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Journal of Algebra 358 (2012) 74–86

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

“Degenerate” 3-dimensional Sklyanin algebras are monomial algebras

S. Paul Smith

Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195, United States

A R T I C L E   I N F O

Article history:
Received 11 January 2012
Available online xxxx
Communicated by Michel Van den Bergh

MSC:
16S38

Keywords:
Monomial algebras
Sklyanin algebras

A B S T R A C T

The 3-dimensional Sklyanin algebras, $S_{a,b,c}$, form a flat family parametrized by points $(a,b,c) \in \mathbb{P}^2 - \mathfrak{D}$ where $\mathfrak{D}$ is a set of 12 points. When $(a,b,c) \in \mathfrak{D}$, the algebras having the same defining relations as the 3-dimensional Sklyanin algebras are called “degenerate Sklyanin algebras”. C. Walton showed they do not have the same properties as the non-degenerate ones. Here we prove that a degenerate Sklyanin algebra is isomorphic to the free algebra on u, v, and w, modulo either the relations $u^2 = v^2 = w^2 = 0$ or the relations $uv = vw = wu = 0$. These monomial algebras are Zhang twists of each other. Therefore all degenerate Sklyanin algebras have the same category of graded modules. A number of properties of the degenerate Sklyanin algebras follow from this observation. We exhibit a quiver Q and an ultramatricial algebra R such that if S is a degenerate Sklyanin algebra then the categories $\mathsf{QGr}\, S$, $\mathsf{QGr}\, kQ$, and $\mathsf{Mod}\, R$, are equivalent. Here $\mathsf{QGr}(-)$ denotes the category of graded right modules modulo the full subcategory of graded modules that are the sum of their finite-dimensional submodules. The group of cube roots of unity, μ_3, acts as automorphisms of the free algebra on two variables, F, in such a way that $\mathsf{QGr}\, S$ is equivalent to $\mathsf{QGr}(F \rtimes \mu_3)$.

1. Introduction

Let k be a field having a primitive cube root of unity ω.

1.1. Let $\mathfrak{D}$ be the subset of the projective plane $\mathbb{P}^2_k$ consisting of the 12 points:

E-mail address: smith@math.washington.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2012.01.039

$$\mathfrak{D} := \{(1,0,0),(0,1,0),(0,0,1)\} \sqcup \{(a,b,c) \mid a^3 = b^3 = c^3\}.$$

The points $(a,b,c) \in \mathbb{P}^2 - \mathfrak{D}$ parametrize the 3-dimensional Sklyanin algebras,

$$S_{a,b,c} = \frac{k\langle x,y,z\rangle}{(f_1,f_2,f_3)},$$

where

$$f_1 = ayz + bzy + cx^2,$$
$$f_2 = azx + bxz + cy^2,$$
$$f_3 = axy + byx + cz^2.$$

In the late 1980s, Artin, Tate, and Van den Bergh [1] and [2], showed that $S_{a,b,c}$ behaves much like the commutative polynomial ring on 3 indeterminates. In many ways it is more interesting because its fine structure is governed by an elliptic curve endowed with a translation automorphism.

1.2. When $(a,b,c) \in \mathfrak{D}$ we continue to write $S_{a,b,c}$ for the algebra with the same generators and relations and call it a degenerate 3-dimensional Sklyanin algebra. This paper concerns their structure.

Walton [8] has shown that the degenerate Sklyanin algebras are nothing like the others: if $(a,b,c) \in \mathfrak{D}$, then $S_{a,b,c}$ has infinite global dimension, is not noetherian, has exponential growth, and has zero divisors, none of which happens when $(a,b,c) \notin \mathfrak{D}$.

Unexplained terminology in this paper can be found in Walton's paper [8].

1.3. Results

Our results show that the degenerate Sklyanin algebras are rather well-behaved, albeit on their own terms.

Theorem 1.1. *Let $S = S_{a,b,c}$ be a degenerate Sklyanin algebra.*

(1) *If $a = b$, then S is isomorphic to $k\langle u,v,w\rangle$ modulo $u^2 = v^2 = w^2 = 0$.*
(2) *If $a \neq b$, then S is isomorphic to $k\langle u,v,w\rangle$ modulo $uv = vw = wu = 0$.*

Corollary 1.2. *If (a,b,c) and (a',b',c') belong to $\mathfrak{D}$, then $S_{a,b,c}$ and $S_{a',b',c'}$ are Zhang twists of one another, and their categories of graded right modules are equivalent,*

$$\mathsf{Gr}\, S_{a,b,c} \equiv \mathsf{Gr}\, S_{a',b',c'}.$$

Of more interest to us is a quotient category, $\mathsf{QGr}\, S$, of the category of graded modules. This has surprisingly good properties. We now state the result and then define $\mathsf{QGr}\, S$.

Theorem 1.3. *Let $S = S_{a,b,c}$ be a degenerate Sklyanin algebra. There is a quiver Q and an ultramatricial algebra R, both independent of $(a,b,c) \in \mathfrak{D}$, such that*

$$\mathsf{QGr}\, S_{a,b,c} \equiv \mathsf{QGr}\, kQ \equiv \mathsf{Mod}\, R$$

where the algebra R is a direct limit of algebras R_n, each of which is a product of three matrix algebras over k, and hence a von Neumann regular ring.

Furthermore, there is an action of μ_3*, the cube roots of unity in k, as automorphisms of the free algebra* $F = k\langle X, Y\rangle$ *such that*

$$\mathsf{QGr}\, S_{a,b,c} \equiv \mathsf{QGr}(F \rtimes \mu_3).$$

1.4. We work with *right modules* throughout this paper.

Let A be a connected graded k-algebra. If $\mathsf{Gr}\, A$ denotes the category of $\mathbb{Z}$-graded right A-modules and $\mathsf{Fdim}\, A$ is the full subcategory of modules that are direct limits of their finite-dimensional submodules, then the quotient category

$$\mathsf{QGr}\, A := \frac{\mathsf{Gr}\, A}{\mathsf{Fdim}\, A}$$

plays the role of the quasi-coherent sheaves on a "non-commutative scheme" that we call $Proj_{nc}\, A$.

Artin, Tate, and Van den Bergh, showed that if $(a, b, c) \in \mathbb{P}^2 - \mathfrak{D}$, then $\mathsf{QGr}\, S$ has "all" the properties enjoyed by $\mathsf{Qcoh}\, \mathbb{P}^2$, the category of quasi-coherent sheaves on the projective plane [1,2].

1.5. Consequences

The ring R in Theorem 1.3 is von Neumann regular because each R_n is. The global dimension of R is therefore equal to 1.

Let $S = S_{a,b,c}$ be a degenerate Sklyanin algebra. Finitely presented monomial algebras are coherent so S is coherent. The full subcategory $\mathsf{gr}\, S$ of $\mathsf{Gr}\, S$ consisting of the finitely presented graded modules is therefore abelian. We write $\mathsf{fdim}\, S$ for the full subcategory $\mathsf{Gr}\, S$ consisting of the finite-dimensional modules; we have $\mathsf{fdim}\, S = (\mathsf{gr}\, S) \cap (\mathsf{Fdim}\, S)$. Since fdim is a Serre subcategory of $\mathsf{gr}\, S$ we may form the quotient category

$$\mathsf{qgr}\, S := \frac{\mathsf{gr}\, S}{\mathsf{fdim}\, S}.$$

The equivalence $\mathsf{QGr}\, S \equiv \mathsf{Mod}\, R$ restricts to an equivalence

$$\mathsf{qgr}\, S \equiv \mathsf{mod}\, R$$

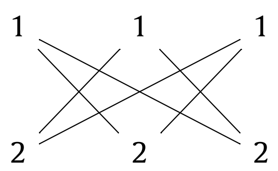
where $\mathsf{mod}\, R$ consists of the finitely presented R-modules.

Modules over von Neumann regular rings are flat so finitely presented R-modules are projective, whence the next result.

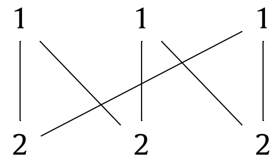
Corollary 1.4. *Every object in* $\mathsf{qgr}\, S$ *is projective in* $\mathsf{QGr}\, S$ *and every short exact sequence in* $\mathsf{qgr}\, S$ *splits.*

The corollary illustrates, again, that the degenerate Sklyanin algebras are unlike the non-degenerate ones but still "nice".

1.6. Since R is only determined up to Morita equivalence there are different Bratteli diagrams corresponding to different algebras in the Morita equivalence class of R. There are, however, two "simplest" ones, namely the stationary Bratteli diagrams that begin



and



These diagrams correspond to the quivers Q and Q' that appear in Section 3.

1.7. *The Grothendieck group* $K_0(\operatorname{qgr} S)$

Let S be a degenerate Sklyanin algebra and $K_0(\operatorname{qgr} S)$ the Grothendieck group of finitely generated projectives in $\operatorname{qgr} S$.

We write $\mathcal{O}$ for S as an object in $\operatorname{qgr} S$. The left ideal of A generated by $u-v$ and $v-w$ is free and $A/A(u-v)+A/(v-w)$ is spanned by the images of 1 and u, so $\mathcal{O} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We show that

$$K_0(\operatorname{qgr} S) = \mathbb{Z}\left[\frac{1}{8}\right] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

with $[\mathcal{O}] = (1, 0, 0)$.

1.8. *The word "degenerate"*

The algebras $S_{a,b,c}$ form a flat family on $\mathbb{P}^2 - \mathfrak{D}$. The family does not extend to a flat family on $\mathbb{P}^2$ because the Hilbert series for a degenerate Sklyanin algebra is different from the Hilbert series of the non-degenerate ones. This is the reason for the quotation marks around "degenerate". It is not unreasonable to think that there might be another compactification of $\mathbb{P}^2 - \mathfrak{D}$ that parametrizes a flat family of algebras that are the Sklyanin algebras on $\mathbb{P}^2 - \mathfrak{D}$. The obvious candidate to consider is $\mathbb{P}^2$ blown up at $\mathfrak{D}$.

2. The degenerate Sklyanin algebras are monomial algebras

Let

$$A = \frac{k\langle u, v, w\rangle}{(u^2, v^2, w^2)} \quad \text{and} \quad A' = \frac{k\langle u, v, w\rangle}{(uv, vw, wu)}. \tag{2.1}$$

The next result uses the blanket hypothesis that the base field k contains a primitive cube root of unity which implies that $\operatorname{char} k \neq 3$.

Theorem 2.1. *Suppose* $(a, b, c) \in \mathfrak{D}$. *Then*

$$S_{a,b,c} \cong \begin{cases} A & \text{if } a = b, \\ A' & \text{if } a \neq b. \end{cases}$$

Proof. The result is a triviality when $(a, b, c) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ so we assume that $a^3 = b^3 = c^3$ for the rest of the proof.

If λ is a non-zero scalar, then $S_{a,b,c} \cong S_{\lambda a,\lambda b,\lambda c}$ so it suffices to prove the theorem when

(1) $(a, b, c) = (1, 1, 1)$,
(2) $(a, b, c) = (1, 1, c)$ with $c^3 = 1$ but $c \neq 1$, and

(3) $a \neq b$ and $abc \neq 0$.

We consider the three cases separately.

(1) Suppose $(a, b, c) = (1, 1, 1)$. Let ω be a primitive cube root of unity. Then

$$(x + y + z)^2 = f_1 + f_2 + f_3,$$
$$\left(x + \omega y + \omega^2 z\right)^2 = f_1 + \omega^2 f_2 + \omega f_3,$$
$$\left(x + \omega^2 y + \omega z\right)^2 = f_1 + \omega f_2 + \omega^2 f_3.$$

Since $\{(1, 1, 1), (1, \omega, \omega^2), (1, \omega^2, \omega)\}$ is linearly independent,

$$\operatorname{span}\left\{x + y + z, x + \omega y + \omega^2 z, x + \omega^2 y + \omega z\right\} = kx + ky + kz.$$

Therefore $S_{1,1,1} \cong A$.

(2) Suppose $c \neq 1$ and $(a, b, c) = (1, 1, c)$. Then

$$\left(x + y + c^{-1} z\right)^2 = c^{-1} f_1 + c^{-1} f_2 + f_3,$$
$$\left(x + c^{-1} y + z\right)^2 = c^{-1} f_1 + f_2 + c^{-1} f_3,$$
$$\left(c^{-1} x + y + z\right)^2 = f_1 + c^{-1} f_2 + c^{-1} f_3.$$

Since

$$\det\begin{pmatrix} 1 & 1 & c^{-1} \\ 1 & c^{-1} & 1 \\ c^{-1} & 1 & 1 \end{pmatrix} = \left(c^{-1} - 1\right)^2\left(c^{-1} + 2\right) \neq 0$$

$\{x + y + c^{-1} z,\ x + c^{-1} y + z,\ c^{-1} x + y + z\}$ is linearly independent. Hence $S_{1,1,c} \cong A$.

(3) Suppose $a \neq b$. Let

$$u = a^{-1} x + b^{-1} y + c^{-1} z,$$
$$v = b^{-1} x + a^{-1} y + c^{-1} z,$$
$$w = abc(x + y) + z.$$

Because $\operatorname{char} k \neq 3$, the hypothesis that $a \neq b$, implies $\{u, v, w\}$ is linearly independent. Furthermore,

$$uv = (abc)^{-1}(f_1 + f_2) + f_3,$$
$$vw = af_1 + bf_2 + cf_3,$$
$$wu = bf_1 + af_2 + cf_3.$$

Hence $S_{a,b,c} \cong A'$. □

Proposition 2.2. *The algebras A and A' are Zhang twists of each other.*

Proof. The defining relations of A' are $uv = vw = wu = 0$ so the map

$$\tau(u) = v, \qquad \tau(v) = w, \qquad \tau(w) = u$$

extends to an algebra automorphism of A'. The Zhang twist of A' by τ is therefore the algebra generated by u, v, w with defining relations

$$u * u = u\tau(u) = uv = 0, \qquad v * v = v\tau(v) = vw = 0, \qquad w * w = w\tau(w) = wu = 0.$$

The Zhang twist of A' by τ is therefore isomorphic to A.

The defining relations of A are $u^2 = v^2 = w^2 = 0$ so the map

$$\theta(u) = w, \qquad \theta(v) = u, \qquad \theta(w) = v$$

extends to an algebra automorphism of A. The Zhang twist of A by θ is therefore the algebra generated by u, v, w with defining relations

$$u * v = u\theta(v) = u^2 = 0, \qquad v * w = v\theta(w) = v^2 = 0, \qquad w * u = w\theta(u) = w^2 = 0.$$

The Zhang twist of A by θ is therefore isomorphic to A'. $\square$

Zhang [10] proved that if a connected graded algebra generated in degree one is a Zhang twist of another one, then their graded module categories are equivalent. This, and Proposition 2.2, implies the next result.

Corollary 2.3. *Let S be a degenerate Sklyanin algebra. There are category equivalences*

$$\mathsf{Gr}\, S \equiv \mathsf{Gr}\, A \equiv \mathsf{Gr}\, A'$$

and

$$\mathsf{QGr}\, S \equiv \mathsf{QGr}\, A \equiv \mathsf{QGr}\, A'.$$

In particular, $\mathsf{Gr}\, S_{a,b,c}$*, and hence* $\mathsf{QGr}\, S_{a,b,c}$*, is the same for all* $(a, b, c) \in \mathfrak{D}$.

Walton [8] determined the Hilbert series of S by exhibiting S as a free module of rank 2 over a free subalgebra. Here is an alternative derivation of the Hilbert series using the description of S as a monomial algebra.

Corollary 2.4. *Let $(a, b, c) \in \mathfrak{D}$. The Hilbert series of $S_{a,b,c}$ is $(1 + t)(1 - 2t)^{-1}$.*

Proof. Let S be a degenerate Sklyanin algebra. Since A is a Zhang twist of A' the Hilbert series of S is the same as that of

$$\frac{k\langle u, v, w\rangle}{(u^2, v^2, w^2)} \cong \frac{k[u]}{(u^2)} * \frac{k[v]}{(v^2)} * \frac{k[w]}{(w^2)}$$

where the right-hand side of this isomorphism is the free product of three copies of $k[\varepsilon]$, the ring of dual numbers. Therefore

$$H_S(t)^{-1} = 3H_{k[\varepsilon]}(t)^{-1} - 2.$$

Since $H_{k[\varepsilon]}(t) = 1 + t$ it follows that $H_S(t)$ is as claimed. □

Here is another simple way to compute $H_S(t)$. A basis for $k\langle u, v, w\rangle/(u^2, v^2, w^2)$ is given by all words in u, v, and w, that do not have uu, vv, or ww, as a subword. The number of such words of length $n \geqslant 1$ is easily seen to be 3.2^{n-1}. Since $3.2^{n-1} = 2^n + 2^{n-1}$ the Hilbert series of this algebra is $(1+t)(1-2t)^{-1}$. The same argument applies to the algebra with relations $uv = vw = wu = 0$.

3. Two quivers

3.1. The main result in [4] is the following.

Theorem 3.1. *If A is a finitely presented monomial algebra there is a finite quiver Q and a homomorphism $f : A \to kQ$ such that $- \bigotimes_A kQ$ induces an equivalence*

$$\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ.$$

One can take Q to be the Ufnarovskii graph of A.

A k-algebra is said to be matricial if it is a finite product of matrix algebras over k. A k-algebra is ultramatricial if it is a direct limit, equivalently a union, of matricial k-algebras.

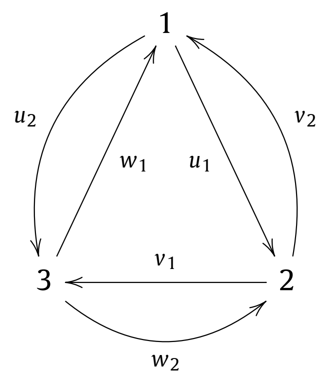
Theorem 3.2. *Let $(a, b, c) \in \mathfrak{D}$ and let $S_{a,b,c}$ be the associated degenerate Sklyanin algebra. There is a quiver Q and an ultramatricial algebra, R, both independent of (a, b, c), for which there is an equivalence of categories*

$$\mathsf{QGr}\, S_{a,b,c} \equiv \mathsf{QGr}\, kQ \equiv \mathsf{Mod}\, R.$$

Proof. Since $S_{a,b,c}$ is a monomial algebra Theorem 3.1 implies that $\mathsf{QGr}\, S_{a,b,c} \equiv \mathsf{QGr}\, kQ$ for some quiver Q. By [6, Thm. 1.2], for every quiver Q there is an ultramatricial k-algebra $R(Q)$ such that $\mathsf{QGr}\, kQ \equiv \mathsf{Mod}\, R(Q)$. Because $\mathsf{QGr}\, S_{a,b,c}$ is the same for all $(a, b, c) \in \mathfrak{D}$, Q and $R(Q)$ can be taken the same for all such (a, b, c). □

3.2. The Ufnarovskii graph for $k\langle u, v, w\rangle/(u^2, v^2, w^2)$ is the quiver

$Q :=$



(3.1)

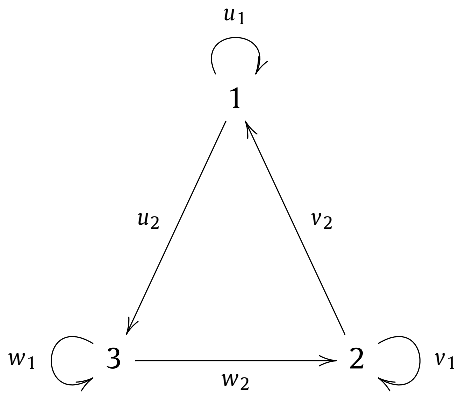
The equivalence $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, kQ$ is induced by the k-algebra homomorphism $f : A \to kQ$ defined by

$$f(u) = u_1 + u_2,$$

$$f(v) = v_1 + v_2,$$

$$f(w) = w_1 + w_2.$$

3.3. The Ufnarovskii graph for $k\langle u, v, w\rangle/(uv, vw, wu)$ is the quiver

$Q' :=$ 

(3.2)

The equivalence $\mathsf{QGr}\, A' \equiv \mathsf{QGr}\, kQ'$ is induced by the k-algebra homomorphism $f' : A' \to kQ'$ defined by

$$f'(u) = u_1 + u_2,$$

$$f'(v) = v_1 + v_2,$$

$$f'(w) = w_1 + w_2.$$

3.4. Direct proof that $\mathsf{QGr}\, kQ$ *is equivalent to* $\mathsf{QGr}\, kQ'$

Since $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, A'$, $\mathsf{QGr}\, kQ$ is equivalent to $\mathsf{QGr}\, kQ'$. This equivalence is not obvious but follows directly from [6, Thm. 1.8] as we will now explain.

Given a quiver Q its n-Veronese is the quiver $Q^{(n)}$ that has the same vertices as Q but the arrows in $Q^{(n)}$ are the paths in Q of length n. By [6, Thm. 1.8], $\mathsf{QGr}\, kQ \equiv \mathsf{QGr}\, kQ^{(n)}$, this being a consequence of the fact that $kQ^{(n)}$ is isomorphic to the n-Veronese subalgebra $(kQ)^{(n)}$ of kQ.

Hence, if the 3-Veronese quivers of Q and Q' are the same, then $\mathsf{QGr}\, kQ$ is equivalent to $\mathsf{QGr}\, kQ'$.

The incidence matrix of the nth Veronese quiver is the nth power of the incidence matrix of the original quiver. The incidence matrices of Q and Q' are

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

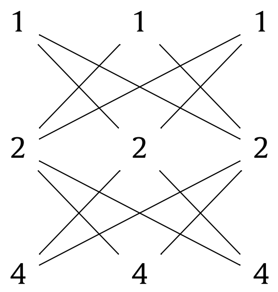
and the third power of each is

$$\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

so $\mathsf{QGr}\, kQ \equiv \mathsf{QGr}\, kQ'$.

3.5. The ultramatricial algebra R in Theorem 3.2

The ring R in Theorem 3.2 is only determined up to Morita equivalence but we can take it to be the algebra that is associated to Q in [6, Thm. 1.2]. That algebra has a stationary Bratteli diagram that begins



(3.3)

and so on. More explicitly, $R = \varinjlim R_n$ where

$$R_n = M_{2^n}(k) \oplus M_{2^n}(k) \oplus M_{2^n}(k)$$

and the map $R_n \to R_{n+1}$ is given by

$$(r, s, t) \mapsto \left(\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \right).$$

Proposition 3.3. *The ring R in Theorem* 3.2 *is left and right coherent, non-noetherian, simple, and von Neumann regular.*

Proof. It is well known that ultramatricial algebras are von Neumann regular and left and right coherent.

The simplicity of R can be read off from the shape of its Bratteli diagram. Let x be a non-zero element of R. There is an integer n such that $x \in R_n$ so $x = (x_1, x_2, x_3)$ where each x_i belongs to one of the matrix factors of R_n. Some x_i is non-zero so, as can be seen from the Bratteli diagram, the image of x in R_{n+2} has a non-zero component in each matrix factor of R_{n+2}. The ideal of R_{n+2} generated by the image of x is therefore R_{n+2}. Hence $RxR = R$.

The ring R is not noetherian because its identity element can be written as a sum of arbitrarily many mutually orthogonal idempotents. □

Because R is ultramatricial it is unit regular [3, Ch. 4] hence directly finite [3, Ch. 5], and it satisfies the comparability axiom [3, Ch. 8].

3.6. Q and Q' are McKay quivers

Section 2 of [7] describes how to associate a McKay quiver to the action of a finite abelian group acting semisimply on a k-algebra. Both Q and Q' can be obtained in this way.

Lemma 3.4. *Let $F = k\langle X, Y\rangle$ be the free algebra with* $\deg X = \deg Y = 1$. *Let μ_3 be the group of* 3*rd roots of unity in k.*

(1) *If $\alpha : \mu_3 \to \mathrm{Aut}_{\mathrm{gr.alg}} F$ is the homomorphism such that $\xi \bullet X = \xi X$ and $\xi \bullet Y = \xi^2 Y$, then $kQ \cong F \rtimes_\alpha \mu_3$.*
(2) *If $\beta : \mu_3 \to \mathrm{Aut}_{\mathrm{gr.alg}} F$ is the homomorphism such that $\xi \bullet X = \xi X$ and $\xi \bullet Y = Y$, then $kQ' \cong F \rtimes_\beta \mu_3$.*

Proof. This is a special case of [7, Prop. 2.1].

Even without Lemma 3.4 one can see that kQ is Morita equivalent to $F \rtimes_\alpha \mu_3$ and kQ' is Morita equivalent to $F \rtimes_\beta \mu_3$ because, as we will explain in the next paragraph, the category of μ_3-equivariant F-modules is equivalent to the category of representations of Q (or Q', depending on the action of μ_3 on F).

Given a μ_3-equivariant F-module M let

$$M_i = \{m \in M \mid \xi \bullet m = \xi^i m \text{ for all } \xi \in \mu_3\}$$

and place M_i at the vertex labeled i in Q or Q'. In Lemma 3.4(1), the clockwise arrows give the action of X on each M_i and the counter-clockwise arrows give the action of Y on each M_i. (In other words, there is a homomorphism $F \to kQ$ given by $X \mapsto u_1 + v_1 + w_1$ and $Y \mapsto u_2 + v_2 + w_2$.) This functor from μ_3-equivariant F-modules to $\mathsf{Mod}\, kQ$ is an equivalence of categories.

The foregoing is "well known" to those to whom it is common knowledge.

Proposition 3.5. *Let k be a field such that μ_3, its 3rd roots of unity, has order 3. Let F be the free k-algebra on two generators placed in degree one. Let $\alpha : \mu_3 \to \mathsf{Aut}_{\mathsf{gr.alg}} F$ be a homomorphism such that F_1 is a sum of two non-isomorphic representations of μ_3. If S is a degenerate Sklyanin algebra, then*

$$\mathsf{QGr}\, S \equiv \mathsf{QGr}(F \rtimes_\alpha \mu_3).$$

3.6.1. Remark

In [5], it is shown that $\mathsf{QGr}\, F$ is equivalent to $\mathsf{Mod}\, T$ where T is the ultramatricial algebra with Bratteli diagram

$$1 \rightrightarrows 2 \rightrightarrows 4 \rightrightarrows 8 \cdots.$$

We do not know an argument that explains the relation between this diagram and that in (3.3) which arises in connection with $\mathsf{QGr}(F \rtimes \mu_3)$.

3.7. The Grothendieck group of qgr *S*

Proposition 3.6. *Let S be a degenerate Sklyanin algebra. Then*

$$K_0(\mathsf{qgr}\, S) \cong \mathbb{Z}\left[\frac{1}{8}\right] \oplus \mathbb{Z} \oplus \mathbb{Z}$$

with $[\mathcal{O}] = (1, 0, 0)$.

Proof. Since $\mathsf{qgr}\, S \equiv \mathsf{mod}\, R$,

$$K_0(\mathsf{qgr}\, S) \cong K_0(R) \cong \varinjlim K_0(R_n)$$

where the second isomorphism holds because $K_0(-)$ commutes with direct limits.

Since R_n is a product of three matrix algebras, $K_0(R_n) \cong \mathbb{Z}^3$. We can pick these isomorphisms in such a way that the map $K_0(R_n) \to K_0(R_{n+1})$ induced by the inclusion $\phi_n : R_n \to R_{n+1}$ in the Bratteli diagram is left multiplication by

$$M := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

The directed system $\cdots \to K_0(R_n) \to K_0(R_{n+1}) \to \cdots$ is therefore isomorphic to

$$\cdots \to \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{M} \cdots.$$

As noted above,

$$M^3 := \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

We have

$$M^3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad M^3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad M^3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Since $\varinjlim K_0(R_n)$ is also the direct limit of the directed system

$$\cdots \to \mathbb{Z}^3 \xrightarrow{M^3} \mathbb{Z}^3 \xrightarrow{M^3} \mathbb{Z}^3 \xrightarrow{M^3} \cdots,$$

$K_0(R)$ is isomorphic to $\mathbb{Z}[\frac{1}{8}] \oplus \mathbb{Z} \oplus \mathbb{Z}$. Together with the observation that $\mathcal{O} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ this completes the proof. $\square$

4. Point modules for $S_{a,b,c}$

4.1. A point module over a connected graded k-algebra A is a graded A-module $M = M_0 \oplus M_1 \oplus \cdots$ such that $\dim_k M_n = 1$ for all $n \geqslant 0$ and $M = M_0 A$.

4.2. If one connected graded algebra is a Zhang twist of another their categories of graded modules are equivalent via a functor that sends point modules to point modules. Thus, to determine the point modules over a degenerate Sklyanin algebra it suffices to determine the point modules over A, the algebra with relations $u^2 = v^2 = w^2 = 0$.

4.3. The letters u, v, w will now serve double duty: they are elements in A and are also an ordered set of homogeneous coordinates on $\mathbb{P}^2 = \mathbb{P}(A_1^*)$, the lines in A_1^*.

If $M = ke_0 \oplus ke_1 \oplus \cdots$ is a point module with $\deg e_n = n$, we define points $p_n \in \mathbb{P}^2$, $n \geqslant 0$, by

$$p_n = (\alpha_n, \beta_n, \gamma_n)$$

where $e_n \cdot u = \alpha_n e_{n+1}$, $e_n \cdot v = \beta_n e_{n+1}$, and $e_n \cdot w = \gamma_n e_{n+1}$. The p_ns do not depend on the choice of homogeneous basis for M. We call $p_0, p_1, \ldots$ the point sequence associated to M.

4.4. Let E be the three lines in $\mathbb{P}^2$ where $uvw = 0$. We call the points that lie on two of those lines *intersection points*. If p is an intersection point the component of E that does not pass through p is called *the line opposite* p. If L is a component of E we call the intersection point that does not lie on L *the point opposite* L.

Lemma 4.1. *Let $(p_0, p_1, \ldots)$ be the point sequence associated to a point module M.*

(1) *Every p_n lies on E.*
(2) *If p_n is an intersection point then p_{n+1} lies on the line opposite p_n and can be any point on that line.*
(3) *If p_n is not an intersection point and lies on L, then p_{n+1} is the point opposite L.*

Proof. (1) If $p_n \notin E$, then $e_n \bullet u$, $e_n \bullet v$, and $e_n \bullet w$ are non-zero scalar multiples of e_{n+1}. But u^2, v^2, and w^2, are zero in A so $e_n \bullet u^2 = e_n \bullet v^2 = e_n \bullet w^2 = 0$ which implies that $e_{n+1}A = ke_{n+1}$ in contradiction of the fact that M is a point module.

(2) Suppose p_n lies on the lines $u = 0$ and $v = 0$. The line opposite p_n is the line $w = 0$. Since $e_n \bullet u = e_n \bullet v = 0$, $e_n \bullet w$ must be a non-zero multiple of e_{n+1} so, since $w^2 = 0$, $e_{n+1} \bullet w = 0$; i.e., $w(p_{n+1}) \neq 0$. The other cases are similar.

(3) Suppose p_n lies on the line $u = 0$ but not on $v = 0$ or $w = 0$. Then $e_n \bullet v = e_n \bullet w = e_{n+1}$ and, because $v^2 = w^2 = 0$, $e_{n+1} \bullet v = e_{n+1} \bullet w = 0$. Hence p_{n+1} is the point opposite the line $u = 0$. The other cases are similar. □

Lemma 4.1 shows that [8, Thm. 1.7] is not correct.

4.5. If s is a non-zero word of length n in the letters u, v, and w, we write $s^\perp$ for the set consisting of the other non-zero words of length n.

A sequence $p_0, p_1, \dots, p_n$ of points in E that arise from a (truncated) point module of length $n+2$ is said to be special if each p_i is an intersection point. A (truncated) point module giving rise to a special sequence is also called special.

Lemma 4.2. *If $n \geqslant 0$, there are $2^n \times 3$ special point sequences $p_0, \dots, p_n$ and $2^n \times 3$ special truncated point modules of length $n+2$ up to isomorphism.*

As noted before, $\dim A_{n+1} = 2^n \times 3$ for all $n \geqslant 1$.

Proposition 4.3. *If $s = s_0 \dots s_n$ is a non-zero word of length $n+1$, there is a special truncated point module N of length $n+2$ such that $N \bullet s \neq 0$ and $N \bullet t = 0$ for every $t \in s^\perp$.*

Proof. We make $N := ke_0 \oplus \cdots \oplus ke_{n+1}$ into a right A-module by having $a \in \{u, v, w\}$ act as follows:

$$e_i \bullet a = \begin{cases} e_{i+1} & \text{if } a = s_i, \\ 0 & \text{if } a \neq s_i \end{cases}$$

for $0 \leqslant i \leqslant n$ and $e_n \bullet u = e_n \bullet v = e_n \bullet w = 0$. Because $s_i \neq s_{i+1}$, u^2, v^2, and w^2, act as zero on N. By construction, $e_0 \bullet s = e_{n+1} \neq 0$ so N is a truncated point module. If $t = t_0 \dots t_n \in s^\perp$, then $t_i \neq s_i$ for some i, whence $e_0 \bullet t = 0$. Of course, $e_i \bullet t = 0$ for $i \geqslant 1$ so $N \bullet t = 0$.

The point sequence associated to N, $p_0, \dots, p_n$, is given by

$$p_i = \begin{cases} (1,0,0) & \text{if } s_i = u, \\ (0,1,0) & \text{if } s_i = v, \\ (0,0,1) & \text{if } s_i = w. \end{cases}$$

In particular, this is a special sequence and N is therefore a special truncated point module. □

Proposition 4.4. *Let $a \in A_n - \{0\}$. There is a truncated special point module N of length $n+1$ such that $N \bullet a \neq 0$.*

Proof. Let

$$I = \{b \in A \mid Nb = 0 \text{ for all truncated point modules } N \text{ of length } n+1\}.$$

To prove the proposition we must show that $I_n = 0$.

Let s be a non-zero word of length n. By Proposition 4.3, there is a truncated point module N of length $n+1$ such that $(\operatorname{Ann} N)_n \subset s^\perp$. Hence $I_n \subset \operatorname{Span}(s^\perp)$. However, L_n is a basis for A_n so the intersection of $\operatorname{Span}(s^\perp)$ as s ranges over L_n is zero. Hence $I_n = 0$. □

Corollary 4.5. *If a is a non-zero element in A there is a special point module M such that $M \bullet a \neq 0$.*

Proposition 4.4 implies that the natural map from A to its point parameter ring (see [8, Sect. 1]) is injective. Hence [8, Thm. 1.9] and [8, Cor. 1.10] are incorrect. Corrections to [8] appear in [9].

Acknowledgments

I thank Chelsea Walton for carefully reading this paper and for several useful conversations about its contents.

References

[1] M. Artin, J. Tate, M. Van den Bergh, Some algebras related to automorphisms of elliptic curves, in: The Grothendieck Festschrift, vol. 1, Birkhäuser, Boston, 1990, pp. 33–85.
[2] M. Artin, J. Tate, M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991) 335–388.
[3] K.R. Goodearl, Von Neumann Regular Rings, Monogr. Stud. Math., vol. 4, Pitman, London–San Francisco–Melbourne, 1979.
[4] C. Holdaway, S.P. Smith, Graded modules over monomial algebras and path algebras of quivers, J. Algebra 353 (2012) 249–260, arXiv:1109.4387.
[5] S.P. Smith, The non-commutative scheme having a free algebra as a homogeneous coordinate ring, arXiv:1104.3822.
[6] S.P. Smith, Category equivalences involving graded modules over path algebras of quivers, Adv. Math., in press, arXiv:1107.3511.
[7] S.P. Smith, Twisted homogeneous coordinate rings and three superpotential algebras, in preparation.
[8] C. Walton, Degenerate Sklyanin algebras and generalized twisted homogeneous coordinate rings, J. Algebra 322 (2009) 2508–2527.
[9] C. Walton, Corrigendum to degenerate Sklyanin algebras and generalized twisted homogeneous coordinate rings, J. Algebra 322 (2009) 2508–2527, arXiv:1112.5211v1.
[10] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. Lond. Math. Soc. (3) 72 (1996) 281–311.