\mathcal{O}_X$ is the image of R, then the existence of a pure resolution for $\mathcal{O}_X$ implies that every indecomposable injective is ∂-pure because every indecomposable injective appears in the minimal resolution of $\mathcal{O}_X$ (see [2]).

Proof. Since indecomposable injectives are ∂-pure, π sends injectives to injectives [16, Corollary 5.4, p. 182]. Therefore, applying π to a minimal injective resolution in $\mathsf{Mod}X$ yields an injective resolution in $\mathsf{Mod}X/\mathsf{T}$ of length no more than gldim$X$. Hence gldim$Z \leqslant$ gldim$X$. Since ∂ is s_j-invariant, s_j preserves T, so descends to an auto-equivalence of Z. Since $\mathsf{Mod}X$ is generated by the $s_j\mathcal{O}_X$, $\mathsf{Mod}Z$ is generated by the $s_j\mathcal{O}_Z$, where $\mathcal{O}_Z = \pi\mathcal{O}_X$. Applying $\mathrm{Hom}_Z(M, -)$ to a short exact sequence $0 \to L \to \bigoplus s_j\mathcal{O}_Z \to N \to 0$, we see that

$$\mathrm{cd}\, M = \max\{i \mid \mathrm{Ext}^i_Z(M, s_j\mathcal{O}_Z) \neq 0 \text{ for some } j\}.$$

But the minimal injective resolution of $\mathcal{O}_X$ and, hence, of each $s_j\mathcal{O}_X$, is ∂-pure, so $s_j\mathcal{O}_Z$ has an injective resolution of length ℓ. Thus $\mathrm{cd}\, M \leqslant \ell$. If M is a Noetherian submodule of the last term in the minimal injective resolution of $\mathcal{O}_Z$, then $\mathrm{Ext}^\ell_Z(M, \mathcal{O}_Z) \neq 0$, whence $\mathrm{cd}\, M = \ell$. □

COROLLARY 3.6. *The Krull and global dimensions of $\mathbb{V}^\ell_n$ equal ℓ.*

Proof. Apply Proposition 3.4 with $\mathsf{Mod}X$ being the category of $\mathbb{Z}^n$-graded modules over the polynomial ring $S = k[X_1, \dots, X_n]$. The set of shifts $S(i)$, $i \in \mathbb{Z}^n$, generates $\mathsf{Mod}X$. Krull dimension is invariant with respect to the shifts (i). The minimal injective resolution of S is known to be pure with respect to Krull dimension (this property holds for commutative Gorenstein rings). It follows that $\mathbb{V}^\ell_n$ has Krull dimension and global dimension ℓ. □

Some of the terminology in the statement of the next result is explained in its proof.

PROPOSITION 3.7. *Let L be the hyperplane in $\mathbb{V}^\ell_n$ defined by the equation $X_n = 0$. The open complement to L is isomorphic to $\mathbb{V}^\ell_{n-1}$.*

Proof. Let $S = k[X_1, \dots, X_n]$ be graded as in Example 3.1 by the group $G = \langle \varepsilon_1, \dots, \varepsilon_n \rangle \cong \mathbb{Z}^n$. Let H be the subgroup of G generated by $\varepsilon_1, \dots, \varepsilon_{n-1}$. The inclusion $S \to S[X_n^{-1}]$ induces a functor $\mathsf{GrMod}_G S \to \mathsf{GrMod}_G S[X_n^{-1}]$, and, hence, a functor between the quotient categories which kills the modules of Krull dimension $\leqslant n - \ell - 1$. The quotient category of $\mathsf{GrMod}_G S[X_n^{-1}]$ is the category corresponding to the open complement of L in $\mathbb{V}^\ell_n$.

The proof of the proposition is analogous to the proof for strongly graded rings. That is, if the subalgebra $B = k[X_1, \dots, X_{n-1}]$ of S is given the inherited H-grading, then there is an equivalence of categories $\mathsf{GrMod}_G S[X_n^{-1}] \to \mathsf{GrMod}_H B$ defined by sending a module M to $M_H := \sum_{h \in H} M_h$; the inverse is defined by sending an H-graded B-module N to $B[X_n, X_n^{-1}] \otimes_B N$ with the tensor product grading. □

Under any reasonable definitions, the previous result says that $\mathbb{V}^1_{n-1}$ is an open subscheme of $\mathbb{V}^1_n$.

4. Regularly Embedded Hypersurfaces

DEFINITION 4.1. A *regularly embedded hypersurface* in a quasi-scheme X is a biclosed subscheme, say $i\colon Y \to X$, for which there is an autoequivalence $\sigma\colon \mathsf{Mod}Y \to \mathsf{Mod}Y$ and a natural equivalence of functors $\sigma \circ R^1 i^! \cong i^*$.

If X is commutative, this is compatible with the usual terminology [7, p. 437]; in that case σ is given by tensoring with the conormal bundle (see Theorem 4.3). Because i^* is a left adjoint, it is right exact, whence $R^2 i^! = 0$; this is saying that Y has codimension one in X, and the existence of σ expresses the regularity of the embedding.

If $P \in \mathsf{Mod}Y$, we will often write P^σ rather than $\sigma(P)$; thus $P^{\sigma^{-1}} \cong R^1 i^! P$. If Y is a commutative scheme then σ is an automorphism of Y as a scheme. If X is also commutative, then σ acts as the identity on the points of Y.

The five term exact sequence arising from the spectral sequence in Proposition 2.4 is

$$\begin{aligned} 0 \;\to\; &\operatorname{Ext}^1_Y(P, i^! Q) \to \operatorname{Ext}^1_X(i_* P, Q) \to \operatorname{Hom}_Y(P, R^1 i^! Q) \\ \to\; &\operatorname{Ext}^2_Y(P, i^! Q) \to \operatorname{Ext}^2_X(i_* P, Q). \end{aligned} \tag{4.1}$$

When Y is a regularly embedded hypersurface $R^2 i^! = 0$, so the E_2 page of the spectral sequence in Proposition 2.4 has only two nonzero rows, and the five term sequence continues to a long exact sequence

$$\begin{aligned} \cdots \;\to\; &\operatorname{Ext}^2_X(i_* P, Q) \to \operatorname{Ext}^1_Y(P, R^1 i^! Q) \\ \to \operatorname{Ext}^3_Y(P, i^! Q) \;\to\; &\operatorname{Ext}^3_X(i_* P, Q) \to \operatorname{Ext}^2_Y(P, R^1 i^! Q) \to \cdots. \end{aligned} \tag{4.2}$$

Rees' Lemma [20, Theorem 9.37] can be obtained from the next result by taking $X = \mathsf{Mod}R$ and $Y = \mathsf{Mod}R/(u)$, where u is a normal regular element in R (Theorem 4.3 shows that Y is a regularly embedded hypersurface).

LEMMA 4.2. *Let Y be a regularly embedded hypersurface in X. Let $P \in \mathsf{Mod}Y$, and $Q \in \mathsf{Mod}X$. If $i^! Q = 0$, then $\operatorname{Ext}^m_Y(P^\sigma, i^* Q) \cong \operatorname{Ext}^{m+1}_X(i_* P, Q)$ for all $m \geqslant 0$.*

Proof. Since $i^! Q = 0$, every third term in the long exact sequence (4.2) is zero, thus giving isomorphisms $\operatorname{Ext}^m_Y(P, R^1 i^! Q) \cong \operatorname{Ext}^{m+1}_X(i_* P, Q)$ for all $m \geqslant 0$. Applying the auto-equivalence σ to the Y-modules on the left-hand side gives the result. □

THEOREM 4.3. *Let R be a ring having an invertible ideal $I \neq R$, i.e. I is a two-sided ideal and there is an R-R-bimodule I^{-1} such that $I \otimes_R I^{-1} \cong I^{-1} \otimes_R I \cong R$ as R-R-bimodules. Then $\mathsf{Mod}R/I$ is a regularly embedded hypersurface in $\mathsf{Mod}R$, and the auto-equivalence is given by $\sigma = - \otimes_{R/I} I/I^2$.*

Proof. Let $i_*\colon \mathsf{Mod}R/I \to \mathsf{Mod}R$ be the restriction of scalars functor, and write i^* and $i^!$ for its left and right adjoints. We will show that there is a natural isomorphism $\operatorname{Ext}^1_R(R/I, M) \otimes_{R/I} I/I^2 \cong M \otimes_R R/I$ of right R/I-modules whenever M is a right R-module.

The projective resolution $0 \to I \to R \to R/I \to 0$ is a complex of R-bimodules, so $\operatorname{Ext}^1_R(R/I, M)$ is isomorphic as a right R-module to the cokernel of the natural map $\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(I, M)$ of right R-modules. Since I^{-1} has a bimodule structure, this cokernel is isomorphic to the cokernel of the natural right R-module homomorphism $M \otimes R \to M \otimes_R I^{-1}$. This cokernel is isomorphic to $M \otimes_R (I^{-1}/R)$ as a right R-module. Hence, to prove the result we need to show that $(I^{-1}/R) \otimes_{R/I} (I/I^2) \cong R/I$ as an R-R-bimodule. We can replace $\otimes_{R/I}$ by $\otimes_R$. Since $I/I^2 \cong I \otimes R/I$ as an R-R-bimodule, we must show that $(I^{-1}/R) \otimes_R I \otimes_R R/I$ is isomorphic to R/I as R-R-bimodules.

Because I is projective as a *right* R-module, applying $I \otimes_R -$ to the exact sequence $0 \to R \to I^{-1} \to I^{-1}/R \to 0$ of R-R-bimodules yields an exact sequence $0 \to I \to I^{-1} \otimes_R I \to (I^{-1}/R) \otimes I \to 0$ of R-R-bimodules. But $I^{-1} \otimes_R I \cong R$, so $(I^{-1}/R) \otimes I \cong R/I$. □

We also note that σ is the restriction of the auto-equivalence $\tau = I \otimes_R -$ on X; that is, $\sigma \cong i^*\tau i_* \cong i^!\tau i_*$.

It is a simple consequence of Theorem 4.3 that normal, regular, nonunits in a ring cut out regularly embedded hypersurfaces. Proposition 7.1 gives an explicit description of the auto-equivalence σ in such a situation.

Remark 4.4. If p is a simple X-module which is not in $\mathsf{Mod}Y$, then $i^*p = i^!p = 0$ (to see that $i^*p = 0$ just use the fact that there is an epimorphism $p \to i^*p$).

PROPOSITION 4.5. *Let $i\colon Y \to X$ be a regularly embedded hypersurface. Let p and q be simple Y-modules.*

(1) *If $p = p^\sigma$, then*

$$\operatorname{Ext}^1_X(p, q) \cong \begin{cases} \operatorname{Ext}^1_Y(p, q) & \text{if } q \neq p, \\ \operatorname{Ext}^1_Y(p, p) \oplus \operatorname{Hom}_Y(p, p) & \text{if } q = p \text{ and } \operatorname{Ext}^2_Y(p, p) = 0. \end{cases}$$

(2) *If $p \neq p^\sigma$, then*

$$\operatorname{Ext}^1_X(p, q) = \begin{cases} \operatorname{Ext}^1_Y(p, q) & \text{if } q \notin \{p, p^\sigma\}, \\ \operatorname{Ext}^1_Y(p, q) \oplus \operatorname{Hom}_Y(p, p) & \text{if } q = p^\sigma \text{ and } \operatorname{Ext}^2_Y(p, q) = 0, \\ \operatorname{Ext}^1_Y(p, p) & \text{if } q = p. \end{cases}$$

Proof. Putting $P = p$ and $Q = q$ in (4.1) gives an exact sequence

$$0 \to \operatorname{Ext}^1_Y(p, q) \to \operatorname{Ext}^1_X(p, q) \to \operatorname{Hom}_Y(p, q^{\sigma^{-1}}) \to \operatorname{Ext}^2_Y(p, q).$$

The result now follows from a case-by-case analysis. □

Remark 4.6. If Y is commutative, then $\operatorname{Ext}^i_Y(p, q) = 0$ if $p \neq q$ and $i > 0$.

LEMMA 4.7. *Let $i\colon Y \to X$ be a regularly embedded hypersurface. If $P \in \mathsf{Mod}Y$ is Noetherian, and q is a simple X-module which is not in $\mathsf{Mod}Y$. Then*

(1) $\mathrm{Ext}_X^m(P, q) = 0$ *for all* $m \geqslant 0$;
(2) $\mathrm{Ext}_X^1(q, P) = 0$.

Proof. (1) By Remark 4.4, $i^*q = i^!q = 0$, so $R^1 i^! q = 0$ also. Hence, the result follows from (4.2) with $Q = q$.

(2) Suppose that $0 \to P \to V \to q \to 0$ is an exact sequence in $\mathsf{Mod}X$. The long exact sequence for $R^* i^!$ is, in part,

$$0 = i^! q \to R^1 i^! P \to R^1 i^! V \to R^1 i^! q \to 0.$$

Applying the auto-equivalence σ yields the short exact sequence $0 \to i^*P \to i^*V \to i^*q \to 0$. But $i^*q = 0$ because $q \notin Y$, and $i^*P \cong P$ because P is an $\mathcal{O}_Y$-module, so $i^*V \cong i^*P \cong P$. However, there is an epimorphism $V \to i^*V$, so composing gives an epimorphism $\pi\colon V \to P$. Since $q \notin Y$, q is not a subquotient of P, whence q is in the kernel of π. Composing with the inclusion $P \to V$ gives an epimorphism from P to itself. Since P is Noetherian, this must be an isomorphism; this allows us to split the original sequence. □

Remark 4.8. Suppose that $i\colon Y \to X$ is a regularly embedded hypersurface. Suppose further that $\mathrm{gldim} X = n < \infty$; that is, $\mathrm{Ext}_X^r(M, N) = 0$ if $r \geqslant n + 1$. It follows from (4.2) that if M and N are Y-modules, then $\mathrm{Ext}_Y^i(M, N) \cong \mathrm{Ext}_Y^{i+2}(M, N^\sigma)$ whenever $i \geqslant n$. Therefore either $\mathrm{gldim} Y = \infty$ or $\mathrm{gldim} Y \leqslant n-1$. Suppose further that $\mathsf{Mod}X = \mathsf{GrMod}A$ where A is a connected $\mathbb{N}$-graded k-algebra, and $\mathsf{Mod}Y = \mathsf{GrMod}A/(z)$ where z is a homogeneous, regular, normal element; then $\mathrm{Ext}_Y^i(M, k) \cong \mathrm{Ext}_Y^{i+2}(M, k^\sigma)$, where k denotes the trivial module; hence the minimal projective resolution for $M \in \mathsf{GrMod}A/(z)$ becomes periodic of period two up to the action of σ. This recovers a result of Jörgensen [14] which is an extension of the analogous periodicity result for Noetherian, regular, local, commutative rings [6].

5. Curves on Quasi-Schemes with a Regularly Embedded Hypersurface

Throughout this section we assume that

- X is a k-linear quasi-scheme, where k is a field;
- Y is a regularly embedded hypersurface in X.

For the purposes of this paper, a *k-valued point* $p \in X$, or simply a *point*, is a simple X-module such that $\mathrm{Hom}_X(p, p) \cong k$.

DEFINITION 5.1. A *curve module* on X is a Noetherian module $C \in \mathsf{Mod}X$ with Krull dimension one. We say that C is *irreducible* if it is 1-critical; that is, if all its proper quotients are Artinian. We say C is a *pure curve module* if $i^!C = 0$. A point $p \in Y$ *lies on* C if there is an epimorphism $C \to p$.

A pure curve module C is in *good position* (with respect to Y) if

- i^*C is semisimple, say $i^*C = p_1 \oplus \cdots \oplus p_n$, where p_i are points in Y,
- $\sigma^i(p_r) = \sigma^j(p_s)$ if and only if $i = j$ and $r = s$,
- the simple X-modules which are quotients of C are $p_1, \ldots, p_n$, and
- $\mathrm{Ext}^1_Y(p, q) = 0$ if p and q are distinct elements in $\{\sigma^i(p_r) | i \in \mathbb{Z}, r = 1, \ldots, n\}$.

If C is in good position, we will write $C \cap Y = \{p_1, \ldots, p_n\}$ for the composition factors of i^*C, and $[C \cap Y] = p_1 + \cdots + p_n$ (the notation should make one think of this as a divisor on C).

Remarks 5.2. (1) Because C is Noetherian, and there is an epimorphism $C \to i^*C$, the semi-simplicity hypothesis means that i^*C is a sum of a *finite* number of simple Y-modules.

(2) The hypothesis that i^*C is semisimple is in effect saying that C meets Y *transversally*.

(3) The condition on the distinctness of the various $\sigma^i(p_r)$ can only be satisfied if σ has infinite order. This means that X is far from being commutative; thus the results in this paper do not apply when, for example, $X = \mathsf{Mod} R$ with R a ring which is a finite module over its center. However, even if R is far from commutative, our results may not say anything. For example, the enveloping algebra of a Heisenberg Lie algebra, U say, has a regularly embedded commutative hypersurface, but the induced auto-equivalence σ is the identity, so there are no curves in good position.

(4) In some applications Y might be the scheme parametrizing the point modules over an $\mathbb{N}$-graded k-algebra (see Example 7.5, where σ is related to the functor which shifts the grading).

(5) The condition that all the simple quotients of C be points lying on Y can often be achieved by choosing Y carefully.

(6) If Y is commutative, then the last condition in the definition is automatic.

PROPOSITION 5.3. *Let C and D be pure curve modules, and suppose that $p \in C \cap Y$. If $0 \to D \to C \to p \to 0$ is exact, then D is in good position if and only if C is. In this case $[D \cap Y] = [C \cap Y] - p + p^\sigma$.*

Proof. Let q be a simple X-module which is not in $\mathsf{Mod} Y$. The last term of the exact sequence

$$0 \to \mathrm{Hom}_X(p, q) \to \mathrm{Hom}_X(C, q) \to \mathrm{Hom}_X(D, q) \to \mathrm{Ext}^1_X(p, q)$$

is zero by Lemma 4.7, so $\mathrm{Hom}_X(C, q) \cong \mathrm{Hom}_X(D, q)$. Therefore all the simple quotients of C lie on Y if and only if the same is true of D.

Since $i^!C = 0$, and $R^2 i^! = 0$, the long exact sequence for $R^* i^!$ is, in part,

$$0 \to i^! p \to R^1 i^! D \to R^1 i^! C \to R^1 i^! p \to 0.$$

Since $i^! p = p$, and $i^* p = p$, applying the auto-equivalence σ gives an exact sequence

$$0 \to p^\sigma \to i^*D \to i^*C \to p \to 0$$

of Y-modules.

Suppose C is in good position. Then $q \neq p^\sigma$ for all $q \in C \cap Y$ and $\operatorname{Ext}^1_Y(q, p^\sigma) = 0$. Also i^*C is semisimple, so for all subquotients V of i^*C, $\operatorname{Ext}^1_Y(V, p^\sigma) = 0$; hence i^*D is semisimple, and it follows from the exact sequence that $[D \cap Y] = [C \cap Y] - p + p^\sigma$.

Suppose D is in good position. Then $q \neq p$ for all $q \in D \cap Y$ and $\operatorname{Ext}^1_Y(p, q) = 0$. Also i^*D is semisimple, so for all subquotients V of i^*D, $\operatorname{Ext}^1_Y(p, V) = 0$; hence i^*C is semisimple, and it follows from the exact sequence that $[D \cap Y] = [C \cap Y] - p + p^\sigma$. □

COROLLARY 5.4. *Let C be a pure curve module in good position, and let $p \in C \cap Y$. If $0 \to D \to C \to p \to 0$ is exact, then D is a pure curve module in good position.*

Proof. It is clear that $\operatorname{Kdim} D = 1$ and that $i^!D = 0$, so D is a pure curve module. By Proposition 5.3, D is in good position. □

COROLLARY 5.5. *Let D be a pure curve module in good position with $p^\sigma \in D \cap Y$. Then there is a unique X-module C fitting into a nonsplit exact sequence $0 \to D \to C \to p \to 0$. Moreover, C is a pure curve module in good position.*

Proof. Since D is pure, $i^!D = 0$, so by Rees' Lemma 4.2,

$$\operatorname{Ext}^1_X(p, D) \cong \operatorname{Hom}_Y(p^\sigma, i^*D); \tag{5.1}$$

the right-hand side of this is isomorphic to k by hypothesis, so there is a unique C fitting into such a nonsplit sequence. Obviously $\operatorname{Kdim} C = 1$. We want to show that C is pure. If not, then it follows from the purity of D and the exact sequence $0 \to i^!D \to i^!C \to i^!p = p$ that $i^!C \cong p$; but then the sequence splits, so we conclude that $i^!C = 0$. Thus C is a pure curve module. By Proposition 5.3, C is in good position. □

THEOREM 5.6. *Let C be a pure irreducible curve module in good position with $[C \cap Y] = p_1 + \cdots + p_n$. There is an associated collection of X-modules $C(i)$, indexed by $i \in \mathbb{Z}^n$, with the following properties*:

(1) $C(0) = C$,
(2) *each $C(i)$ is a pure curve module in good position,*
(3) $[C(i) \cap Y] = \sigma^{-i_1}(p_1) + \cdots + \sigma^{-i_n}(p_n)$,
(4) $\operatorname{Hom}_X(C(i), C(j)) = \begin{cases} k & \text{if } i \leqslant j, \\ 0 & \text{otherwise,} \end{cases}$
(5) *there are monomorphisms $\psi^i_j \colon C(i) \to C(j)$ for all $i \leqslant j$ such that $\psi^h_j = \psi^i_j \circ \psi^h_i$ whenever $h \leqslant i \leqslant j$;*
(6) *the images of the morphisms $C(i) \to C(j)$ for $i \leqslant j$ give all submodules of $C(j)$.*

Proof. (1) and (2). First we construct for each $i \leqslant 0$ a submodule $C(i) \subset C$. Define $C(-\varepsilon_s)$ to be the kernel of 'the' nonzero map $C \to p_s$. To see that we may unambiguously define the other $C(i)$ for $|i| \leqslant -2$ by

$$C(i-\varepsilon_s) := C(i)(-\varepsilon_s) = \ker(C(i) \to \sigma^{-i_s}(p_s)),$$

we need only show that there is no ambiguity in constructing $C(-\varepsilon_r - \varepsilon_s)$ when $r \neq s$: that is, we must show that $C(-\varepsilon_r)(-\varepsilon_s) \cong C(-\varepsilon_s)(-\varepsilon_r)$. Since $C(-\varepsilon_r)$ and $C(-\varepsilon_s)$ are distinct maximal submodules of C, $C(-\varepsilon_r) \cap C(-\varepsilon_s)$ is the kernel of the maps $C(-\varepsilon_r) \to p_s$ and $C(-\varepsilon_s) \to p_r$, so

$$C(-\varepsilon_r)(-\varepsilon_s) \cong C(-\varepsilon_s) \cap C(-\varepsilon_r) \cong C(-\varepsilon_s)(-\varepsilon_r).$$

By Corollary 5.4, each $C(i)$ is a pure curve module in good position with $[C(i) \cap Y]$ as claimed.

Now we use Corollary 5.5 to construct for each $i \geqslant 0$ an X-module $C(i)$ containing a copy of C. It will follow from Corollary 5.5 that each $C(i)$ is a pure curve module in good position with $[C(i) \cap Y]$ as claimed. We construct the $C(i)$'s by induction on $|i|$. Define $C(\varepsilon_s)$ to be the unique X-module fitting into a nonsplit exact sequence

$$0 \to C \to C(\varepsilon_s) \to \sigma^{-1}(p_s) \to 0.$$

To see that we may unambiguously define the other $C(i)$ for $|i| \geqslant 2$ by a similar method, we need only show that there is no ambiguity in constructing $C(\varepsilon_r + \varepsilon_s)$ when $r \neq s$: we must show that $C(\varepsilon_r)(\varepsilon_s)$ and $C(\varepsilon_s)(\varepsilon_r)$, each of which is obtained by applying Corollary 5.5 twice, are isomorphic. Set $N = C(\varepsilon_r)(\varepsilon_s)$; there is a nonsplit extension

$$0 \to C(\varepsilon_r) \to N \to \sigma^{-1}(p_s) \to 0.$$

Since C is in good position

$$\operatorname{Ext}^1_X(\sigma^{-1}(p_s), \sigma^{-1}(p_r)) = \operatorname{Ext}^1_Y(\sigma^{-1}(p_s), \sigma^{-1}(p_r)) = 0,$$

whence $N/C \cong \sigma^{-1}(p_r) \oplus \sigma^{-1}(p_s)$. Hence C is contained in a submodule $C' \subset N$ such that $C'/C \cong \sigma^{-1}(p_s)$. Since N has no simple submodules, the sequence

$$0 \to C \to C' \to \sigma^{-1}(p_s) \to 0$$

is nonsplit, whence $C' \cong C(\varepsilon_s)$. Likewise, the sequence

$$0 \to C' \to N \to \sigma^{-1}(p_r) \to 0$$

is nonsplit so $N \cong C'(\varepsilon_r) = C(\varepsilon_s)(\varepsilon_r)$, whence $C(\varepsilon_r)(\varepsilon_s) \cong C(\varepsilon_s)(\varepsilon_r)$ as required. We may now unambiguously define $C(i)$ by induction on $|i|$ for all $i \geqslant 0$.

Now for a general $i \in \mathbb{Z}^n$, we choose $j \geqslant 0$ such that $i + j \geqslant 0$, and define $C(i) := C(i+j)(-j)$. We must show this is independent of the choice of j. If j' also works, let $h = j \cup j'$; it suffices to show that

$$C(i+j)(-j) \cong C(i+h)(-h);$$

using an induction argument, we can reduce to the case $h = j + \varepsilon_s$, so we need only show that $C(i + j)(-j) \cong C(i + j + \varepsilon_s)(-j - \varepsilon_s)$. But the latter is, by definition, isomorphic to $C(i + j)(\varepsilon_s)(-j)(-\varepsilon_s)$, which is isomorphic to $C(i + j)(\varepsilon_s)(-\varepsilon_s)(-j)$ by the earlier arguments. However, it is easy to see from the earlier arguments that $(\varepsilon_s)(-\varepsilon_s)$ does not change the isomorphism class. Thus, $C(i)$ is defined for all $i \in \mathbb{Z}^n$. This completes the proof of (1) and (2).

(3) It is straightforward to use the formula for the divisors in Proposition 5.3 to show that $[C(i) \cap Y]$ is as described.

(6) To see that the nonzero submodules of C are precisely the $C(i)$ for $i \leqslant 0$, suppose that $0 \neq D \subset C$. Since C is irreducible, C/D has finite length. Choose a maximal submodule of C containing D, say D'. Because C is in good position, all the simple quotients of C lie on Y; thus C/D' is one of the $p_s \in C \cap Y$, and $D' = \ker(C \to p_s) = C(-\varepsilon_s)$ for some s. The length of D'/D is less than that of C/D, so an induction argument on length shows that $D \cong C(i)$ for some $i \leqslant 0$. More generally, $C(i)$ is a submodule of $C(j)$ whenever $i \leqslant j$, and these are all the nonzero submodules.

(4) It follows from (3) that $C(i) \ncong C(j)$ if $i \neq j$. Hence there is only one possibility for the image of a nonzero map $C(i) \to C(j)$. Thus, if $\operatorname{Hom}_X(C(i), C(j)) \neq 0$, it is isomorphic to $\operatorname{Hom}_X(C(i), C(i))$. Consider the case $i = 0$, the general case being similar. Because the simple quotients of C are pairwise nonisomorphic, any endomorphism $C \to C$ induces an automorphism of each point $p_s \in C \cap Y$. But $\operatorname{End}_X p_s = k$, by hypothesis. Therefore $\operatorname{Hom}_X(C(i), C(j))$ is either zero or isomorphic to k. It is clear from the construction that there is a monomorphism $C(i) \to C(j)$ if $i \leqslant j$. It follows from (6) and the description of the divisors $[C(i) \cap Y]$ that there is no such monomorphism if $i \nleqslant j$.

(5) Let I be an injective envelope of C in $\mathsf{Mod} X$. If $j \geqslant 0$, then $C(j)$ is an essential extension of C, so $C(j)$ embeds in I; for any $i \in \mathbb{Z}^n$ there exists $j \geqslant 0$ such that $C(i)$ embeds in $C(j)$, so every $C(i)$ embeds in I.

Now we show that I cannot contain two distinct submodules, each isomorphic to the same $C(i)$. Suppose to the contrary that U and V were such submodules. Because C contains no proper direct sums, and is essential in I, I contains no proper direct sums. Therefore $U \cap V \neq 0$, and this is isomorphic to some $C(j)$ with $j < i$. There exists $s \in \{1, \ldots, n\}$ such that $i \geqslant j + \varepsilon_s$, so we may replace U and V by their submodules isomorphic to $C(j + \varepsilon_s)$. In other words, we may assume that $i = j + \varepsilon_s$ for some s. Therefore there is a nonsplit exact sequence

$$0 \to U \to U + V \to V/U \cap V \to 0,$$

with $U \cong V \cong C(i)$, and $V/U \cap V$ isomorphic to a simple quotient of $C(i)$. But

$$\operatorname{Ext}^1_X(V/U \cap V, U) = 0$$

by (5.1), so the sequence splits, contradicting the fact that I has no simple submodules. We conclude that no such U and V can exist.

For each j choose a monomorphism $\theta_j\colon\ C(j) \to I$. By the uniqueness result in the previous paragraph, if $i \leqslant j$ there is a unique $\varphi_j^i\colon\ C(i) \to C(j)$ such that $\theta_j\varphi_j^i = \theta_i$. It follows that $\varphi_j^i\varphi_i^h = \varphi_j^h$ whenever $h \leqslant i \leqslant j$. □

Remark 5.7. If $i, j, k \in \mathbb{Z}^n$ with $i, j \leqslant k$, then there are unique submodules of $C(k)$ isomorphic to $C(i)$ and $C(j)$, so for the moment let us identify $C(i)$ and $C(j)$ with these submodules. Then $C(i) + C(j) = C(i \cup j)$, where

$$i \cup j := (\max\{i_1, j_1\}, \dots, \max\{i_n, j_n\}),$$

and $C(i) \cap C(j) = C(i \cap j)$, where

$$i \cap j := (\min\{i_1, j_1\}, \dots, \min\{i_n, j_n\}).$$

Hence, there is an exact sequence

$$0 \to C(i \cap j) \to C(i) \oplus C(j) \to C(i \cup j) \to 0. \tag{5.2}$$

Notation and terminology. Let C be a pure irreducible curve module in good position, and let n denote the length of i^*C; that is, n is the number of points in $C \cap Y$. We are now going to use the letter C to denote the quasi-scheme

$$\mathsf{Mod}C \;=\; \text{the full subcategory of } \mathsf{Mod}X \text{ consisting of subquotients of direct sums of the modules } C(i),\, i \in \mathbb{Z}^n.$$

We will write $\mathcal{O}_C$ for the module C itself, and write $\mathcal{O}(i) := C(i)$ to emphasize the fact that these modules behave like twists of an ample line bundle. By a *point of C* we mean a point $p \in C(i) \cap Y$ for some i.

COROLLARY 5.8. *Let $C = \mathsf{Mod}C$ be defined as above. Then C is a closed sub-quasischeme of X. If Y is a commutative scheme, then $C \to X$ is not biclosed.*

Furthermore, the points $p \in C$ are biclosed in C if we define $\mathsf{Mod}p$ *to be all direct sums of p in* $\mathsf{Mod}C$ (*equivalently, in* $\mathsf{Mod}X$).

Proof. By Lemma 1.6.4, $\mathsf{Mod}C$ is a closed subcategory of $\mathsf{Mod}X$. By 1.6.5, C is a closed sub-quasischeme.

To show that C is not biclosed in X we must show that the inclusion, say $g_*\colon\ \mathsf{Mod}C \to \mathsf{Mod}X$, does not have a left adjoint. Thus, we must show g_* does not commute with products. By Corollary 6.2 below, C has infinitely many points. If their product in $\mathsf{Mod}X$ were in $\mathsf{Mod}C$, then all submodules of it, in particular $\mathcal{O}_Y$, would also be in $\mathsf{Mod}C$. But every point of Y is a quotient of $\mathcal{O}_Y$, so all these would be in $\mathsf{Mod}C$ also. This contradicts Corollary 6.2. Hence g_* has no left adjoint.

By [27, Proposition 3.4.3.2], $\mathsf{Mod}p$ is closed in $\mathsf{Mod}C$. We will show it is biclosed by proving that $\mathsf{Mod}p$ is closed under products in $\mathsf{Mod}C$. This follows from Theorem 6.9 and an easy computation. □

Remark 5.9. It is easy to see that $\mathsf{Mod}C$ is a closed subcategory of the category $\mathsf{C}(C)$ defined in the introduction. If $\mathsf{Mod}Y$ contains all simple X-modules (which can often be achieved), then $\mathsf{Mod}C = \mathsf{C}(C)$. In general, it is unclear if $\mathsf{Mod}C$ equals $\mathsf{C}(C)$.

6. The Curve C and the Structure of $\mathsf{Mod}C$

In this section we will prove that if C has the properties listed in Theorem 5.6 then $\mathsf{Mod}C$ is equivalent to $\mathbb{V}^1_n$.

The proof will involve showing that C behaves like $\mathbb{P}^1$. As a first hint of this, recall that on $\mathbb{P}^1$ there are exact sequences $0 \to \mathcal{O}(-\ell-m) \to \mathcal{O}(-m)\oplus\mathcal{O}(-\ell) \to \mathcal{O} \to 0$ for all integers $\ell, m > 0$; analogously, since $\mathcal{O}_C$ is the sum of $\mathcal{O}(-m\varepsilon_s)$ and $\mathcal{O}(-\ell\varepsilon_r)$ whenever $r \neq s$, there is an exact sequence $0 \to \mathcal{O}(-\ell\varepsilon_r - m\varepsilon_s) \to \mathcal{O}(-\ell\varepsilon_r) \oplus \mathcal{O}(-m\varepsilon_s) \to \mathcal{O}_C \to 0$. The next result also shows that C behaves like $\mathbb{P}^1$.

PROPOSITION 6.1. *Let C be a pure irreducible curve module in good position. Every Noetherian C-module is a quotient of a finite direct sum of $\mathcal{O}(i)$'s.*

Proof. Since every C-module is a subquotient of a direct sum of $\mathcal{O}(j)$'s, it suffices to show that an arbitrary submodule K of a finite direct sum $N = \bigoplus \mathcal{O}(j)$ is a quotient of a direct sum of various $\mathcal{O}(i)$'s. This is what we will prove.

It is convenient to reindex the terms in the sum, say $N = N_1 \oplus \cdots \oplus N_r$ with each N_t isomorphic to $\mathcal{O}(b_t)$ for some $b_t \in \mathbb{Z}^n$.

Step 1. We can assume that no N_t is contained in K. If K contains N_t, then $K = N_t \oplus K'$, where $K' = K \cap N'$, $N' = N_1 \oplus \cdots \oplus \widehat{N_t} \oplus \cdots \oplus N_r$, and $N/K \cong N'/K'$, so it suffices to prove the result for $K' \subset N'$.

Step 2. It suffices to prove the result when $K \cap N_t \neq 0$ for all t. If $K \cap N_t = 0$, then K is a submodule of N/N_t which is again a sum of various $\mathcal{O}(j)$'s, so we could replace N by N/N_t.

Step 3. It suffices to prove the result when N/K is simple. By Step 2, we may assume that $K \cap N_t \neq 0$ for all t. Since each N_t is 1-critical, N/K has finite length. There is a chain of submodules

$$N = K_0 \supset K_1 \supset \cdots \supset K_\ell = K$$

with each K_s/K_{s-1} simple, so a downwards induction argument shows that K_s is a quotient of a finite direct sum of $\mathcal{O}(i)$'s.

Step 4. The result is true when N/K is simple. Suppose there is an exact sequence

$$0 \to K \to \oplus N_t \xrightarrow{\alpha} p \to 0$$

with p a point on Y, and α the sum of morphisms $\alpha_t\colon N_t \to p$. Write $L_t := \ker\alpha_t$; since $N_t \not\subset K$, L_t is a maximal submodule of N_t. Thus $\alpha_t \neq 0$.

Define $b = b_1 \cup \cdots \cup b_r$, where $b_1, \ldots, b_r \in \mathbb{Z}^n$ are defined by $N_t \cong \mathcal{O}(b_t)$. By Theorem 5.6.4, there is a monomorphism $\varphi_t\colon N_t \to \mathcal{O}(b)$ for each t, and by Remark 5.7, $\mathcal{O}(b) = \sum_t \varphi_t N_t$.

The points on any submodule of $\mathcal{O}(b)$ are in the σ-orbits of the points on $\mathcal{O}(b)$ so, if $[\mathcal{O}(b) \cap Y] = q_1 + \cdots + q_n$, p is in the orbit of some q_t, say q_1 for simplicity. For each $t = 1, \ldots, r$, write $b_t = (b_{t1}, \ldots, b_{tn})$; since the morphisms $\alpha_t\colon N_t \to p$ are nonzero, $b_{11} = \cdots = b_{r1}$, and as $\mathcal{O}(b) = \sum_t \varphi_t N_t$, $\mathcal{O}(b) = \mathcal{O}(b_{11}, \ldots)$ by Remark 5.7. Hence, there is a morphism $\gamma\colon \mathcal{O}(b) \to p$, and we may adjust the φ_t's by suitable scalar multiples so that $\alpha_t = \gamma \circ \varphi_t$ for all p.

Now $L_1 \oplus \cdots \oplus L_r$ is a submodule of K and the quotient has length $r-1$. Each L_t is isomorphic to some $\mathcal{O}(i)$, so our plan for showing that K is a quotient of a direct sum of $\mathcal{O}(i)$'s is to find submodules $Q_2, \ldots, Q_r$ of K, each isomorphic to some $\mathcal{O}(i)$, such that the chain

$$L_1 \oplus \cdots \oplus L_r \subset L_1 \oplus \cdots \oplus L_r + Q_2 \subset L_1 \oplus \cdots \oplus L_r + Q_2 + Q_3 \subset \cdots$$

is strictly increasing, from which it follows that

$$K = L_1 \oplus \cdots \oplus L_r + Q_2 + \cdots + Q_r.$$

We define Q_t to be the kernel of the morphism $\varphi_1 \oplus \varphi_t\colon N_1 \oplus N_t \to \mathcal{O}(b)$. Since φ_t is a monomorphism, Q_t is isomorphic to $\varphi_1 N_1 \cap \varphi_t N_t$, which is a submodule of $\mathcal{O}(b)$, and therefore isomorphic to some $\mathcal{O}(i)$. Suppose, contrariwise, that the chain is not strictly increasing. Thus, for some t,

$$\begin{aligned} Q_t &\subset L_1 \oplus \cdots \oplus L_r + Q_2 + \cdots + Q_{t-1} \\ &\subset N_1 \oplus \cdots \oplus N_{t-1} \oplus L_t \oplus \cdots \oplus L_r, \end{aligned}$$

whence $Q_t \subset N_1 \oplus L_t$. Hence, if π_1 and π_t are the projections of $N_1 \oplus N_t$ onto N_1 and N_t, then $\alpha_t \pi_t(Q_t) = 0$; however,

$$(\alpha_1 \pi_1 \oplus \alpha_t \pi_t)(Q_t) = \gamma(\varphi_1 \pi_1 \oplus \varphi_t \pi_t)(Q_t) = 0,$$

so $\alpha_1 \pi_1(Q_t) = 0$ also, whence $Q_t \subset L_1 \oplus N_t$. Therefore $Q_t \subset L_1 \oplus L_t$. It follows that there is an epimorphism

$$\varphi_1 N_1 + \varphi_t N_t \cong N_1 \oplus N_t / Q_t \to N_1/L_1 \oplus N_t/L_t \cong p \oplus p,$$

contradicting the fact that submodules of $\mathcal{O}(b)$ are curve modules in good position. We conclude that the chain of submodules is strictly increasing, from which it follows that K is a quotient of a direct sum of $\mathcal{O}(i)$'s. $\square$

COROLLARY 6.2. *The simple C-modules, or the points on C, are*

$$\{\sigma^r(p_s) \mid r \in \mathbb{Z}, 1 \leqslant s \leqslant n\}.$$

Proof. By Proposition 6.1, a simple C-module is a quotient of some $\mathcal{O}(i) = C(i)$, so the result follows from Theorem 5.6.3. $\square$

PROPOSITION 6.3. *The Noetherian 1-critical C-modules are the $\mathcal{O}(i)$'s.*

Proof. Let N be a Noetherian 1-critical C-module. By Proposition 6.1, there is a nonzero morphism $\mathcal{O}(i) \to N$ for some i. Since $\mathcal{O}(i)$ and N are 1-critical this is a monomorphism, yielding a short exact sequence $0 \to \mathcal{O}(i) \to N \to T \to 0$ with T of finite length. Now choose a submodule D of N such that D contains $\mathcal{O}(i)$ and $D/\mathcal{O}(i)$ is simple. Then D, being 1-critical also, is a nonsplit extension of $\mathcal{O}(i)$. By the proof of Theorem 5.6, $D \cong \mathcal{O}(i)(\varepsilon_s) \cong \mathcal{O}(i+\varepsilon_s)$ for some s. An induction argument on the length of T shows that $N \cong \mathcal{O}(j)$ for some $j \geqslant i$, with $|j-i|$ equal to the length of T. □

The next result can be interpreted as saying that vector bundles on C can be filtered by line bundles.

COROLLARY 6.4. *If $\mathcal{M}$ is a Noetherian C-module having no nonzero Artinian submodules, then $\mathcal{M}$ has a filtration by submodules, say $\mathcal{M} = \mathcal{M}_t \supset \cdots \supset \mathcal{M}_0 = 0$, such that each $\mathcal{M}_s/\mathcal{M}_{s-1}$ is isomorphic to some $\mathcal{O}(i)$.*

Proof. This is a standard argument. First $\mathcal{M}$ has Krull dimension one because it is a subquotient of a finite direct sum of $\mathcal{O}(j)$'s. Pick a submodule $\mathcal{N}$ which is maximal in $\mathcal{M}$ subject to the condition that $\mathcal{M}/\mathcal{N}$ has no Artinian submodule, except zero. Then $\mathcal{M}/\mathcal{N}$ is 1-critical, so isomorphic to some $\mathcal{O}(i)$ by the previous result. Downwards induction completes the proof; such a descending chain must stop because $\operatorname{Kdim}\mathcal{M} = 1$. □

PROPOSITION 6.5. *If p is a point on C, then*

(1) $\operatorname{Ext}^1_C(p,q) = \begin{cases} k & \textit{if } q = p^\sigma, \\ 0 & \textit{otherwise}; \end{cases}$

(2) $\operatorname{Ext}^1_C(\mathcal{O}(i), p) = 0$ *for all* $i \in \mathbb{Z}^n$.

Proof. (1) Suppose that $0 \to q \to V \to p \to 0$ is a nonsplit extension of C-modules. By Theorem 6.1, there is an epimorphism $\psi\colon \bigoplus \mathcal{O}(j) \to V$. For some j, $\psi(\mathcal{O}(j)) \not\subset q$, whence $\mathcal{O}(j) \to V$ is an epimorphism because V is nonsplit. Thus $V \cong \mathcal{O}(j)/\mathcal{O}(j)(-\varepsilon_r - \varepsilon_s)$ for some r and s. If $r \neq s$ this quotient is semisimple. Thus $r = s$, and it follows that $q = p^\sigma$.

(2) Suppose that $0 \to p \to V \to \mathcal{O}(i) \to 0$ is a nonsplit extension of C-modules. Since $\mathsf{Mod}C$ is a full subcategory this sequence is also nonsplit in $\mathsf{Mod}X$. By Proposition 6.1, there is an epimorphism $\varphi\colon \mathcal{Q} = \bigoplus \mathcal{O}(j) \to V$. If we call the number of summands the *rank* of $\mathcal{Q}$, we may choose $\mathcal{Q}$ of minimal rank.

Choose an arbitrary rank two submodule of $\mathcal{Q}$, say $\mathcal{O}(a) \oplus \mathcal{O}(b)$, and write $W = \varphi(\mathcal{O}(a) \oplus \mathcal{O}(b))$. If $p \cap W = 0$, then W embeds in $\mathcal{O}(i)$, so is isomorphic to some $\mathcal{O}(h)$, so we could replace $\mathcal{O}(a) \oplus \mathcal{O}(b)$ by $\mathcal{O}(h)$, contradicting the minimality of the rank; hence $p \subset W$. Our immediate goal is to show that the sequence $0 \to p \to W \to W/p \to 0$ splits. Since W/p embeds in $\mathcal{O}(i)$ it is isomorphic to some $\mathcal{O}(h)$. To show the sequence splits it suffices to show that W contains a copy of $\mathcal{O}(h)$. If $p \subset \varphi(\mathcal{O}(a))$, then $p = \varphi(\mathcal{O}(a))$ because $\mathcal{O}(a)$ and

$\mathcal{O}(h)$ are 1-critical; therefore $\mathcal{O}(b) \to W \to W/p \cong \mathcal{O}(h)$ is an epimorphism, from which it follows that $\varphi(\mathcal{O}(b)) \cong \mathcal{O}(h)$; hence the sequence splits. So we may assume that $p \cap \varphi(\mathcal{O}(a)) = 0$, and similarly $p \cap \varphi(\mathcal{O}(b)) = 0$. Now the morphism $\mathcal{O}(a) \to W \to W/p \cong \mathcal{O}(h)$ is monic, making $\mathcal{O}(a)$ a submodule of $\mathcal{O}(h)$; similarly, $\mathcal{O}(b)$ is a submodule of $\mathcal{O}(h)$. Since $\mathcal{O}(h)$ is the sum of these two submodules, the epimorphism $\mathcal{O}(a) \oplus \mathcal{O}(b) \to W/p \cong \mathcal{O}(h)$ arises from an exact sequence

$$0 \to \mathcal{O}(a \cap b) \to \mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathcal{O}(a \cup b) = \mathcal{O}(h) \to 0,$$

as in (5.2) above. The kernel of the epimorphism $\mathcal{O}(a) \oplus \mathcal{O}(b) \to W$ is a submodule of $\mathcal{O}(a \cap b)$, so is isomorphic to some $\mathcal{O}(d)$, and $\mathcal{O}(a \cap b)/\mathcal{O}(d) \cong p$. Therefore

$$d = a \cap b - \varepsilon_s$$

for some s. We may, without loss of generality, assume that $b_s \geqslant a_s$, whence

$$d = (a - \varepsilon_s) \cap b;$$

this implies that $(a - \varepsilon_s) \cup b = a \cup b = h$, so there is an exact sequence

$$0 \to \mathcal{O}(d) \to \mathcal{O}(a - \varepsilon_s) \oplus \mathcal{O}(b) \to \mathcal{O}(h) \to 0.$$

But this yields a monomorphism

$$\mathcal{O}(h) \cong \frac{\mathcal{O}(a - \varepsilon_s) \oplus \mathcal{O}(b)}{\mathcal{O}(d)} \longrightarrow \frac{\mathcal{O}(a) \oplus \mathcal{O}(b)}{\mathcal{O}(d)} \cong W,$$

from which it follows that the sequence $0 \to p \to W \to W/p \to 0$ splits. Explicitly, $\varphi(\mathcal{O}(a) \oplus \mathcal{O}(b)) \cong p \oplus \mathcal{O}(a \cup b)$.

The foregoing proves that the sequence $0 \to p \to V \to \mathcal{O}(i) \to 0$ splits if the rank of $\mathcal{Q}$ is two. Of course, the rank of $\mathcal{Q}$ cannot be one. Now suppose that the rank of $\mathcal{Q}$ is at least three, and write $\mathcal{Q} = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \oplus \cdots$. The previous paragraph showed that

$$\varphi(\mathcal{O}(a) \oplus \mathcal{O}(b)) = p \oplus \mathcal{O}(h),$$

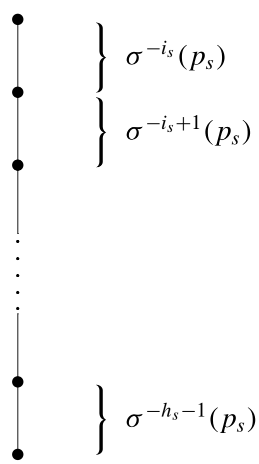
for some h. For some $g \in \mathbb{Z}^n$ there is an epimorphism $\mathcal{O}(g) \to p$. Define $\mathcal{Q}' := \mathcal{O}(g) \oplus \mathcal{O}(h) \oplus \mathcal{O}(c) \oplus \cdots$, and define $\psi\colon \mathcal{Q}' \to V$ by declaring that ψ agrees with φ on $\mathcal{O}(c) \oplus \cdots$, $\psi(\mathcal{O}(g)) = p$, and $\psi(\mathcal{O}(h)) = \mathcal{O}(h)$. Therefore

$$\begin{aligned} \varphi(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) &= p + \mathcal{O}(h) + \varphi(\mathcal{O}(c)) \\ &= p + \psi(\mathcal{O}(h) \oplus \mathcal{O}(c)). \end{aligned}$$

The same argument as before shows that the minimality of the rank implies that $p \subset \psi(\mathcal{O}(h) \oplus \mathcal{O}(c))$. Hence $\psi(\mathcal{O}(h) \oplus \mathcal{O}(c) \oplus \cdots) = V$, contradicting the minimality of the rank. We conclude that the rank of $\mathcal{Q}$ is two, so the sequence $0 \to p \to V \to \mathcal{O}(i) \to 0$ splits, as required. □

The fact that $\mathrm{Ext}^1_C(p, p)$ is zero is a noncommutative phenomenon: such an Ext group is nonzero for all points on a commutative curve. In the commutative case σ is invisible – it is the identity. The fact that $\bigoplus_{q\in C} \mathrm{Ext}^1_C(p, q)$ is one-dimensional is saying that C is smooth. The fact that $\mathrm{Ext}^1_C(\mathcal{O}(i), p) = 0$ is like the fact that $H^1(Z, p) = 0$ for points on a commutative curve Z. One more curve-like feature is apparent at this stage: it is a straightforward consequence of Proposition 6.5 that every Noetherian C-module is isomorphic to a direct sum of a torsion and a torsion-free C-module, if torsion = Artinian. The next result says that locally $\mathsf{Mod}C$ is like the category of modules over a principal ideal domain.

COROLLARY 6.6. *Every finite length C-module is a direct sum of uniserial modules. In particular, if $h \leqslant i \in \mathbb{Z}^n$, then $\mathcal{O}(i)/\mathcal{O}(h)$ is a direct sum of indecomposable uniserial modules, each of which looks like*



provided $i_s > h_s$.

Proof. By Proposition 6.5, if $p \in C$, there is a unique point $q \in C$ such that $\mathrm{Ext}^1_C(p, q) \neq 0$, namely $q = p^\sigma$. Hence any finite length C-module, M say, decomposes as a direct sum $M = M_1 \oplus \cdots \oplus M_n$ where each M_s has all its composition factors belonging to the orbit of p_s. Moreover, because $\dim_k \mathrm{Ext}^1_C(p, p^\sigma) = 1$, each M_s splits as a direct sum of indecomposables which look like the diagram above. Finally, the decomposition of $\mathcal{O}(i)/\mathcal{O}(h)$ follows from Theorem 5.6.3. $\square$

Fix a point $p \in C$. The category of finite length C-modules having composition factors in the orbit of p is equivalent to the category of finite length objects in $\mathsf{GrMod}k[t]$, where $\deg t = 1$. The Auslander–Reiten quiver of this category is $\mathbb{Z}A_\infty$.

Before computing $\mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$ we must describe the injective envelope of $\mathcal{O}(j)$.

PROPOSITION 6.7. *Let E denote the injective envelope of $\mathcal{O}_C$ in $\mathsf{Mod}C$. Then*

(1) *E is the direct limit of the directed system $(\mathcal{O}(i), \psi^i_j)$ described in Theorem 5.6,*
(2) *the universal morphisms ψ_i: $\mathcal{O}(i) \to E$ are monomorphisms,*
(3) *E is the injective envelope in* $\mathsf{Mod}C$ *of all the $\mathcal{O}(i)$,*
(4) *the $\mathcal{O}(i)$'s give all the nonzero Noetherian submodules of E,*
(5) $\mathrm{Hom}_C(E, E) \cong k$.

Proof. Since $\mathcal{O}(i) \to \mathcal{O}(j)$ are essential extensions for all $i \leqslant j$ (see Theorem 5.6.5), E is also an injective envelope of $\mathcal{O}(i)$ for all i.

Let $0 \neq N$ be a Noetherian submodule of E. Then there is an epimorphism φ: $\bigoplus \mathcal{O}(i) \to N$ from a suitable finite direct sum of $\mathcal{O}(i)$'s. We can assume that $\varphi(\mathcal{O}(i)) \neq 0$ for all terms in the sum. Since $\mathcal{O}_C$ is essential in E, E has no nonzero Artinian submodules, whence $\varphi(\mathcal{O}(i)) \cong \mathcal{O}(i)$. Since $\mathcal{O}_C$ is indecomposable, so is E, so $\varphi(\mathcal{O}(i)) \cap \varphi(\mathcal{O}(j)) \neq 0$ for all i, j appearing in the direct sum. It follows that $N/\varphi(\mathcal{O}(i))$ has finite length, whence N is 1-critical. By Proposition 6.3, $N \cong \mathcal{O}(i)$ for some i. By Theorem 5.6.6, E cannot contain two distinct submodules, each isomorphic to the same $\mathcal{O}(i)$. Therefore, the set of all Noetherian submodules of E is $\{\mathcal{O}(i) \mid i \in \mathbb{Z}^n\}$. Thus E is the direct limit of the $\mathcal{O}(i)$'s as described.

It only remains to prove (5). Let θ: $E \to E$ be a nonzero morphism. It is a monomorphism because every proper Noetherian quotient of E is Artinian. Since E contains a unique copy of any of its submodules, θ sends $\mathcal{O}_C \subset E$ to itself. Replacing θ by a scalar multiple of itself, we may assume that its restriction to $\mathcal{O}_C$ is the identity. However, if $i \geqslant 0$, and $\theta \in \mathrm{Hom}_X(\mathcal{O}(i), \mathcal{O}(i))$ restricts to the identity on $\mathcal{O}_C$, then θ is the identity on $\mathcal{O}(i)$. Since E is the direct limit of the $\mathcal{O}(i)$'s, it follows that θ is the identity on E. $\square$

The next result should be compared to the fact that $H^1(\mathbb{P}^1, \mathcal{O}(j))$ is zero if $j \geqslant -1$, and k^{-j-2} otherwise. In Section 8 we give a proof of the next result based on a description of the minimal graded injective resolution of the $\mathbb{Z}^n$-graded polynomial ring $k[x_1, \ldots, x_n]$.

PROPOSITION 6.8. *Let C be a pure irreducible curve in good position. Then*

$$\dim \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) = \max\{0, |\{s \mid i_s \geqslant j_s + 1\}| - 1\}.$$

Proof. Let E denote the injective envelope of $\mathcal{O}_C$ in $\mathsf{Mod}C$. Since $\mathcal{O}(j)$ embeds in E, there is an exact sequence

$$\begin{aligned} \cdots \to\ & \mathrm{Hom}_C(\mathcal{O}(i), E) \to \mathrm{Hom}_C(\mathcal{O}(i), E/\mathcal{O}(j)) \\ \to\ & \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) \to 0. \end{aligned}$$

If $i \leqslant j$, the map $\mathrm{Hom}_C(\mathcal{O}(i), \mathcal{O}(j)) \to \mathrm{Hom}_C(\mathcal{O}(i), E)$ is surjective, so

$$\mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) \cong \mathrm{Hom}_C(\mathcal{O}(i), E/\mathcal{O}(j)),$$

and this is zero because no simple quotient of $\mathcal{O}(i)$ can be a quotient of any $\mathcal{O}(k)$ with $k > j$.

From now on suppose that $i \nleqslant j$. Thus

$$\dim \operatorname{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) = \dim \operatorname{Hom}_C(\mathcal{O}(i), E/\mathcal{O}(j)) - 1.$$

The image of any morphism $\mathcal{O}(i) \to E/\mathcal{O}(j)$ is of the form $N/\mathcal{O}(j)$ where N is a Noetherian submodule of E containing $\mathcal{O}(j)$. Thus, $N \cong \mathcal{O}(k)$ for some $k \geqslant j$. Hence, $\operatorname{Hom}_C(\mathcal{O}(i), E/\mathcal{O}(j))$ arises from pairs $h, k \in \mathbb{Z}^n$ such that $h \leqslant i$ and $j \leqslant k$, and

$$\mathcal{O}(i)/\mathcal{O}(h) \cong \mathcal{O}(k)/\mathcal{O}(j) \neq 0.$$

By Corollary 6.6, $\mathcal{O}(i)/\mathcal{O}(h)$ and $\mathcal{O}(k)/\mathcal{O}(j)$ are direct sums of uniserial modules, and they are isomorphic if and only if the intervals $[h_s + 1, i_s]$ and $[j_s + 1, k_s]$ are equal for all s. Define $h', k' \in \mathbb{Z}^n$ by

$$(h'_s, k'_s) = \begin{cases} (i_s - 1, j_s + 1) & \text{if } i_s < j_s + 1, \\ (j_s, i_s) & \text{if } i_s \geqslant j_s + 1. \end{cases}$$

Then $\mathcal{O}(i)/\mathcal{O}(h') \cong \mathcal{O}(k')/\mathcal{O}(j)$ and, given any other isomorphism $\mathcal{O}(i)/\mathcal{O}(h) \cong \mathcal{O}(k)/\mathcal{O}(j)$, we have $h \geqslant h'$ and $k \leqslant k'$, so there are morphisms

$$\mathcal{O}(i)/\mathcal{O}(h') \to \mathcal{O}(i)/\mathcal{O}(h) \to \mathcal{O}(k)/\mathcal{O}(j) \to \mathcal{O}(k')/\mathcal{O}(j).$$

Now $\mathcal{O}(i)/\mathcal{O}(h')$ is a direct sum of $|\{s \mid i_s \geqslant j_s + 1\}|$ nonzero uniserial modules, and there are no nonzero homomorphisms between any two of these summands, so

$$\begin{aligned} \dim_k \operatorname{Hom}_C(\mathcal{O}(i), E/\mathcal{O}(j)) &= \dim_k \operatorname{Hom}_C(\mathcal{O}(i)/\mathcal{O}(h'), \mathcal{O}(k')/\mathcal{O}(j)) \\ &= |\{s \mid i_s \geqslant j_s + 1\}|. \end{aligned}$$

Subtracting one gives the result. □

It is now possible to prove that $E/\mathcal{O}_C$ is injective, but we prefer to see this as a consequence of the next theorem: the theorem shows that $\operatorname{gldim} C = 1$, which implies that $E/\mathcal{O}_C$ is injective.

THEOREM 6.9. *Let C be a pure irreducible curve in good position, and let n denote the cardinality of $C \cap Y$. Then $C \cong \mathbb{V}^1_n$, via an equivalence of categories sending $\mathcal{O}_C$ to $\mathcal{O}_{\mathbb{V}}$.*

Proof. Let $S = k[X_1, \dots, X_n]$ be the polynomial ring with its $\mathbb{Z}^n$ grading as in Example 3.1. Let $\pi\colon \mathsf{GrMod}S \to \mathsf{GrMod}S/\{\operatorname{Kdim} \leqslant n-2\}$ be the quotient functor. It is standard that π is exact.

Define the functor $F\colon \mathsf{Mod}C \to \mathsf{GrMod}S$ by

$$F = \bigoplus_{i \in \mathbb{Z}^n} \operatorname{Hom}_C(\mathcal{O}(i), -).$$

The action of S on $F(N)$ is defined as follows. The monomorphism $\psi^i_{i+\varepsilon_s}\colon \mathcal{O}(i) \to \mathcal{O}(i + \varepsilon_s)$ induces a map $X_s\colon \operatorname{Hom}_C(\mathcal{O}(i + \varepsilon_s), N) \to \operatorname{Hom}_C(\mathcal{O}(i), N)$; summing over all $i \in \mathbb{Z}^n$ gives an action of X_s on $F(N)$. The commutativity of the maps ψ^i_j ensures that the actions of X_s and X_r on $F(N)$ commute, so $F(N)$ becomes an S-module. The $\mathbb{Z}^n$-grading on FN is defined by $(FN)_{-i} := \mathrm{Hom}_C(\mathcal{O}(i), N)$, thus giving FN the structure of a graded S-module.

We will prove that $\pi \circ F$ is an equivalence of categories. By Proposition 1.3, it suffices to show this is an equivalence between the subcategories of Noetherian objects.

Since F is left exact, so is πF. Unfortunately F is not right exact: if $0 \to U \to V \to W \to 0$ is exact in $\mathsf{Mod} C$, then there is an exact sequence

$$0 \to FU \to FV \to FW \to \bigoplus_i \mathrm{Ext}^1_C(\mathcal{O}(i), U) \to \cdots$$

of graded S-modules. To show that πF is right exact it suffices to show that π sends $\bigoplus_i \mathrm{Ext}^1_C(\mathcal{O}(i), U)$ to zero, and for this it suffices to show it is a direct limit of modules of GK-dimension $\leqslant n-2$. Of course, GK-dimension equals Krull dimension for S-modules. Since U has a finite filtration by submodules with slices isomorphic to various $\mathcal{O}(j)$'s and various points $p \in C$, we need only show that $\bigoplus_i \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$ has GK-dimension $\leqslant n-2$. (By Proposition 6.5, $\mathrm{Ext}^1_C(\mathcal{O}(i), p) = 0$ for all points $p \in C$.)

Proposition 6.8 showed that $\mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) = 0$ if $i_s < j_s$ for all except one s. The action of X_s^m on $\bigoplus_i \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$ is obtained from the natural maps

$$\mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j)) \to \mathrm{Ext}^1_C(\mathcal{O}(i - m\varepsilon_s), \mathcal{O}(j)).$$

Therefore, for large m, $(X_1 X_2 \cdots X_{n-1})^m$ annihilates $\mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$; the same is true for other products $X_1 \cdots \hat{X}_s \cdots X_n$. Thus, every element of

$$\bigoplus_i \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$$

is annihilated by some power of the ideal generated by

$$X_1 X_2 \cdots X_{n-1}, \ldots, X_1 \cdots \hat{X}_s \cdots X_n, \ldots, X_2 X_3 \cdots X_n.$$

But the quotient of the polynomial ring by this ideal has GK-dimension $n-2$, so $\bigoplus \mathrm{Ext}^1_C(\mathcal{O}(i), \mathcal{O}(j))$ has GK-dimension $\leqslant n-2$.

Therefore πF is exact.

To prove πF is faithful it suffices to show that if p is a point on C, then $\mathrm{Kdim} F(p) \geqslant n-1$. If $p = \sigma^m(p_s)$, where $p_s \in C \cap Y$, then $\mathrm{Hom}_C(\mathcal{O}(i), p) \neq 0$ for all i such that $i_s = m$; the set of such i is a sublattice of $\mathbb{Z}^n$ isomorphic to $\mathbb{Z}^{n-1}$, so it follows that $\mathrm{GKdim} F(p) = n-1$. Thus πF is faithful.

Now we show that πF is full. First, we show that $FC \cong S$. It follows from the definition of F that FC is the direct sum of the one-dimensional subspaces $(FC)_i = \mathrm{Hom}_C(\mathcal{O}(-i), C)$ indexed by $i \geqslant 0$, so it suffices to show that FC is generated as an S-module by $(FC)_0$. This is obvious from the way the action of the X_s's is defined. Thus, $\pi F\mathcal{O}_C = \pi FC = \mathcal{O}_{\mathbb{V}}$. It follows that each $\mathcal{O}_{\mathbb{V}}(i)$ is in the image of πF. These are generators for the category of $\mathcal{O}_{\mathbb{V}}$-modules. It is easy to see that $F(\psi^0_{\varepsilon_s}))\colon S \to S(\varepsilon_s)$ coincides with multiplication by X_s. It follows that π sends $\mathrm{Hom}_C(\mathcal{O}(i), \mathcal{O}(j))$ isomorphically to $\mathrm{Hom}_{\mathrm{Gr}S}(S(i), S(j)) \cong \mathrm{Hom}_{\mathbb{V}^1_n}(\mathcal{O}_{\mathbb{V}}(i), \mathcal{O}_{\mathbb{V}}(j))$. It follows from the exactness of πF that it is full. □

A general result on equivalences of categories is proved in Section 9.

7. Applications

In this section we apply our results to some well-known examples.

The first application is to two affine surfaces containing a regularly embedded curve. Specifically, we have a Noetherian domain R with global dimension and Krull dimension two, and a normal, regular, nonunit $u \in R$ such that that R/uR is a commutative ring of Krull dimension one. The first such R is the enveloping algebra of the two-dimensional non-Abelian Lie algebra. The second is the coordinate ring of the quantum affine plane, namely $k_q[x, y]$ with defining relation $yx = qxy$, where $0 \neq q \in k$ is not a root of unity.

PROPOSITION 7.1. *Let R be a ring containing a regular, normal, nonunit u. Then*

(1) $\mathsf{Mod}R/(u)$ *is a regularly embedded hypersurface in* $\mathsf{Mod}R$,
(2) *the auto-equivalence on* $\mathsf{Mod}R/(u)$ *is induced by the auto-equivalence τ on* $\mathsf{Mod}R$ *defined as follows: if M is an R-module, then $\tau M = M$ as an Abelian group, endowed with the new R-action*

$$m * a = m.(uau^{-1}),$$

for $a \in R$ and $m \in M$.

Proof. The hypotheses in Theorem 4.3 hold with $I = uR$. Now $M \otimes_R I$ may be identified with M, via the map $m \otimes ru \leftrightarrow mr$; if the right R-action on $M \otimes_R I$ is transferred to M via this bijection, then we may identify τM with M endowed with the R-action as described. □

EXAMPLE 7.2. Let $U = k[x, y]$ be the enveloping algebra of the two-dimensional solvable Lie algebra, with defining relation $xy - yx = y$. Set $I = yU = Uy$. Define $X = \mathsf{Mod}U$ and $Y = \mathsf{Mod}U/I$. Then Y is a regularly embedded commutative hypersurface in X.

In particular, $Y \cong \mathbb{A}^1$, the points on it being the simple modules $U/(y, x - \lambda)$, $\lambda \in k$. If $p = U/(y, x - \lambda)$, then $p^\sigma = U/(y, x - \lambda + 1)$.

The simplest curves in X arise from the 'line modules' $C = U/(\alpha x+\beta y+\gamma)U$. Now $i^*C = U/(\alpha x + \beta y + \gamma)U + yU$; if $\alpha = 0$, then $i^*C = 0$, or equivalently $[C \cap Y] = \emptyset$, so suppose that $\alpha \neq 0$. Then i^*C is a single point p. By the results in Sections 5 and 6, the curve associated to C is isomorphic to $\mathbb{V}^1_1 \cong \mathsf{GrMod}k[t]$.

More generally, consider a curve module $C = U/aU$. Write $a = yr + g$ where $g \in k[x]$. Then $i^*C \cong k[x]/(g)$, so C is in good position if and only if g has no multiple zeroes and no two of these zeroes differ by an integer. We do not know a simple criterion for C to be 1-critical. If it is, then the associated curve is isomorphic to $\mathbb{V}_n^1$, where $n = \deg g$.

EXAMPLE 7.3. Let $R = k[x, y]$ with defining relation $yx = qxy$, where $0 \neq q \in k$ is not a root of unity. Let $I = (xy)$. Since q is not a root of unity, the only finite-dimensional simple R-modules are the one-dimensional modules over the commutative ring R/I. Let $X = \mathsf{Mod}R$, and $Y = \mathsf{Mod}R/I$. Then Y is a regularly embedded commutative hypersurface in X. (We could also take Y to be $\mathsf{Mod}R/(x)$, or $\mathsf{Mod}R/(y)$.)

Let p be a point on Y. Either x or y vanishes at p. If $p = R/(x, y - \alpha)$ then $p^\sigma = R/(x, y - q^{-1}\alpha)$, and if $p = R/(x - \alpha, y)$ then $p^\sigma = R/(x - q\alpha, y)$. Since q is not a root of unity all points of Y except $(0, 0)$ have infinite σ-orbit.

The simplest curves are the 'line modules' $C = R/(\alpha x + \beta y + \gamma)R$. If $\gamma = 0$, then C is not in good position because C passes through the point $(0, 0) = R/(x, y)$ which is fixed by σ. So suppose that $\gamma \neq 0$. Then C is in good position, and the curve it determines is isomorphic to $\mathbb{V}_2^1$, except if $\alpha\beta = 0$, in which case it is isomorphic to $\mathbb{V}_1^1$. More generally, suppose that $C = R/fR$ is 1-critical. Then $[C \cap Y]$ is corresponds to the points in $\mathsf{Mod}R/(xy, f)$. Let $f_x \in k[x]$ and $f_y \in k[y]$ be such that $R/(x, f) = R/(x, f_y)$ and $R/(y, f) = R/(y, f_x)$. Then C is in good position if both f_x and f_y have distinct zeroes, both have nonzero constant term, and no two zeroes lie in the same σ-orbit. Suppose this happens. Then $[C \cap Y]$ consists of $n = \deg f_x + \deg f_y$ points. The associated curve is isomorphic to $\mathbb{V}_n^1$.

Our next application is to the quantum $\mathbb{P}^2$'s of Artin, Tate, and Van den Bergh [3]. It follows from Proposition 3.4, and is implicit in [3], that these quasi-schemes have global and Krull dimension two, so they are 'smooth surfaces'. Before applying our results we show that the scheme parametrizing the point modules is a regularly embedded curve.

PROPOSITION 7.4. *Let A be a $\mathbb{Z}$-graded algebra, and let z be a homogeneous regular normal nonunit. Then* $\mathsf{Tails}A/(z)$ *is a regularly embedded hypersurface in* $\mathsf{Tails}A$. *Moreover, if z is central of degree d, then σ is naturally equivalent to the degree shift $(-d)$.*

Proof. Set $B = A/(z)$. Let g_*: $\mathsf{GrMod}B \to \mathsf{GrMod}A$ be the inclusion of the full subcategory of graded A-modules annihilated by z. By Theorem 4.3 and Proposition 7.1, $\mathsf{GrMod}B$ is a regularly embedded hypersurface in $\mathsf{GrMod}A$. However, the proof of Proposition 7.1 must be slightly modified since we are working in the graded category. Explicitly, $\nu \circ R^1 g^! \cong g^*$ where ν is the auto-equivalence $-\bigotimes_A Az$. Explicitly, $\nu M = M(-d)$ as a graded Abelian group, and the A-action on νM is defined by $m * a = m(z^{-1}az)$.

Set $X = \mathsf{Tails}A$ and $Y = \mathsf{Tails}B$. There is a commutative diagram

$$\begin{tikzcd} \mathsf{GrMod}B \arrow[r, "g_*"] \arrow[d, "\pi_1"'] & \mathsf{GrMod}A \arrow[d, "\pi_2"] \\ \mathsf{Mod}Y \arrow[r, "i_*"'] & \mathsf{Mod}X \end{tikzcd}$$

where π_1 and π_2 are the quotient functors. The existence of i_* follows from the universal property of the quotient category $\mathsf{GrMod}B/\mathsf{Tors}$.

We refer to Artin and Zhang [5] for details about the functors π_i. Each π_i has a right adjoint, say ω_i, and $\pi_i\omega_i$ is naturally equivalent to the identity functor. By [5],

$$\omega_2\pi_2 M \cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geqslant n}, M),$$

and similarly for $\omega_1\pi_1$.

The auto-equivalence ν sends finite-dimensional modules to finite-dimensional modules so induces an automorphism, τ say, on $\mathsf{Mod}Y$ such that $\tau\pi_1 = \pi_1\nu$. We wish to show that i_* has a left adjoint i^* and a right adjoint $i^!$ such that $\tau \circ R^1 i^! \cong i^*$. Define $i^! := \pi_1 g^! \omega_2$, and $i^* := \pi_1 g^* \omega_2$. Before showing these are adjoints to i_* we make some preliminary calculations.

Since $g^! = \underline{\mathrm{Hom}}_A(B, -)$, we have

$$\begin{aligned} g^!\omega_2\pi_2 M &= \underline{\mathrm{Hom}}_A(B, \varinjlim \underline{\mathrm{Hom}}_A(A_{\geqslant n}, M)) \\ &= \varinjlim \underline{\mathrm{Hom}}_B(B_{\geqslant n}, \underline{\mathrm{Hom}}_A(B, M)) \\ &\cong \omega_1\pi_1 g^! M, \end{aligned}$$

so $g^!\omega_2\pi_2 \cong \omega_1\pi_1 g^!$. It follows that

$$i^!\pi_2 = \pi_1 g^!\omega_2\pi_2 \cong \pi_1\omega_1\pi_1 g^! \cong \pi_1 g^!.$$

If N is a graded B-module, then

$$\begin{aligned} g_*\omega_1\pi_1 N &= g_* \varinjlim \underline{\mathrm{Hom}}_B(B_{\geqslant n}, N) \\ &= \varinjlim \underline{\mathrm{Hom}}_B(B_{\geqslant n}, N) \\ &\cong \varinjlim \underline{\mathrm{Hom}}_A(A_{\geqslant n}, N) \\ &= \varinjlim \underline{\mathrm{Hom}}_A(A_{\geqslant n}, g_* N) \\ &= \omega_2\pi_2 g_* N, \end{aligned}$$

so $g_*\omega_1\pi_1 \cong \omega_2\pi_2 g_*$.

Now we check that $i^!$ and i^* are adjoints to i_* as claimed. Let $\mathcal{M}$ be a X-module, and let $\mathcal{N}$ be a Y-module. Define $M = \omega_2\mathcal{M}$ and $N = \omega_1\mathcal{N}$. Then

$$\begin{aligned}
\operatorname{Hom}_Y(\mathcal{N}, i^!\mathcal{M}) &= \operatorname{Hom}_Y(\pi_1 N, \pi_1 g^! \omega_2 \mathcal{M}) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}B}(N, \omega_1 \pi_1 g^! M) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}B}(N, g^! \omega_2 \pi_2 M) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}A}(g_* N, \omega_2 \pi_2 M) \\
&\cong \operatorname{Hom}_X(\pi_2 g_* N, \pi_2 M) \\
&\cong \operatorname{Hom}_X(i_* \pi_1 N, \pi_2 M) \\
&= \operatorname{Hom}_X(i_* \mathcal{N}, \mathcal{M}),
\end{aligned}$$

showing that $i^!$ is right adjoint to i_*. Also

$$\begin{aligned}
\operatorname{Hom}_Y(i^*\mathcal{M}, \mathcal{N}) &= \operatorname{Hom}_Y(\pi_1 g^* \omega_2 \mathcal{M}, \mathcal{N}) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}B}(g^* \omega_2 \mathcal{M}, \omega_1 \mathcal{N}) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}A}(\omega_2 \mathcal{M}, g_* \omega_1 \pi_1 N) \\
&\cong \operatorname{Hom}_{\mathrm{Gr}A}(\omega_2 \mathcal{M}, \omega_2 \pi_2 g_* N) \\
&\cong \operatorname{Hom}_X(\pi_2 \omega_2 \mathcal{M}, \pi_2 g_* N) \\
&\cong \operatorname{Hom}_X(\mathcal{M}, i_* \pi_1 N) \\
&= \operatorname{Hom}_X(\mathcal{M}, i_* \mathcal{N}),
\end{aligned}$$

showing that i^* is left adjoint to i_*.

The following computation, using the exactness of the π_i's, completes the proof.

$$\begin{aligned}
\tau \circ (R^1 i^!) &\cong \tau \circ (R^1 i^!) \circ \pi_2 \omega_2 \\
&\cong \tau \circ (R^1 (i^! \pi_2)) \circ \omega_2 \\
&\cong \tau \circ (R^1 (\pi_1 g^!)) \circ \omega_2 \\
&\cong \tau \pi_1 \circ (R^1 g^!) \circ \omega_2 \\
&\cong \pi_1 \nu \circ (R^1 g^!) \circ \omega_2 \\
&\cong \pi_1 g^* \omega_2 \\
&= i^*.
\end{aligned}$$

□

EXAMPLE 7.5. Let A be one of the three-dimensional regular algebras in [3], set $X = \mathsf{Tails}A$, and $Y = \mathsf{Tails}A/(g)$ where $g \in A_3$ is the cubic normal element vanishing on the point modules (equivalently, cutting out the point scheme). By [3], Y is a commutative scheme, isomorphic to a cubic divisor, E say, in $\mathbb{P}(A_1^*) \cong \mathbb{P}^2$. Thus our results apply. It follows from the previous result that $p^\sigma = p(-3)$. (Thus σ is *not* the same as the automorphism labelled σ in [4]; our σ is their σ^{-3}.)

Consider a line module M, and let $\mathcal{M}$ be its image in $\mathsf{Tails}A$. Then $\mathcal{M}$ meets Y at 3 points counted with multiplicity, say p_1, p_2, p_3. If there are no equalities among the various $\{p_i(3j) \mid 1 \leqslant i \leqslant 3,\ j \in \mathbb{Z}\}$, then $\mathcal{M}$ is in good position, thus giving a curve isomorphic to $\mathbb{V}_3^1$. This cannot happen if σ has finite order.

Recall that line modules M correspond bijectively to lines ℓ in $\mathbb{P}^2$, the points of πM lying on Y being given by the points of $\ell \cap E$. Thus a general line module will be in good position. Those not in good position are, up to shifting by $i \in \mathbb{Z}^3$, those which fail to meet E at three distinct points.

The next lemma shows that for quasi-schemes having a certain duality, the auto-equivalence σ has another interpretation.

LEMMA 7.6. *Suppose that* $\operatorname{gldim} X = 2$, *that* $i\colon Y \to X$ *is a regularly embedded hypersurface and that* Y *is commutative. Further, suppose that for every point* $p \in Y$ *there is a point* $p' \in Y$ *such that* $\operatorname{Ext}_X^{2-i}(p,-)^* \cong \operatorname{Ext}_X^i(-,p')$ *for* $0 \leqslant i \leqslant d$. *If* $p = p^\sigma$ *or* $p \neq p^{\sigma^2}$, *then* $p' = p^\sigma$. *Otherwise* $p' \in \{p, p^\sigma\}$.

Proof. We have $\operatorname{Ext}_X^1(p,q) \cong \operatorname{Ext}_X^1(q,p')^*$ for all $q \in Y$. Since $0 \neq \operatorname{Ext}_X^1(p',p')^* \cong \operatorname{Ext}_X^1(p,p')$, $p' \in \{p, p^\sigma\}$ by Proposition 4.5. If $p = p^\sigma$ the result follows, so suppose that $p \neq p^\sigma$. Thus $0 \neq \operatorname{Ext}_X^1(p,p^\sigma) \cong \operatorname{Ext}_X^1(p^\sigma,p')^*$, so $p' \in \{p^\sigma, p^{\sigma^2}\}$. Therefore $p' \in \{p,p^\sigma\} \cap \{p^\sigma, p^{\sigma^2}\}$. If $p \neq p^{\sigma^2}$, it follows that $p' = p^\sigma$. □

The lemma does not explain the relationship between p' and p^σ when $p = p^{\sigma^2} \neq p^\sigma$. But this case does not arise for points on a curve in good position.

Lemma 7.6 applies to the three examples just considered, either as an inefficient tool to compute σ, or as an alternative interpretation of σ showing its fundamental nature. For example, if U is the enveloping algebra in Example 7.2, and $X = \mathsf{Mod}U$, then there is a duality $\operatorname{Ext}_X^{2-i}(p,-)^* \cong \operatorname{Ext}_X^i(-,p')$. Since $\operatorname{gldim} U = 2$, the contravariant functor $\operatorname{Ext}_X^2(p,-)^*$ is left exact, so is equivalent to $\operatorname{Hom}_X(-,p')$ where $p' = \operatorname{Ext}_X^2(p,U)^*$. There is a projective resolution

$$0 \to U \to U^2 \to U \to p = U/(y, x-\lambda) \to 0,$$

where the first map is $r \mapsto ((x-\lambda+1)r, -yr)$, and the second is $(a,b) \mapsto ya + (x-\lambda)b$. Thus $\operatorname{Ext}_X^2(p,U) \cong U/(x-\lambda+1, y)$ as a left U-module, so $p' \cong U/(x-\lambda+1,y)$ as a right U-module. In particular, $p' \neq p$, so $p' = p^\sigma$.

EXAMPLE 7.7. Let A denote the four-dimensional Sklyanin algebra, $A = A(E,\tau)$ where E is an elliptic curve and τ is a translation on E. *We assume that* τ *is not of finite order.* Thus the center of A is a polynomial ring $k[\Omega_1, \Omega_2]$ on two degree two elements. We will denote by Ω a nonzero linear combination of Ω_1 and Ω_2. Let $X = \mathsf{Tails}A/(\Omega)$ and $Y = \mathsf{Tails}A/(\Omega_1,\Omega_2)$. Then Y is a regularly embedded hypersurface in X (and $\sigma = (-2)$), and Y is isomorphic to E, a commutative scheme so our results apply. For example, if M is a line module annihilated by Ω, let $\mathcal{M}$ denote the X-module associated to it. If the two point modules which M maps onto correspond to points $p, q \in E$ which are not in the same τ-orbit, then $\mathcal{M}$ is in good position, so the corresponding quasi-scheme C is isomorphic to $\mathbb{V}_2^1$.

Not only is every line module annihilated by a nonzero central element, but so is every graded module of GK-dimension two [22]. Thus, if M is a critical graded module of GK-dimension two, then M is annihilated by a central Ω of degree two (because the annihilator of a critical module is prime). Geometrically, this says that every irreducible curve in the Sklyanin $\mathbb{P}^3$ lies on one of the quadric hypersurfaces containing E. Thus, M yields a curve module lying on X for a suitable choice of

Ω, and if this curve module is in good position with respect to E, it determines a curve isomorphic to some $\mathbb{V}_n^1$. If e is the multiplicity of M, then $M/\Omega_1 M + \Omega_2 M$ is of GK-dimension one and multiplicity $2e$, so its image in $\mathsf{Tails}A/(\Omega)$ has length $2e$; thus $n = 2e$.

There are further applications of our results to other quantum $\mathbb{P}^3$'s, namely those containing a commutative quadric hypersurface as a regularly embedded hypersurface [23–25], and [26].

8. Properties of $\mathbb{V}_n^1$

This section discusses the ways in which the quasi-schemes $\mathbb{V}_n^1$ defined in Example 3.1 are similar to, and differ from, the projective line $\mathbb{P}^1$. Recall, the construction: let $S = k[X_1, \dots, X_n]$ be the commutative polynomial ring with the $\mathbb{Z}^n$ grading defined by $\deg X_s = \varepsilon_s$. Define

$$\mathsf{Mod}\mathbb{V}_n^1 := \mathsf{GrMod}S/\{\mathrm{Kdim} \leqslant n-2\}$$

to be the category of $\mathbb{Z}^n$-graded modules modulo the full subcategory of modules having Krull dimension $\leqslant n-2$. We write $\mathcal{O}$ for the image of S in $\mathsf{Mod}\mathbb{V}_n^1$. By Corollary 3.6, the Krull dimension and global dimension of $\mathbb{V}_n^1$ is one, as for $\mathsf{Mod}\mathbb{P}^1$.

The shifts $\mathcal{O}(i)$, $i \in \mathbb{Z}^n$, are like line bundles on $\mathbb{V}_n^1$. The morphisms between them are similar to the $\mathbb{P}^1$ case: by Theorem 5.6, there is a monomorphism $\mathcal{O}(i) \to \mathcal{O}(j)$ exactly when $i \leqslant j$, and the images give all nonzero submodules of $\mathcal{O}(j)$.

A $\mathbb{V}_n^1$-module is *torsion* if it is a direct limit of Artinian modules. The torsion-free Noetherian $\mathbb{V}_n^1$-modules are analogous to vector bundles. By Corollary 6.4, every vector bundle has a filtration by line bundles, and by Proposition 6.5, every Noetherian module is a direct sum of its torsion submodule and a vector bundle. Over $\mathbb{P}^1$ every vector bundle is a direct sum of line bundles. We show in Proposition 8.2 below that this is true for $\mathbb{V}_n^1$ if n is one or two but, as the next example shows, this is not true if $n \geqslant 3$.

EXAMPLE 8.1. Let $n = 3$. Define $L = Sx_1 + Sx_2 + Sx_3/Sx_3$. Consider the exact sequence

$$0 \to K \to S(-\varepsilon_1) \oplus S(-\varepsilon_2) \xrightarrow{\theta} L \to 0,$$

where θ is the obvious surjection. Applying the quotient functor π produces an exact sequence in the quotient category, and then applying the right adjoint ω to π produces an exact sequence

$$0 \to \omega\pi K \to S(-\varepsilon_1) \oplus S(-\varepsilon_2) \to \omega\pi L$$

in $\mathsf{GrMod}S$. Since $\operatorname{projdim} K = 1$, depth $K = 2$ by the Auslander–Buchsbaum formula. Hence $\omega\pi K \cong K$. If πK, which is a submodule of $\mathcal{O}(-\varepsilon_1) \oplus \mathcal{O}(-\varepsilon_2)$, were isomorphic to a direct sum of shifts of $\mathcal{O}$, then K would be free, so projdim $L = 1$. However, by computing the Hilbert series of L, we see that projdim $L > 1$. Therefore πK is not a direct sum of line bundles.

PROPOSITION 8.2. *If $n \in \{1, 2\}$, then every vector bundle on $\mathbb{V}_n^1$ is a direct sum of line bundles.*

Proof. This is a triviality if $n = 1$ because it just says that every finitely generated torsion-free Noetherian $k[x]$-module is free. So suppose that $n = 2$. Let $\mathcal{M} = \pi M$ be a torsion-free Noetherian module, where M is a finitely generated S-module. We may assume that M has no submodule of Krull dimension $\leqslant 1$. Let $K = \operatorname{Fract} S$. Then the natural map $M \to K \otimes_S M$ is injective, so M embeds in a free S-module. Hence, we have an exact sequence $0 \to \mathcal{M} \to \bigoplus \mathcal{O}(i) \to \mathcal{N} \to 0$ for some collection of i's in $\mathbb{Z}^n$. Applying ω, the right adjoint to π, we get an exact sequence $0 \to M \xrightarrow{\theta} \bigoplus S(i) \to \omega\mathcal{N}$. The cokernel of θ is a submodule of $\mathcal{N}$, so has no finite-dimensional submodule. Hence, by the Auslander–Buchsbaum formula for S, the projective dimension of coker θ is $\leqslant 1$. It follows that projdim $M = 0$, whence M is a sum of shifts of S; hence M is a sum of shifts of $\mathcal{O}$. $\square$

The cohomology groups $H^1(\mathbb{V}_n^1, \mathcal{O}(j)) = \operatorname{Ext}^1_{\mathbb{V}_n^1}(\mathcal{O}, \mathcal{O}(j))$ were computed in Proposition 6.8. Now we give a proof of this using a result which has wider applicability.

LEMMA 8.3. *Let A be a G-graded k-algebra. Make $A[t] = A \otimes_k k[t]$ a $G \times \mathbb{Z}$-graded algebra through the tensor product grading. View $k[t^{-1}]$ as a $k[t]$-module via the vector space isomorphism $k[t^{-1}] \cong k[t, t^{-1}]/tk[t]$.*

If E is an (indecomposable) injective in $\mathsf{GrMod}_G A$, then $E \otimes_k k[t, t^{-1}]$, and $E \otimes_k k[t^{-1}]$ are (indecomposable) injectives in $\mathsf{GrMod} A[t]$. Every (indecomposable) injective in $\mathsf{GrMod} A[t]$ is isomorphic to one of these up to shifting by degree.

Proof. Let I be the injective envelope of $E \otimes k[t, t^{-1}]$ in $\mathsf{GrMod} A[t]$. Since $E \otimes k[t, t^{-1}]$ is t-torsion-free, so is I. Hence, t acts bijectively on I, and $I \cong I_{*,0} \otimes k[t, t^{-1}]$. Since E is injective, the inclusion $E \to I_{*,0}$ splits as an A-module; hence the inclusion $E \otimes k[t, t^{-1}] \to I_{*,0} \otimes k[t, t^{-1}]$ splits as an $A[t]$-module. This implies that $E = I_{*,0}$, and that $I = E \otimes k[t, t^{-1}]$.

Let J denote the injective envelope of $E \otimes k[t^{-1}]$ in $\mathsf{GrMod} A[t]$. Since $(E \otimes k[t^{-1}])_{*,0} = 0$, so too is $J_{*,>0} = 0$. Thus J is t-torsion. We now prove by induction that $J_{*,n} = E \otimes t^n$ for all $n \leqslant 0$. If $E \neq J_{0,*}$, then as A-modules, $J_{0,*} = E \oplus M$; however, t annihilates $J_{*,0}$, so M is an $A[t]$-module, whence $E \otimes k[t^{-1}]$ is not essential in J, a contradiction. Now suppose that $n < 0$, and the result is true for $n+1$. Suppose that M is a nonzero A-submodule of $J_{*,n}$ such that $M \cap Et^n = 0$. By the induction hypothesis, $Mt \subset Et^{n+1}$. Since the map $t\colon Et^n \to Et^{n+1}$ is bijective, there is a homogeneous element $m \in J_{*,n} \backslash Et^n$ such that $mt = 0$. Then mA is an $A[t]$-submodule of J which has zero intersection with $E \otimes k[t^{-1}]$, contradicting the fact that $E \otimes k[t^{-1}]$ is essential in J. Hence $E \otimes k[t^{-1}] = J$.

Now let I be a nonzero indecomposable injective in $\mathsf{GrMod}A[t]$. Notice that $t\colon I \to I$ is surjective.

If I has no t-torsion, then $t\colon I \to I$ is bijective, whence $I \cong I_{*,0} \otimes k[t, t^{-1}]$. If $I_{*,0}$ is not injective in $\mathsf{GrMod}A$, let E be its injective envelope. Then $E \otimes k[t, t^{-1}] = I \oplus I'$. Since I' is t-torsion-free, $I' = I'_{*,0} \otimes k[t, t^{-1}]$, whence $E = I_{*,0} \oplus I'_{*,0}$, a contradiction.

The other case is that I has t-torsion. Choose $0 \neq m \in I_{g,n}$ such that $mt = 0$. Then Am is an $A[t]$ submodule of I. Since I is indecomposable, Am is an essential submodule, whence the submodule $I_{*,>n}$ is zero. The same argument shows that $I_{*,<n}$ is t-torsion-free, whence $t\colon I_{*,<n} \to I = I_{*,\leqslant n}$ is bijective. Thus $I \cong I_{*,n} \otimes k[t^{-1}]$. It is also immediate that $I_{*,n}$ is an idecomposable injective A-module. □

PROPOSITION 8.4. *Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring with the $\mathbb{Z}^n$-grading given by* $\deg x_i = \varepsilon_i$*. Let $e = (e_1, \ldots, e_n)$ be an n-tuple, where each e_j is either ± 1 or -1, and define the graded S-module*

$$M_e = k[x_1^{e_1}] \otimes \cdots \otimes k[x_n^{e_n}],$$

where $k[x_r^{\pm 1}] = k[x_r, x_r^{-1}]$, and $k[x_r^{-1}] = k[x_r, x_r^{-1}]/x_r k[x_r]$. Define $|e|$ to be the number of e_rs equal to -1.

(1) *Every indecomposable injective in* $\mathsf{GrMod}S$ *is isomorphic to a shift of some M_e.*
(2) *The minimal injective resolution of S in* $\mathsf{GrMod}S$ *is*

$$\begin{aligned} 0 \to S \ \to \ & M_{(\pm 1, \pm 1, \ldots, \pm 1)} \to \bigoplus_{|e|=1} M_e \to \cdots \\ & \to \bigoplus_{|e|=n-1} M_e \to M_{(-1,\ldots,-1)} \cong S^* \to 0. \end{aligned} \tag{8.1}$$

Proof. (1) This follows from the lemma by induction on n, the case $n = 0$ reducing to the field k.

(2) By (1), every term in the resolution is injective. Now take the tensor product of the deleted injective resolutions

$$0 \to k[x_r] \to k[x_r, x_r^{-1}] \to k[x_r^{-1}](1) \to 0.$$

□

COROLLARY 8.5. *In* $\mathsf{Mod}\mathbb{V}_n^1$, $\underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{O})^*$ *is isomorphic to the kernel of the map*

$$\bigoplus_{r=1}^{n} S[x_1^{-1}, \ldots, x_{r-1}^{-1}, x_{r+1}^{-1}, \ldots, x_n^{-1}]x_r \to S[x_1^{-1}, \ldots, x_n^{-1}], \tag{8.2}$$

where the individual maps are inclusions. In particular, the dimension of each homogeneous component is

$$\dim_k \underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{O})^*_{(i_1,\ldots,i_n)} = \max\{0, |\{s \mid i_s > 0\}| - 1\}.$$

Proof. Let $\pi\colon \mathsf{GrMod}_{\mathbb{Z}^n} S \to \mathsf{Mod}\mathbb{V}_n^1$ be the quotient functor. Applying π to (8.1) gives the injective resolution of $\mathcal{O}$ in $\mathsf{Mod}\mathbb{V}_n^1$. Since $\operatorname{gldim}\mathbb{V}_n^1 = 1$, we can truncate the resolution after the first step. Applying $\underline{\operatorname{Hom}}_S(S, -)$ to the truncated resolution, we see that

$$\underline{\operatorname{Ext}}^1(\mathcal{O}, \mathcal{O}) = \operatorname{coker}\left(S[x_1^{-1}, \ldots, x_n^{-1}] \to \bigoplus_{|e|=1} M_e \right).$$

Thus $\underline{\operatorname{Ext}}^1(\mathcal{O}, \mathcal{O})^*$ is the kernel of the $\underline{\operatorname{Hom}}_k(-, k)$ dual of this. This is precisely the statement of the corollary. The dimensions of the homogeneous components may be computed by observing that the components of $S[x_1^{-1}, \ldots, x_n^{-1}]$ are all one-dimensional. Hence, the dimension of the degree i component of the kernel is one less than the dimension of the degree i component of the left hand term in (8.2). Details are left to the reader. □

We now consider Serre duality. For $\mathbb{P}^1$ there are functorial isomorphisms

$$H^i(\mathbb{P}^1, \mathcal{F})^* \cong \operatorname{Ext}^{1-i}(\mathcal{F}, \omega^\circ),$$

where $\omega^\circ \cong \mathcal{O}(-2)$. Recall that $\mathcal{O}(-2)$ is the image under π of $\underline{\operatorname{Ext}}^1(\mathcal{O}, \mathcal{O})^* \cong A(-2)$. For all $\mathbb{V}_n^1$, we set $\omega^\circ = \pi\underline{\operatorname{Ext}}^1(\mathcal{O}, \mathcal{O})^*$. This module is described in the previous corollary; for $n = 2$, it is isomorphic to $A(-1, -1)$, so $\omega^\circ \cong \mathcal{O}(-1, -1)$, and the same sort of proof as in [13] or [28] will show there is a functorial isomorphism $H^i(\mathbb{V}_2^1, \mathcal{F})^* \cong \operatorname{Ext}^{1-i}(\mathcal{F}, \omega^\circ)$.

Now suppose that $n \geqslant 3$. The first thing to observe is that $\underline{\operatorname{Ext}}^1(\mathcal{O}, \mathcal{O})$ is not a finitely generated S-module because it is not concentrated in degree $\geqslant j$ for any $j \in \mathbb{Z}^n$. Nevertheless, the arguments in [13] and [28] show that $H^1(\mathbb{V}_n^1, \mathcal{F})^* \cong \operatorname{Hom}(\mathcal{F}, \omega^\circ)$.

The following result on the K-theory of $\mathbb{V}_n^1$ was prompted by questions of Idun Reiten. We thank her for her interest.

PROPOSITION 8.6. *Let* $K_0(\mathbb{V}_n^1)$ *denote the Grothendieck group of* $\mathsf{mod}\mathbb{V}_n^1$, *and let* Pic *denote the Picard group as defined in* [15]. *Then*

(1) $K_0(\mathbb{V}_n^1) \cong \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]/(q)$ *where* $q = (1 - T_1) \cdots (1 - T_n)$, *and*
(2) $\operatorname{Pic}(\mathbb{V}_n^1) = \mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{Z}^n$ *where* $\mathfrak{m} = (1 - T_1, \ldots, 1 - T_n)$.

Proof. The Hilbert series of $k[X_1, \ldots, X_n]$ with its $\mathbb{Z}^n$-grading is $\prod_{i=1}^n (1 - T_i)^{-1}$, so the argument in [15, Theorem 2.4] gives $K_0(\mathbb{V}_n^1)$. The proof of (2) is straightforward because the Krull dimension of an S-module can be read off from its Hilbert series (see [15] where this idea is used). □

9. An Equivalence of Categories

The main result in this section can be used to give an alternative proof of Theorem 6.9. The result is analogous to [5, Theorem 4.5(1)]. That result concerns a k-linear

Abelian category C, an object $\mathcal{O} \in \mathsf{C}$, and an auto-equivalence σ of C; it is supposed, among other things, that $\{s^n\mathcal{O} \mid n < 0\}$ is a set of generators for C. Section 6 shows the need for a version of this result which assumes somewhat less: in Section 6 we have a set of generators $\{\mathcal{O}(i) \mid i \in \mathbb{Z}^n\}$ and we must prove an equivalence of categories result similar to [5, Theorem 4.5(1)] before we can conclude that each (i) extends to an auto-equivalence of $\mathsf{Mod}C$.

We consider the following situation. Let C be a k-linear Noetherian category, with a generating set $\mathsf{O} = \{\mathcal{O}(g) \mid g \in G\}$ of Noetherian objects indexed by a group G. Let

$$\mathsf{Sum}\,\mathsf{O} = \text{the full subcategory of } \mathsf{C} \text{ consisting of finite direct sums of the } \mathcal{O}(g)\text{s}.$$

A *shift* on O is a set $S = \{s_g \mid g \in G\}$ of k-linear automorphisms of $\mathsf{Sum}\,\mathsf{O}$ satisfying

(i) $s_e = \mathrm{id}$ where e is the identity in G;
(ii) $s_g s_h = s_{hg}$ for all $g, h \in G$;
(iii) $s_g(\mathcal{O}(h)) = \mathcal{O}(hg)$ for all $g, h \in G$.

Given such data, we may associate to each $\mathcal{M} \in \mathsf{C}$ a graded k-space

$$\Gamma(\mathcal{M}) := \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{M}) := \bigoplus_{g\in G} \mathrm{Hom}(\mathcal{O}(g^{-1}), \mathcal{M}).$$

We may make

$$A := \Gamma(\mathcal{O}) = \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O}) = \bigoplus_{g\in G} \mathrm{Hom}(\mathcal{O}(g^{-1}), \mathcal{O});$$

a G-graded k-algebra with multiplication

$$x \cdot y = x s_{g^{-1}}(y),$$

where $x \in \mathrm{Hom}(\mathcal{O}(g^{-1}), \mathcal{O})$ and $y \in \mathrm{Hom}(\mathcal{O}(h^{-1}), \mathcal{O})$. We may define a right graded A-module action on $\Gamma(\mathcal{M})$ by $m \cdot y = m s_{g^{-1}}(y)$, where $m \in \mathrm{Hom}(\mathcal{O}(g^{-1}), \mathcal{M})$ and $y \in \mathrm{Hom}(\mathcal{O}(h^{-1}), \mathcal{O})$. A morphism $\mathcal{M} \to \mathcal{N}$ gives a morphism of G-graded A-modules $\Gamma(\mathcal{M}) \to \Gamma(\mathcal{N})$. Thus, we have a functor $\Gamma = \underline{\mathrm{Hom}}(\mathcal{O}, -)\colon \mathsf{C} \to \mathsf{GrMod}A$. Our goal is a generalization of [5, Theorem 4.5(1)] which shows that Γ induces an equivalence between C and an appropriate quotient category of $\mathsf{GrMod}A$.

Aiming for generality, but still following the ideas in [5], we let B be a graded subalgebra of A. The restriction functor $\mathsf{GrMod}A \to \mathsf{GrMod}B$ is exact, so its composition with Γ gives a functor $F\colon \mathsf{C} \to \mathsf{GrMod}B$. Since F is built from Hom-functors it is left exact. We denote the right derived functors of F by $\underline{\mathrm{Ext}}^i(\mathcal{O}, -)$; for $\mathcal{M} \in \mathsf{C}$, we may view $\underline{\mathrm{Ext}}^i(\mathcal{O}, \mathcal{M})$ as a graded right A-module or B-module. We can compute these right derived functors via injective resolutions because C

consists of the Noetherian objects in the locally Noetherian Grothendieck category $\overline{C} = \mathsf{Lex}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}k)$ (Proposition 1.3).

Recall that a dense subcategory $\mathsf{T} \subset \mathsf{GrMod}B$ is *localizing* if the quotient functor $\pi\colon \mathsf{GrMod}B \to \mathsf{GrMod}B/\mathsf{T}$ has a right adjoint; we denote the right adjoint by ω, and note that $\pi\omega \cong \mathrm{id}$. Given such a T, objects in it are called *torsion*, and objects having no nonzero torsion subobjects are said to be *torsion-free*.

We denote by $\mathsf{grmod}B$ the full subcategory of $\mathsf{GrMod}B$ consisting of the Noetherian modules.

THEOREM 9.1. *Let* C *be a Noetherian k-linear category with a generating set* $\mathsf{O} = \{\mathcal{O}(g) | g \in G\}$ *and shifts* $S = \{s_g\}$ *on* O. *Let* $A = \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O})$ *and let B be a G-graded subalgebra of A. Suppose that* $\mathsf{T} \subset \mathsf{GrMod}B$ *is a localizing subcategory such that*

(1) *every $A(g)/B(g)$ is torsion*;
(2) *each* $\underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{M})$ *is torsion*;
(3) *if* $0 \neq \mathcal{M} \in \mathsf{C}$, *then* $\underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{M})$ *is not torsion*;
(4) *the torsion submodule of B/XB is Noetherian for every finite set of homogeneous elements $X \subset B$.*

Then B is right Noetherian, and there is an equivalence of categories

$$\mathsf{C} \cong \mathsf{grmod}B/\mathsf{T} \cap \mathsf{grmod}B.$$

Proof. Our proof is follows that in [5, Theorem 4.5(1)].

Let $\pi\colon \mathsf{GrMod}B \to \mathsf{GrMod}B/\mathsf{T}$ be the quotient functor. It is exact. We want to prove that $\pi F\colon \mathsf{C} \to \mathsf{grmod}B/\mathsf{T} \cap \mathsf{grmod}B$ is an equivalence.

Step 1. πF is exact.

Since Γ is defined in terms of $\mathrm{Hom}(\mathcal{O}(g), -)$, it is left exact. If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{L} \to 0$ is exact, there is an exact sequence

$$0 \to F\mathcal{M} \to F\mathcal{N} \to F\mathcal{L} \to \underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{M}). \tag{9.1}$$

By hypothesis $\underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{M})$ is torsion, so πF is exact.

Step 2. πF is faithful.

Since πF is exact, we must show that $\pi F(\mathcal{M}) \neq 0$ if

$\mathcal{M} \neq 0$. This follows from the hypothesis that $\underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{M})$ is not torsion if $\mathcal{M} \neq 0$.

Step 3. If $\mathcal{M} \subset \mathcal{N}$, then $F\mathcal{N}/F\mathcal{M}$ is torsionfree. In particular, $F\mathcal{N}$ is torsion-free.

By (9.1), $F\mathcal{N}/F\mathcal{M}$ is a submodule of $F\mathcal{L}$, so it suffices to show $F\mathcal{L}$ is torsion-free. Suppose $x \in \mathrm{Hom}(\mathcal{O}(g), \mathcal{L}) \subset F\mathcal{L}$ is such that xB is torsion. Since xA/xB is a homomorphic image of a torsion module $A(g)/B(g)$, xA is torsion. The image of $F(x)\colon F(\mathcal{O}(g)) \to F(\mathcal{L})$ is xA, which is torsion, so $\pi F(x) = 0$. The map $x\colon \mathcal{O}(g) \to \mathcal{L}$ factors as an epimorphism followed by a monomorphism, say

$\mathcal{O}(g) \to \mathcal{I} \to \mathcal{L}$. By Step 1, $\pi F(x)$ is a composition of an epimorphism followed by a monomorphism $\pi F(\mathcal{O}(g)) \to \pi F \mathcal{I} \to \pi F \mathcal{L}$. Therefore $\pi F(\mathcal{I}) = 0$ so, by Step 2, $\mathcal{I} = 0$, whence $x = 0$.

Step 4. B is right Noetherian as a G-graded B-module.

Let N be a G-graded right ideal of B. A homogeneous element $x \in N_g$ is map $\mathcal{O}(g) \to \mathcal{O}$. For a finite subset $X \subset N$, there is a map from $\mathcal{P}_X := \bigoplus_{x \in X} \mathcal{O}(g_x)$ to $\mathcal{O}$; write $\mathcal{N}_X$ for its image. The image of $F\mathcal{P}_X \to F\mathcal{O}$ is XA which is a submodule of $F(\mathcal{N}_X)$. Since $\mathcal{O}$ is Noetherian, there is a finite set X such that $\mathcal{N}_X$ is maximal among such subobjects. Set $\mathcal{N} = \mathcal{N}_X$ and $\mathcal{P} = \mathcal{P}_X$.

Applying F to the factorization $\mathcal{P} \to \mathcal{N} \to \mathcal{O}$ where the first map is epic and the second monic, we see that $F\mathcal{N}/XA$ is torsion. Since $\mathcal{N}$ is maximal, $N \subset F\mathcal{N}$. But XA/XB is torsion, hence $N/XB \subset F\mathcal{N}/XB$ is torsion. By hypothesis, $N/XB \subset B/XB$ is Noetherian, whence N is finitely generated.

Step 5. For each $\mathcal{M}$ there is an epimorphism (x_i): $\bigoplus \mathcal{O}(g) \to \mathcal{M}$, such that $F\mathcal{M}/(\sum_i x_i B)$ is torsion.

There is an epimorphism (x_i) because O is a set of generators. Because πF is exact, applying F to this epimorphism gives a map $\bigoplus A(g) \to F\mathcal{M}$ which is an epimorphism up to torsion; that is, $F\mathcal{M}/\sum x_i A$ is torsion. But each $A(g)/B(g)$ is torsion, so $F\mathcal{M}/\sum x_i B$ is torsion.

Now we define a functor $\mathsf{C} \to \mathsf{grmod} B/\mathsf{T} \cap \mathsf{grmod} B$ by sending $\mathcal{M}$ to $\pi F(\mathcal{M}) = \pi(\sum_i x_i B)$, where the x_i are as in Step 5. This functor is well-defined, and πF is the composition of it with the inclusion $\mathsf{grmod} B/\mathsf{T} \cap \mathsf{grmod} B \to \mathsf{GrMod} B/\mathsf{T}$.

Step 6. For every finitely generated G-graded right B-module M, there is $\mathcal{M} \in \mathsf{C}$ such that $\pi F(\mathcal{M}) \cong \pi M$.

Choose finite sums P_1, P_0 of shifts of B and an exact sequence

$$P_1 \to P_0 \to M \to 0.$$

Let $\mathcal{P}_i$ be the sum of shifts of $\mathcal{O}$ corresponding to P_i. Since the map $P_1 \to P_0$ is determined by elements in B, which are also elements in A, there is a corresponding map $\mathcal{P}_1 \to \mathcal{P}_0$. By definition, $\pi F(\mathcal{P}_i) \cong \pi(P_i)$ for $i = 0, 1$. Let $\mathcal{M}$ be the cokernel of the map $\mathcal{P}_1 \to \mathcal{P}_0$. Then the sequence $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0$ yields an exact sequence $\pi P_1 \to \pi P_0 \to \pi F(\mathcal{M}) \to 0$. Therefore $\pi F(\mathcal{M}) \cong \pi M$.

Step 7. Let $M = F\mathcal{M}$ and let $M \to M'$ be an injective B-module homomorphism. If M' is torsionfree, then M'/M is torsionfree.

Let $x \in M'_g$ be such that $xB + M/M$ is torsion. We can view x as a map $B(g) \to M'$. Let $L = \{b \in B \mid xb \in M\}$. Then $(B/L)(g)$ is torsion. Applying π to the exact sequence $P_1 \to P_0 \to L(g) \to 0$ where P_i are finite sums of shifts of B, we obtain an exact sequence

$$\pi P_1 \to \pi P_0 \to \pi B(g) \to 0 \tag{9.2}$$

in $\mathsf{GrMod} B/\mathsf{T}$, because $\pi(L(g)) = \pi B(g)$. Let (f_{ij}) and (g_i) denote the maps from $P_1 \to P_0$ and $P_0 \to L(g) \subset B(g)$ respectively. Then $\sum_i g_i f_{ij} = 0$ for all j. We can lift (9.2) to an exact sequence $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{O}(g) \to 0$ because πF is full on the objects and morphisms in (9.2); we denote the lifted maps by (f_{ij}) and (g_i) also.

Let $h\colon P_0 \to M$ be the composition of (g_i) and x. Then we can lift the sequence $P_1 \to P_0 \to M$ to $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M}$. Hence the map $\mathcal{P}_0 \to \mathcal{M}$ factors through $(g_i)\colon \mathcal{P}_0 \to \mathcal{O}(g)$. Now we have a sequence $F\mathcal{P}_1 \to F\mathcal{P}_0 \to M$ and $P_i \subset F\mathcal{P}_i$ for $i = 0, 1$. Therefore h factors through $(g_i)\colon P_0 \to B(g)$; thus there is a map $y\colon B \to M$ such that $(g_i)x = h = (g_i)y$, and it follows that $(g_i)(x - y) = 0$. Since the image of (g_i) is $L(g)$ and $B(g)/L(g)$ is torsion, the image of $x - y$ is torsion. But M' is torsionfree, so $x = y$ and $xB = yB \subset M$. Therefore M'/M is torsionfree.

Step 8. πF is an equivalence $\mathsf{C} \to \mathsf{grmod}B/\mathsf{T} \cap \mathsf{grmod}B$.

By Steps 1, 2, and 6, it suffices to show that the functor is full on morphisms. Since T is localizing, π has a right adjoint $\omega\colon \mathsf{GrMod}B/\mathsf{T} \to \mathsf{GrMod}B$ and $\pi\omega \cong$ id. Since O is a generating set, it suffices to show that

$$\pi F\colon \operatorname{Hom}_{\mathsf{C}}(\mathcal{O}(g), \mathcal{M}) \to \operatorname{Hom}_{\mathsf{GrMod}B/\mathsf{T}}(\pi F(\mathcal{O}(g)), \pi F(\mathcal{M}))$$

is an isomorphism for all $\mathcal{M} \in \mathsf{C}$ and all $g \in G$. If we set $M = F\mathcal{M}$, then

$$M = \bigoplus_{g\in G} \operatorname{Hom}_{\mathsf{C}}(\mathcal{O}(g), \mathcal{M}).$$

However, by Step 7, $\omega\pi M \cong M$, so

$$\begin{aligned} \bigoplus_{g\in G} \operatorname{Hom}_{\mathsf{C}}(\mathcal{O}(g), \mathcal{M}) &\cong \omega\pi M \\ &\cong \bigoplus_{g\in G} \operatorname{Hom}_{\mathsf{GrMod}B}(B(g), \omega\pi M) \\ &\cong \bigoplus_{g\in G} \operatorname{Hom}_{\mathsf{GrMod}B/\mathsf{T}}(\pi(B(g)), \pi M). \\ &= \bigoplus_{g\in G} \operatorname{Hom}_{\mathsf{GrMod}B/\mathsf{T}}(\pi F(\mathcal{O}(g)), \pi F(\mathcal{M})) \end{aligned}$$

as required. □

The next result shows that the conditions on T are unavoidable if one seeks a result of this type. A localizing category is *stable* if it is closed under injective envelopes in the ambient category.

PROPOSITION 9.2. *Let* T *be a stable localizing subcategory of* $\mathsf{GrMod}A$. *Then for every nonzero object* $\mathcal{M}$ *in* $\mathsf{GrMod}A/\mathsf{T}$, $\underline{\operatorname{Hom}}(\mathcal{A}, \mathcal{M})$ *is nonzero and torsionfree and* $\underline{\operatorname{Ext}}^i(\mathcal{A}, \mathcal{M})$ *is torsion for all* $i > 0$.

Proof. Compare the minimal injective resolutions of $\mathcal{M}$ and $\omega\mathcal{M}$. □

In the situation of Theorem 9.1, there is at most one localizing subcategory T with properties (1)–(4). Moreover, by Step 3 in the proof $\underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{M})$ is torsionfree for all $0 \neq \mathcal{M} \in \mathsf{C}$. If, in the situation of Theorem 9.1, T is sent to itself by the degree shift functors (g) on $\mathsf{GrMod}B$, then the shifts s_g on O can be extended to C.

Often, as in the earlier part of this paper, the subcategory T may be defined by using a dimension function. Let ∂ be an exact dimension function on $\mathsf{GrMod}B$. Suppose that

(a) $\partial(F\mathcal{M}) > d$ for all $0 \neq \mathcal{M} \in \mathsf{C}$, and
(b) $\partial(\underline{\mathrm{Ext}}^1(\mathcal{O}, \mathcal{M})) \leqslant d$ for all $\mathcal{M}$ in C,
(c) $\partial(A(g)/B(g)) \leqslant d$.

Then the full subcategory of $\mathsf{GrMod}B$ consisting of modules M such that $\partial M \leqslant d$ is a localizing subcategory satisfying conditions (2) and (3) in Theorem 9.1.

We record some useful consequences of Theorem 9.1.

COROLLARY 9.3. *Let* C *be a Noetherian k-linear category with a generating set* $O = \{\mathcal{O}(g) \mid g \in G\}$, *and shifts* $S = \{s_g\}$ *on* $\mathsf{Sum}\,\mathrm{O}$. *Let* $A = \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O})$. *If each* $\mathcal{O}(g)$ *is projective, then A is right Noetherian, and* $\mathsf{C} \cong \mathsf{grmod}A$.

Proof. By hypothesis, $\mathrm{Ext}^1(\mathcal{O}(g), -) = 0$, so T is zero. Now take $B = A$, and apply the theorem. In this case the shifts s_g can be extended from O to the whole category C. □

COROLLARY 9.4. *Let B be a graded Noetherian algebra. Let* $\mathsf{T} \subset \mathsf{GrMod}B$ *be a stable localizing category, and set* $\mathsf{C} = \mathsf{grmod}B/\mathsf{T} \cap \mathsf{grmod}B$. *Write* $\mathcal{B}(g)$ *for the image of* $B(g)$ *in* C. *Suppose there is a shift on* $\mathsf{Sum}\{\mathcal{B}(g) \mid g \in G\}$. *If each* $\mathcal{B}(g)$ *is projective, then* $\mathsf{grmod}B/\mathsf{T} \cap \mathsf{grmod}B \cong \mathsf{grmod}A$, *where* $A = \underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{B})$.

If $A = B$ is right Noetherian, we then have the following.

COROLLARY 9.5. *Let* C *be a Noetherian k-linear category with a generating set* $\mathrm{O} = \{\mathcal{O}(g) \mid g \in G\}$, *and shifts* $S = \{s_g\}$ *of* $\mathsf{Sum}\,\mathrm{O}$. *Let* $A = \underline{\mathrm{Hom}}(\mathcal{O}, \mathcal{O})$. *Suppose that* $\mathsf{T} \subset \mathsf{GrMod}A$ *is a stable localizing subcategory satisfying conditions* (2)–(3) *of Theorem* 9.1. *If A is right Noetherian, then* $\mathsf{C} \cong \mathsf{grmod}A/\mathsf{T} \cap \mathsf{grmod}A$.

EXAMPLE 9.6. Let R be a right Noetherian connected graded ring of global dimension $n < \infty$. Let s be the degree shift functor on $\mathsf{GrMod}R$. Let $\mathcal{O} = \bigoplus_{i=0}^{l-1} s^i R$ and $\mathcal{O}(i) = s^{li}(\mathcal{O})$. Then $B = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(\mathcal{O}(-i), \mathcal{O})$ is right Noetherian of global dimension n because B is locally finite and $\mathsf{GrMod}B \cong \mathsf{GrMod}R$.

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