#29

Differential Operators on The Flag Variety and The Conze Embedding

T. J. Hodges and S. P. Smith

Abstract

Let G be a connected complex semi-simple Lie group with Borel subgroup B containing a maximal torus T and unipotent radical N. Let $\mathfrak{g},\mathfrak{b},\mathfrak{h},\mathfrak{n}$ denote the corresponding Lie algebras and denote by $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$, denote by $M(\lambda)$ the Verma module of highest weight $\lambda-\rho$ (ρ is the half-sum of the positive roots). Write $D_\lambda = U(\mathfrak{g})/\mathrm{Ann}\, M(\lambda)$. Let $n = \dim N$, and denote by A_n the ring of regular differential operators on (complex) affine n-space.

A. Beilinson and J. N. Bernstein (C. R. Acad. Sci. 292 (1981) 15 - 18) have constructed, for each $\lambda \in \mathfrak{h}^*$, a sheaf, $\mathcal{D}_\lambda$ of twisted differential operators, on the flag variety G/B, such that $D_\lambda = \Gamma(G/B,\mathcal{D}_\lambda)$. Let w_0 be the longest element of the Weyl group, and denote by Bw_0B the large Bruhat cell. Then Bw_0B is isomorphic to affine n-space (as a subvariety of G/B) and $\Gamma(Bw_0B,\mathcal{D}_\lambda) \cong A_n$. The restriction map

$$j_\lambda : \Gamma(G/B,\mathcal{D}_\lambda) \to \Gamma(Bw_0B,\mathcal{D}_\lambda)$$

gives an embedding of D_λ into A_n.

Denote by $\mathfrak{n}^-$ the nilpotent subalgebra of $\mathfrak{g}$ opposite $\mathfrak{n}$. N. Conze (Bull. Soc. Math. France 102 (1974) 379 - 415; Zentralblatt für Mathematik (1975) 298.17012) showed that

the action of $U(\underset{\sim}{g})$ on $M(\lambda)$ induces an action of $U(\underset{\sim}{g})$ on $S(\underset{\sim}{n}^-)$, such that $U(\underset{\sim}{g})$ acts as regular differential operators on $\underset{\sim}{n}^-$. Consequently one has a map $U(\underset{\sim}{g}) \to A_n$ (realising A_n as regular differential operators on $\underset{\sim}{n}^-$) with kernel Ann $M(\lambda)$. Denote by $i_\lambda : D_\lambda \to A_n$ the induced embedding.

In this paper it is shown that these two embeddings of D_λ into A_n are essentially the same. More precisely the following is established: let $i_{w_o\lambda} : D_\lambda \to A_n$ denote the Conze embedding obtained through the action of $U(\underset{\sim}{g})$ on $M(w_o\lambda)$ and hence on $S(\underset{\sim}{n}^-)$; then there exists an automorphism τ of $\underset{\sim}{g}$ (extending to an automorphism of D_λ), and an automorphism ψ of A_n such that $i_{w_o\lambda} = \psi j_\lambda \tau$.

Department of Mathematics,
University of Utah,
Salt Lake City,
Utah 84112,
USA.

Mathematics Institute,
University of Warwick,
Coventry CV4 7AL,
United Kingdom.

Differential Operators on the Flag Variety and the Conze Embedding

T.J. Hodges and S.P.Smith

1. Introduction

Let G be a connected complex semi-simple Lie group with Borel subgroup B containing a maximal torus T and unipotent radical N. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$, $\mathfrak{n}$ denote the corresponding Lie algebras and denote by $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$ denote by $M(\lambda)$ the Verma module of highest weight $\lambda-\rho$ (where ρ is the half-sum of the positive roots). Put $D_\lambda = U(\mathfrak{g})/\operatorname{ann} M(\lambda)$. In [BB] Beilinson and Bernstein construct a sheaf $\mathcal{D}_\lambda$ of twisted differential operators on the flag variety X=G/B such that $D_\lambda = \Gamma(X,\mathcal{D}_\lambda)$. If $V \subset X$ is an open affine subset isomorphic to affine n-space A^n ($n = \dim X = \dim N$) then $\Gamma(V,\mathcal{D}_\lambda) \simeq A_n$ the n-th Weyl algebra. The restriction map $D_\lambda \to \Gamma(V,\mathcal{D}_\lambda)$ gives an embedding of D_λ in A_n. In [C] Conze shows that the action of $U(\mathfrak{g})$ on $M(\lambda)$ induces an action of $U(\mathfrak{g})$ on $S(\mathfrak{n}^-)$, the symmetric algebra of $\mathfrak{n}^-$, such that $U(\mathfrak{g})$ acts as differential operators of finite order. Consequently one has a map $U(\mathfrak{g}) \to A_n$ (realising A_n as the ring of differential operators on $S(\mathfrak{n}^-)$), with kernel $\operatorname{ann} M(\lambda)$. In Theorem 4.4 we describe the relationship between these two embeddings of D_λ in A_n.

If $V=Bw_0B$ denotes the large Bruhat cell then up to an automorphism of $U(\mathfrak{g})$ and an automorphism of A_n these two embeddings coincide. The precise relationship is as follows. Let $j_\lambda\colon D_\lambda \to A_n$ be the embedding obtained from the restriction map

$D_\lambda \to \Gamma(V, \mathcal{D}_\lambda)$; let w_o denote the longest element of the Weyl group, and let $i_{w_o\lambda}$: $D_{w_o\lambda} \to A_n$ denote the Conze embedding obtained through the action of $\mathfrak{g}$ on $M(w_o\lambda)$; let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an automorphism such that $\tau(kX_\alpha) = kX_{w_o\alpha}$ and $\tau(H_\alpha) = H_{w_o\alpha}$ for all roots α $(k=C)$; τ extends to an automorphism of $U(\mathfrak{g})$ and induces an automorphism of D_λ which we also denote by τ; denote by ψ the canonical automorphism of A_n given by $\psi(p_j) = q_j$ and $\psi(q_j) = -p_j$; then $i_{w_o\lambda} = \psi j_\lambda \tau$.

One consequence of this and the equivalence of categories established in [BB] is that if λ is dominant regular then the Conze embedding $D_{w_o\lambda} \to A_n$ makes A_n flat as a right $D_{w_o\lambda}$ - module.

The proof of these results proceeds by examining the action of $\mathfrak{g}$ on $\Gamma(V,\mathcal{O})$ (where $\mathcal{O}$ is the structure sheaf of X) induced by the map $\mathfrak{g} \to D_\lambda$ and the restriction $D_\lambda \to \Gamma(V, \mathcal{D}_\lambda)$. As N acts simply transitively on V we may identify $\Gamma(V,\mathcal{O})$ with k[N], the ring of regular functions on the affine algebraic group N. This action of $\mathfrak{g}$ on k[N] induces an action of $\mathfrak{g}$ on $k[N]^\circ$, the Hopf dual of k[N] consisting of those distributions on N supported at the identify. It is well known that $k[N]^\circ$ is isomorphic to $U(\mathfrak{n})$ the algebra of right invariant differential operators on N. We show that this action of $\mathfrak{g}$ on $k[N]^\circ$ makes $k[N]^\circ$ isomorphic to $M(\lambda)$, the co-unit being a highest weight vector. We are then able to show that the action of $\mathfrak{g}$ on k[N], which we are identifying with $\Gamma(V,\mathcal{O})$, is such that k[N] is isomorphic to the dual of $M(\lambda)$ (not the full dual but the module of $\mathfrak{h}$-finite functionals on $M(\lambda)$). This appears in §2.

In Section 3, we examine an arbitrary finite dimensional nilpotent Lie algebra $\mathfrak{n}$ over a field k of characteristic zero. If N is the unipotent algebraic group with Lie algebra $\mathfrak{n}$ then the natural action of $\mathfrak{n}$ on k[N] as right invariant derivations gives an embedding of $U(\mathfrak{n})$ into a Weyl algebra; denote this embedding by $i_2 : U(\mathfrak{n}) \to A$. The symmetrisation map $\omega: S(\mathfrak{n}) \longrightarrow U(\mathfrak{n})$ allows the left regular representation of $U(\mathfrak{n})$ on itself to be transferred to an action of $U(\mathfrak{n})$ on $S(\mathfrak{n})$. As $U(\mathfrak{n})$ acts on $S(\mathfrak{n})$ as differential operators of finite order, if B denotes the ring of differential operators on $S(\mathfrak{n})$, we obtain an embedding $i_1 : U(\mathfrak{n}) \longrightarrow B$. We show in Theorem 3.7 that there is an isomorphism $\psi : A \longrightarrow B$ such that $i_1 = \psi i_2$.

In Section 4, the result of §3 is applied to the subalgebra $\mathfrak{n}$ of $\mathfrak{g}$. Let A denote the ring of differential operators on $\Gamma(V,\mathcal{O})$. When we identify $\Gamma(V,\mathcal{O})$ with k[N] the action of $\mathfrak{g}$ on $\Gamma(V,\mathcal{O})$ is such that $\mathfrak{n}$ has its natural action (as right invariant differential operators) on k[N]. Hence the map $j_\lambda : D_\lambda \longrightarrow A$, when restricted to $U(\mathfrak{n})$ coincides with the map $i_2 : U(\mathfrak{n}) \longrightarrow A$ described above. If B denotes the ring of differential operators on $S(\mathfrak{n})$ then the result of §3 says that $\psi j_\lambda : D_\lambda \longrightarrow B$ when restricted to $U(\mathfrak{n})$ coincides with $i_1 : U(\mathfrak{n}) \longrightarrow B$. More importantly, the map $\psi j_\lambda \tau : D_\lambda \longrightarrow B$ (where we now identify B with the ring of differential operators on $S(\mathfrak{n}^-)$) has the property that when restricted to $U(\mathfrak{n}^-)$, the action of $\psi(\mathfrak{n}^-)$ induced on $S(\mathfrak{n}^-)$ is the same as that induced by the symmetrisation map $\omega : S(\mathfrak{n}^-) \longrightarrow U(\mathfrak{n}^-)$. To show that $\psi j_\lambda \tau = i_{w_o \lambda}$, it is then just a matter of checking that the actions of $\mathfrak{n}$ and $\mathfrak{h}$ on $S(\mathfrak{n}^-)$ are such that $\mathfrak{n}.1 = 0$ and that $H.1 = (w_o\lambda - \rho)(H).1$

for $H \in \underset{\sim}{h}$. This is straightforward after §2.

2. The Beilinson-Bernstein construction and the action of $\mathfrak{g}$ on $\mathcal{O}$

2.1 For $u \in U(\mathfrak{g})$ denote by u^t the image of u under the anti-automorphism given by $X_\alpha^t = X_{-\alpha}$ for $\alpha \in R$ (R is the set of roots), and $H^t = H$ for $H \in \mathfrak{h}$. If M is a $U(\mathfrak{g})$-module, $M^* = \mathrm{Hom}_k(M,k)$ is given a $U(\mathfrak{g})$-module structure by $(u.\theta)(m) = \theta(u^t m)$ for $u \in U(\mathfrak{g})$, $\theta \in M^*$, $m \in M$. Define the <u>dual of M</u>, $\delta(M)$, to be the subspace of M^* consisting of those functionals which generate a finite-dimensional $U(\mathfrak{h})$-module. It is easy to check that $\delta(M)$ is a $U(\mathfrak{g})$-module.

2.2 The following characterisation of the dual of a Verma module is better suited to our purposes (it is probably well known but does not seem to appear explicitly in the literature).

LEMMA. *Let I denote the ideal of $U(\mathfrak{n}^-)$ generated by $\mathfrak{n}^-$. Then $\delta(M(\lambda))$ consists of those functionals in $M(\lambda)^*$ which vanish on $I^n M(\lambda)$ for some n.*

Proof. Write $M = M(\lambda)$. It is standard that $\delta(M) = \bigoplus_\mu (M_\mu)^*$ where M_μ consists of the elements of M of weight μ, and $(M_\mu)^*$ is identified with the subspace of M^* consisting of those functionals on M vanishing on $\bigoplus_{\nu \neq \mu} M_\nu$. Because M is a free $U(\mathfrak{n}^-)$-module and $\bigcap I^n = 0$, we also have $\bigcap I^n M = 0$. Hence, as M_μ is finite dimensional, we have

$M_\mu \cap I^n M = 0$ for some n. But $I^n M$ is a sum of weight spaces (because I^n and M are). So an element of $(M_\mu)^*$ vanishes on $I^n M$.

Conversely, if $\theta \varepsilon M^*$ vanishes on some $I^n M$, then $\theta \varepsilon \delta(M)$ for the following reason: as $I^n M$ is a sum of weight spaces, it has a complement M' in M which is also a sum of weight spaces, hence $\theta \varepsilon \Sigma (M_\mu)^*$ where the sum is over the finite set $\{\mu | M_\mu \not\subset M'\}$ (the set is finite as $U(\underset{\sim}{n}^-)/I^n$ is finite dimensional).

2.3 Denote by k[N] the ring of regular functions on the affine algebraic group N, and let $\underline{m}$ denote the ideal of k[N] of those functions vanishing at the identity $e \varepsilon N$. Denote by Δ and ε respectively the co-multiplication and co-unit of k[N]; that is, $\varepsilon : K[N] \to k$ is the algebra map with kernel $\underline{m}$, or $\varepsilon(f) = f(e)$.

The Hopf dual $k[N]^\circ$ of k[N] is defined as the algebra of functionals on k[N] which vanish on some power of $\underline{m}$. The multiplication in $k[N]^\circ$ is defined by $\phi\theta = (\phi \otimes \theta)\Delta$ for $\theta, \phi \varepsilon k[N]^\circ$; that is $\phi\theta(f) = \sum_{(f)} \phi(f_{(1)})\theta(f_{(2)})$ where $\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$ in Sweedler's notation.

We view $U(\underset{\sim}{n})$ as the algebra of right invariant differential operators on N. There is an algebra anti-isomorphism (see, for example [W, p.99]) $i : k[N]^\circ \to U(\underset{\sim}{n})$ given by $i(\theta) = (\theta \otimes id)\Delta$; $(\theta \otimes id)\Delta$ is the differential operator given by $(\theta \otimes id)\Delta(f) = \sum_{(f)} \theta(f_{(1)}) f_{(2)}$ for $f \varepsilon k[N]$. Through this anti-isomorphism the action of $U(\underset{\sim}{n})$ on itself by right multiplication can be transferred to give $k[N]^\circ$ the structure of a right $U(\underset{\sim}{n})$- module : for $d \in U(\underset{\sim}{n})$, $\theta \in k[N]^\circ$ define $\theta d = i^{-1}(i(\theta)d)$ where $i(\theta)d$ is the product in $U(\underset{\sim}{n})$.

LEMMA. <u>The action of</u> $U(\underset{\sim}{n})$ <u>on</u> $k[N]^\circ$ <u>given by</u> $(\theta d)(f) = \theta(d(f))$ <u>for</u> $\theta \in k[N]^\circ$, $d \in U(\underset{\sim}{n})$, $f \in k[N]$ <u>makes</u> $k[N]^\circ$ <u>a free right</u> $U(\underset{\sim}{n})$-<u>module generated</u> by ε.

<u>Proof.</u> The action of $U(\underset{\sim}{n})$ on $k[N]^\circ$ described in the statement of the Lemma coincides with that described just prior to the Lemma. To see this, first observe that if $d = i(\phi) = (\phi \otimes id)\Delta$ then $i(\theta)d = i(\theta)i(\phi) = i(\phi\theta)$. Hence $\theta d = \phi\theta$ and $(\theta d)(f) = (\phi\theta)(f) = \sum_{(f)} \phi(f_{(1)})\theta(f_{(2)}) = \theta(\sum_{(f)}\phi(f_{(1)})f_{(2)}) = \theta\big((\phi\otimes id)\Delta(f)\big) = \theta\big(d(f)\big)$.

Thus, as $U(\underset{\sim}{n})$ is a free right $U(\underset{\sim}{n})$-module generated by 1, $k[N]^\circ$ is a free right $U(\underset{\sim}{n})$-module generated by $i^{-1}(1)$; but $i(\varepsilon) = (\varepsilon\otimes id)\Delta = 1$, hence the result.

2.4 Let Der $k[N]$ denote the module of k-linear derivations on $k[N]$ and think of $\underset{\sim}{n} \subset$ Der $k[N]$ as the space of right invariant derivations.

PROPOSITION. <u>The map</u> $k[N] \otimes \underset{\sim}{n} \to$ Der $k[N]$ <u>is an isomorphism of</u> $k[N]$-<u>modules</u>

<u>Proof.</u> [H, Theorem 3.1, p.37].

We point this out because it explains why $\mathcal{D}_\lambda$ is a sheaf of twisted differential operators (see [BB] for the definition). The point is that if N is any irreducible affine algebraic group with Lie algebra $\mathfrak{n}$, the subalgebra of $\mathrm{End}_k k[N]$ generated by k[N] and $\mathfrak{n}$ coincides with that generated by k[N] and Der k[N]. But as N is smooth, this is just the ring of differential operators on N. In other words, the smash product $k[N] \# U(\mathfrak{n})$ is isomorphic to the ring of differential operators on N (in particular, if N is unipotent this is a Weyl algebra).

2.5 The construction of the sheaf $\mathcal{D}_\lambda$ is described in [BB]. We recall the details and then describe in some detail the local structure of $\mathcal{D}_\lambda$ over the large Bruhat cell $V = Bw_oB$.

First let $\mathcal{O} \otimes U(\mathfrak{g})$ be the sheaf of k-algebras with multiplication such that $\mathcal{O} \otimes 1$ is a subsheaf isomorphic to $\mathcal{O}$, $1 \otimes U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})$, $(f \otimes 1)(g \otimes u) = fg \otimes u$, and for $X \in \mathfrak{g}$, $[1 \otimes X, f \otimes 1] = X(f) \otimes 1$ where X(f) is obtained by considering X as a global vector field on G/B. Notice that $[f \otimes X, g \otimes Y] = fX(g) \otimes Y + fg \otimes [X,Y] - gY(f) \otimes X$ for $X, Y \in \mathfrak{g}$.

Consider $\mathcal{O} \otimes \mathfrak{g}$ as a subsheaf of $\mathcal{O} \otimes U(\mathfrak{g})$, and denote by α the map $\mathcal{O} \otimes \mathfrak{g} \to T_{G/B}$ into the tangent bundle. Denote by $\mathfrak{b}^\circ$ the kernel of α and put $\mathfrak{n}^\circ = [\mathfrak{b}^\circ, \mathfrak{b}^\circ]$. The geometric fibre of $\mathfrak{b}^\circ$ at $x \in G/B$ is $\mathfrak{b}_x = (\mathrm{Ad}\, x)\mathfrak{b}$, the subalgebra of $\mathfrak{g}$ consisting of those vector fields vanishing at x. The geometric fibre of $\mathfrak{n}^\circ$ at x is $\mathfrak{n}_x = (\mathrm{Ad}x)\,\mathfrak{n}$. The factor bundle $\mathfrak{b}^\circ/\mathfrak{n}^\circ$ is trivial and isomorphic to $X \times \mathfrak{h}$, and hence one has a map $\mathfrak{b}^\circ \to \mathcal{O} \otimes \mathfrak{h}$ with kernel $\mathfrak{n}^\circ$. For $\lambda \in \mathfrak{h}^*$ denote by λ° the induced map $\mathfrak{b}^\circ \to \mathcal{O}$.

For $\lambda \in \mathfrak{h}^*$ denote by $\mathcal{I}_\lambda$ the sheaf of ideals generated by $\zeta - (\lambda-\rho)^\circ(\zeta)$ for $\zeta \in \mathfrak{b}^\circ$; then $\mathcal{D}_\lambda$ is defined as $\mathcal{O} \otimes U(\mathfrak{g})/\mathcal{I}_\lambda$.

2.6 Put $A = \Gamma(V, \mathcal{D}_\lambda)$, and $\mathcal{O}_V = \Gamma(V, \mathcal{O})$. Then $A \simeq \mathcal{O}_V \otimes U(\mathfrak{g})/I_\lambda$ where $I_\lambda = \Gamma(V, \mathcal{I}_\lambda)$. The map $\mathcal{O} \otimes \mathfrak{g} \to T_{G/B}$ induces a map $\mathcal{O}_V \otimes \mathfrak{g} \to \mathrm{Der}\,\mathcal{O}_V$ which we denote by α also. The kernel of this map is $\mathfrak{b}_V^\circ$ and $\mathfrak{n}_V^\circ = [\mathfrak{b}_V^\circ, \mathfrak{b}_V^\circ]$.

LEMMA. (i) _The restriction of_ α _to_ $\mathcal{O}_V \otimes \mathfrak{n}$ _is an isomorphism from_ $\mathcal{O}_V \otimes \mathfrak{n}$ _onto_ $\mathrm{Der}\,\mathcal{O}_V$.

(ii) $\mathcal{O}_V \otimes \mathfrak{g} = \mathfrak{b}_V^\circ \oplus (\mathcal{O}_V \otimes \mathfrak{n})$. _Let_ $p: \mathcal{O}_V \otimes \mathfrak{g} \to \mathfrak{b}_V^\circ$ _be the projection._

(iii) _There is a surjective map_ $\mathfrak{b}_V^\circ \to \mathcal{O}_V \otimes \mathfrak{h}$ _with kernel_ $\mathfrak{n}_V^\circ$.

(iv) _If_ $H \in \mathfrak{h}$ _the image of_ $p(1 \otimes H)$ _in_ $\mathfrak{b}_V^\circ/\mathfrak{n}_V^\circ = \mathcal{O}_V \otimes \mathfrak{h}$ _is_ $1 \otimes H$.

Proof. All this is implicit in the construction of $\mathcal{D}_\lambda$ given in [BB].

(i) follows from Proposition 2.4 and the identification of $\mathcal{O}_V$ with $k[N]$.

(ii) As $\mathcal{O}_V$ is a polynomial algebra, $\mathrm{Der}\,\mathcal{O}_V$ is a free

$\mathcal{O}_V$-module, hence $\mathcal{O}_V \otimes \underset{\sim}{g}$ splits as a direct sum.

(iii) and (iv) follow from the fact that $\underset{\sim}{b}^\circ/\underset{\sim}{n}^\circ \simeq \mathcal{O} \otimes \underset{\sim}{h}$.

All the maps mentioned in the Lemma are both $\mathcal{O}_V$-module maps and Lie algebra homorphisms.

2.7 For $\lambda \varepsilon \underset{\sim}{h}^*$ denote by λ° the composition $\underset{\sim}{b}_V^\circ \to \mathcal{O}_V \otimes \underset{\sim}{h} \to \mathcal{O}_V$. Then I_λ is the ideal of $\mathcal{O}_V \otimes U(\underset{\sim}{g})$ generated by $\zeta - (\lambda-\rho)^\circ(\zeta)$ for $\zeta \varepsilon \underset{\sim}{b}_V^\circ$. We will identify $\mathcal{O}_V$ and $\underset{\sim}{g}$ with their images in A.

LEMMA. (i) A _is generated by_ $\mathcal{O}_V$ _and_ $\underset{\sim}{n}$, _and_ $A \simeq A_n$, the n-th Weyl algebra.

(ii) $A/A\underset{\sim}{n} \simeq \mathcal{O}_V$ _as_ $\mathcal{O}_V$_-modules_.

(iii) _The embedding of_ $\underset{\sim}{g}$ _in_ A _gives an action of_ $\underset{\sim}{g}$ _on_ $A/A\underset{\sim}{n}$ _which transferred to_ $\mathcal{O}_V$ _becomes_

$$X.f = X(f) + (\lambda-\rho)^\circ p(1 \otimes X)f \text{ for } X\varepsilon\underset{\sim}{g},\ f\varepsilon\ \mathcal{O}_V.$$

Proof. A is generated by $\mathcal{O}_V$ and $\underset{\sim}{g}$, hence by $\mathcal{O}_V$, $\underset{\sim}{b}_V^\circ$ and $\underset{\sim}{n}$ (by Lemma 2.6(ii)); however the image $\underset{\sim}{b}_V^\circ$ in A lies in $\mathcal{O}_V$ (because of the definnition of I_λ). Hence A is generated by $\mathcal{O}_V$ and $\underset{\sim}{n}$. The fact that $A \simeq A_n$ follows from the discussion in §2.4. Hence (i).

The easiest way to see (ii) is to realise A_n as $k[q_1,\ldots,q_n,p_1,\ldots,p_n]$ where $k[q_1,\ldots,q_n] = \mathcal{O}_V$ and $p_j = \partial/\partial q_j$. Then as $\mathcal{O}_V \otimes \underset{\sim}{n} \to \mathrm{Der}\,\mathcal{O}_V$ is an isomorphism, $A\underset{\sim}{n} = Ap_1 + \ldots + Ap_n$ so $A/A\underset{\sim}{n} = k[q_1,\ldots,q_n] = \mathcal{O}_V$.

For $a\varepsilon A$ let $\overline{a}$ denote the image in $A/A\underset{\sim}{n}$. For $X\varepsilon\underset{\sim}{g}$, $f\varepsilon\mathcal{O}_V$

the action of $\mathfrak{g}$ on $\mathcal{O}_V$ is given by $X.f = \overline{(1 \otimes X)(f \otimes 1)} = \overline{[1 \otimes X, f \otimes 1]} + \overline{f \otimes X} = \overline{X(f) \otimes 1} + \overline{f \otimes X} = X(f) + \overline{f \otimes X}$. We have $1 \otimes X = p(1 \otimes X) + b$ for some $b \in \mathcal{O}_V \otimes \mathfrak{n}$, so in A, $1 \otimes X = (\lambda - \rho)^\circ p(1 \otimes X) + b$ and hence $\overline{f \otimes X} = \overline{(\lambda-\rho)^0 p(1 \otimes X)f}$ (because the image of b in A lies in $A\mathfrak{n}$). Hence (iii).

2.8 It is the action of $\mathfrak{g}$ on $\mathcal{O}_V$ given in Lemma 2.7 (iii) that we want to understand.

Through the identification of $\mathcal{O}_V$ and $k[N]$ we transfer this action of $\mathfrak{g}$ on $\mathcal{O}_V$ to an action on $k[N]$ and use the Hopf algebra structure of $k[N]$ to transfer this to an action of $\mathfrak{g}$ on $k[N]^\circ$. We will show in Theorem 2.10 that $k[N]^\circ \cong M(\lambda)$.

LEMMA. <u>If</u> $k[N]^*$ <u>is given a</u> $\mathfrak{g}$<u>-module structure through</u> $(u.\theta)(f) = \theta(u^t f)$ <u>for</u> $u \in U(\mathfrak{g})$, $\theta \in k[N]^*$, $f \in k[N]$ <u>then</u> $k[N]^\circ$ <u>is a submodule.</u>

<u>Proof</u>. Let $X \in \mathfrak{g}$, $\theta \in k[N]^\circ$ such that $\theta(\underline{m}^s) = 0$. We want to show $X.\theta$ vanishes on some power of $\underline{m}$.

Suppose $X \in \mathfrak{b}$ and $f \in \underline{m}^{s+1}$. Then $(X.\theta)(f) = \theta(X^t.f) = \theta(X^t(f) + (\lambda-\rho)^\circ p(1 \otimes X^t)f) = \theta(X^t(f))$ (using the fact that $\theta(\underline{m}^s) = 0$ and $f \in \underline{m}^s$). But $X^t \in \mathfrak{b}^-$ and vector fields in $\mathfrak{b}^-$ vanish at w_0B which identifies with $e \in N$ under the identification of N and V, so $X^t(k[N]) \subset \underline{m}$. By induction $X^t(\underline{m}^{s+1}) \subset \underline{m}^s$ so $\theta(X^t(f)) = 0$. We have shown that if $X \in \mathfrak{b}$ then $(X.\theta)(\underline{m}^{s+1}) = 0$ and hence $X.\theta \in k[N]^\circ$.

Suppose $X \in \mathfrak{n}^-$. Then $X^t \in \mathfrak{n}$ and $(\lambda-\rho)^\circ p(1 \otimes X^t) = 0$ hence for any $f \in k[N]$, $(X.\theta)(f) = \theta(X^t(f))$. This action of X^t

gives an action of $\mathfrak{n}$ on $k[N]^\circ$ which agrees with that in Lemma 2.3 and as $k[N]^\circ$ is a $U(\mathfrak{n})$-module under this action we have $X.\theta \in k[N]^\circ$.

So $k[N]^\circ$ is indeed closed under this action of $\mathfrak{g}$.

2.9 Before proving that $k[N]^\circ \simeq M(\lambda)$, where the $\mathfrak{g}$-module action on $k[N]^\circ$ is that given is §2.8, the following technical results are required.

LEMMA. <u>Inside</u> $\mathcal{O}_V \otimes \mathfrak{g}$ <u>one has</u>

(a) $1 \otimes \mathfrak{n}^- \subset \underline{m} \otimes \mathfrak{n} + \underline{m} \otimes \mathfrak{h} + \mathfrak{n}_V^\circ$

(b) $1 \otimes \mathfrak{h} \subset \underline{m} \otimes \mathfrak{n} + \mathfrak{b}_V^\circ$

<u>Proof</u> (a) As $\mathfrak{n}$ acts ad-nilpotently on $\mathfrak{g}$ there is a chain of subspaces $0 = N_0 \subset N_1 \ldots \subset N_m = \mathfrak{n}^-$ with the property that $[\mathfrak{n}, N_j] \subset \mathfrak{b} + N_{j-1}$. Moreover each of these subspaces has a basis consisting of weight vectors (e.g. the basis of N_1 consists of those $X_\alpha, \alpha \in R^-$, such that $(\alpha + R^+) \cap R^- = \phi$ etc).

We will show that $1 \otimes N_j \subset \underline{m} \otimes \mathfrak{n} + \mathfrak{n}_V^\circ + \underline{m} \otimes (\mathfrak{b} + N_{j-1})$ and by induction on j, the first part of the Lemma will follow. Pick $X \in N_j$ a weight vector and $H \in \mathfrak{h}$ such that $[X,H] = X$. As $\mathcal{O}_V x \mathfrak{g} = \mathfrak{b}_V^\circ \oplus (\mathcal{O}_V \otimes \mathfrak{n})$ there exist $a, b \in \mathcal{O}_V \otimes \mathfrak{n}$ such that $1 \otimes X + a$ and $1 \otimes H + b$ are both in $\mathfrak{b}_V^\circ$. In fact as $X, H \in \mathfrak{b}$ and vanish at $w_0 B$ so must a,b vanish at $w_0 B$ (because elements of $\mathfrak{b}_V^\circ$ vanish on all of V and so at $w_0 B$ in particular); but the only elements of $\mathrm{Der}\,\mathcal{O}_V$ vanishing at $w_0 B$ are those in $\underline{m}\mathrm{Der}\,\mathcal{O}_V$, so a and b belong to $\underline{m} \otimes \mathfrak{n}$.

Now $[1\otimes X + a,\ 1\otimes H+b]\,\varepsilon\,[\mathfrak{b}_V^\circ, \mathfrak{b}_V^\circ] = \mathfrak{n}_V^\circ$, but this bracket equals $1\otimes X + [1\otimes X, b] + [a, 1\otimes H] + [a,b]$. The last two terms both belong to $\underline{m}\otimes\mathfrak{n}$. If $b = \Sigma f_i \otimes X_i$ with $f_i\,\varepsilon\,\underline{m}, X_i\,\varepsilon\,\mathfrak{n}$ then $[1\otimes X, b] = \Sigma X(f_i)\otimes X_i + f_i \otimes [X, X_i]$. The first term is in $\underline{m}\otimes\mathfrak{n}$, and the second belongs to $\underline{m}\otimes(\mathfrak{b} + N_{j-1})$. Putting all these facts together we have shown that $1\otimes X\ \varepsilon\ \mathfrak{n}_V^\circ + \underline{m}\otimes\mathfrak{n} + \underline{m}\otimes(\mathfrak{b} + N_{j-1})$ as required.

(b) If $H\varepsilon\mathfrak{h}$, we can pick $b\,\varepsilon\,\underline{m}\otimes\mathfrak{n}$ such that $1\otimes H + b\ \varepsilon\,\mathfrak{b}_V^\circ$ (by what was said above). Hence (b).

COROLLARY. <u>If</u> $X\varepsilon\mathfrak{n}-$, <u>then</u> $(\lambda-\rho)^\circ p(1\otimes X)\ \varepsilon\ \underline{m}$

<u>Proof</u>. As $1\otimes X\ \varepsilon\ \underline{m}\otimes\mathfrak{n} + \underline{m}\otimes\mathfrak{h} + \mathfrak{n}_V^\circ$, and p has kernel $\mathcal{O}_V\otimes\mathfrak{n} \supset \underline{m}\otimes\mathfrak{n}$, and $(\lambda-\rho)^\circ(\mathfrak{n}_V^\circ) = 0$ (by Lemma 2.6) we have $(\lambda-\rho)^\circ p(1\otimes X) \in (\lambda-\rho)^\circ p(\underline{m}\otimes\mathfrak{h})$. The result follows from the fact that $(\lambda-\rho)^\circ$ and p are both $\mathcal{O}_V$-module maps.

2.10 THEOREM. <u>As a</u> $\mathfrak{g}$<u>-module</u> $k[N]^\circ \simeq M(\lambda)$ <u>and the co-unit</u> ε <u>is a highest weight vector</u>.

<u>Proof</u>. It is necessary to show that (i) $U(\mathfrak{n}-).\varepsilon = k[N]^\circ$ and that $k[N]^\circ$ is free as a $U(\mathfrak{n}-)$-module; (ii) $\mathfrak{n}.\varepsilon = 0$; (iii) for $H\varepsilon\mathfrak{h}$, $(H-(\lambda-\rho)(H)).\varepsilon = 0$

(i) If $X\varepsilon\mathfrak{n}-$ then $(X.\theta)(f) = \theta(X^t.f) = \theta(X^t(f) + (\lambda-\rho)^\circ p(1\otimes X^t)f)$, but $X^t\varepsilon\ \mathfrak{n}$, so $p(1\otimes X^t) = 0$, and $(X.\theta)(f) = \theta(X^t(f))$. Now apply Lemma 2.3.

(ii) If $X\varepsilon\mathfrak{n}$ then $(X.\varepsilon)(f) = \varepsilon(X^t(f) + (\lambda-\rho)^\circ p(1\otimes X^t)f)$. But ε vanishes on $\underline{m}$ and $X^t(f)\varepsilon\underline{m}$ because $X^t\varepsilon\mathfrak{b}^-$ vanishes at w_oB. So $(X.\varepsilon)(f) = \varepsilon((\lambda-\rho)^\circ p(1\otimes X^t)f)$. Applying Corollary

2.9, one has $(\lambda-\rho)^\circ p(1 \otimes X^t) \in \underline{m}$ so $\varepsilon((\lambda-\rho)^\circ p(1 \otimes X^t)f)=0$. Hence $X.\varepsilon = 0$.

(iii) If $H\varepsilon \underset{\sim}{h}$ then $H^t = H$ so

$(H.\varepsilon)(f) = \varepsilon(H(f) + (\lambda-\rho)^\circ p(1 \otimes H)f) = \varepsilon(H(f) + (\lambda-\rho)(H)f)$. But $H\varepsilon \underset{\sim}{b}^-$ so H vanishes at w_oB and $H(f)\varepsilon \underline{m}$. Hence $\varepsilon(H(f))=0$ and $(H.\varepsilon)(f) = (\lambda-\rho)(H)\varepsilon(f)$ as required.

2.11 THEOREM. <u>As</u> $\underset{\sim}{g}$-<u>modules</u>, $\mathcal{O}_V \simeq \delta(M(\lambda))$.

<u>Proof</u>. Identify $\mathcal{O}_V$ with $k[N]$. The previous theorem gives a pairing $k[N]xM(\lambda)\to k$ given by $\langle f,m\rangle = \theta(f)$ where $m\varepsilon M(\lambda)$ corresponds to $\theta\varepsilon k[N]^\circ$ in the isomorphism of Theorem 2.10. In other words, if $e\varepsilon M(\lambda)$ is the highest weight vector corresponding to ε in the isomorphism above, then for $u\varepsilon U(\underset{\sim}{n}^-)$ we have $\langle f,ue\rangle = (u.\varepsilon)(f) = \varepsilon(u^t.f)$.

By Lemma 2.2, $\delta(M(\lambda))$ consists of those functionals which vanish on $I^nM(\lambda)$ for some n, where I is the ideal of $U(\underset{\sim}{n}-)$ generated by $\underset{\sim}{n}^-$. The anti-automorphism, $u\to u^t$, restricts to an anti-isomorphism from $U(\underset{\sim}{n}^-)$ to $U(\underset{\sim}{n})$ and $I^t = J$ where J is the ideal of $U(\underset{\sim}{n})$ generated by $\underset{\sim}{n}$. Using this anti-isomorphism $M(\lambda)$ may be viewed as a right $U(\underset{\sim}{n})$-module, and $\delta(M(\lambda)$ may be characterized as the space of functionals which vanish on $M(\lambda)J^n$ for some n.

It is standard that the functionals on $U(\underset{\sim}{n})$ which vanish on some power of J form an algebra isomorphic to $k[N]$, and the pairing $k[N] \times U(\underset{\sim}{n}) \to k$ is given by $(f,u) = \varepsilon(u(f))$ for $f\varepsilon k[N]$, $u\varepsilon U(\underset{\sim}{n})$. Identifying $U(\underset{\sim}{n})$ with $M(\lambda)$ we must show these two pairings are the same. We must show that $\langle f, ue\rangle = (f, u^t)$ for $u \in U(\mathfrak{n}^-)$ and $f \in k[N]$. But $\langle f, ue\rangle = \varepsilon(u^t . f)$, so for $X \in \mathfrak{n}^-$, $\langle f, Xe\rangle = \varepsilon(X^t(f) + (\lambda - \rho)\circ p(1 \otimes X^t)f) = \varepsilon(X^t(f))$ as $X^t \in \mathfrak{n}$. As $\varepsilon(X^t(f)) = (f, X^t)$ we have $\langle f, Xe\rangle = (f, X^t)$ for all $X \in \mathfrak{n}^-$ and hence $\langle f, ue\rangle = (f, u^t)$ for all $u \in U(\mathfrak{n}^-)$.

3. Realising the enveloping algebra of a nilpotent Lie algebra as a subalgebra of a Weyl algebra.

3.1 Let $\mathfrak{n}$ be any nilpotent Lie algebra of dimension n over a field k of characteristic zero.

Let B denote the ring of differential operators on $S(\mathfrak{n})$. Define an action of $U(\mathfrak{n})$ on $S(\mathfrak{n})$ by $u.a = \omega^{-1}(u\omega(a))$ for $u \in U(\mathfrak{n})$, $a \in S(\mathfrak{n})$. This gives an embedding $i_1 \colon U(\mathfrak{n}) \to B$. Let N be the unipotent algebraic group with Lie algebra $\mathfrak{n}$ and let A denote the ring of differential operators on $k[N]$, the co-ordinate ring of N. The action of $U(\mathfrak{n})$ on $k[N]$ as right invariant differential operators gives an embedding $i_2 \colon U(\mathfrak{n}) \to A$. We show there is an isomorphism $\psi \colon A \to B$ such that $\psi i_2 = i_1$.

3.2 We will use Δ to denote the co-multiplication on both $U(\mathfrak{n})$ and $k[N]$; it will be clear from the context which algebra Δ is acting on and no confusion should arise.

Let $\langle \mathfrak{n} \rangle$ denote the ideal of $U(\mathfrak{n})$ generated by $\mathfrak{n}$, and let $U(\mathfrak{n})^\circ$ denote the subalgebra of $U(\mathfrak{n})^*$ consisting of those functionals which vanish on some power of $\langle \mathfrak{n} \rangle$. Then $U(\mathfrak{n})^\circ \simeq k[N]$ as algebras. We shall identify $U(\mathfrak{n})^\circ$ and $k[N]$ and denote the pairing $U(\mathfrak{n}) \times k[N] \to k$ by $\langle , \rangle$ viz. $\langle u, f \rangle = f(u)$ for $u \in U(\mathfrak{n})$, $f \in k[N] = U(\mathfrak{n})^\circ$. There is a natural map $U(\mathfrak{n}) \to \mathrm{End}_k k[N]$ given by $u \mapsto (\check{u} \otimes \mathrm{id})\Delta$ where $\check{u}$ is the element of $U(\mathfrak{n})$ obtained through the anti-isomorphism of $U(\mathfrak{n})$ induced by $\check{X} = -X$ for $X \in \mathfrak{n}$. Hence if $u \in U(\mathfrak{n})$ and $f \in k[N]$ then

$u(f) = \sum_{(f)} \langle \check{u}, f_{(1)} \rangle f_{(2)}$. So if $v \varepsilon U(\underset{\sim}{n})$ then $u(f)$ is the functional on $U(\underset{\sim}{n})$ given by $u(f)(v)$ $= \sum_{(f)} \langle \check{u}, f_{(1)} \rangle \langle v, f_{(2)} \rangle = (\Delta f)(\check{u} \otimes v) = f(\check{u}v)$.

A right invariant operator T on k[N] is one with the property that $\Delta T = (T \otimes id)\Delta$, and it is an easy matter to check that the action of $U(\underset{\sim}{n})$ on k[N] defined above makes $U(\underset{\sim}{n})$ act as right invariant differential operators on k[N].

Denote by $i_2 : U(\underset{\sim}{n}) \to A$ the map given by $i_2(u)(f) = u(f)$.

3.3 Choose a chain $\underset{\sim}{n} = \underset{\sim}{n}_1 \supset \underset{\sim}{n}_2 \supset \ldots \supset \underset{\sim}{n}_n \supset \underset{\sim}{n}_{n+1} = 0$ of ideals in $\underset{\sim}{n}$ such that $[\underset{\sim}{n}, \underset{\sim}{n}_j] \subset \underset{\sim}{n}_{j+1}$. Pick $X_j \varepsilon\, \underset{\sim}{n}_j \setminus \underset{\sim}{n}_{j+1}$, so $X_1, \ldots, X_n$ is a basis for $\underset{\sim}{n}$. We shall describe the generators of k[N] in a somewhat unorthodox manner (see [H] for a description of the usual generators of k[N]). While this has the disadvantage that some work is required to show that k[N] actually is generated by these elements, later on these generators will be easier to work with (being related to the symmetrisation map, whereas the usual generators are not). Set $Y_j = \omega^{-1}(X_j)$, so that $S(\underset{\sim}{n}) = k[Y_1, \ldots, Y_n]$. For each multi-index $J = (j_1, \ldots, j_n)$ set $Y^J = Y_1^{j_1} \ldots . Y_n^{j_n}$. So the $\omega(Y^J)$ form a basis for $U(\underset{\sim}{n})$ and this basis contains $X_1, \ldots, X_n$. Define $q_i \varepsilon U(\underset{\sim}{n})^*$ by $q_i(X_i) = 1$ and $q_i(\omega(Y^J)) = 0$ for all other basis elements $\omega(Y^J)$. We will show that $U(\underset{\sim}{n})^\circ$ $= k[q_1, \ldots, q_n]$.

3.4 LEMMA. For each m, $\langle \underset{\sim}{n} \rangle^m$ is spanned by a subset of $\{\omega(Y^J)\}$.

Proof. We use the ideas of [S], and the notation in [D, §2.8.12] (which contains a useful account of the results in [S]). Let T denote the tensor algebra of $\underset{\sim}{n}$, and let $\theta: T \to \underset{\sim}{n}$ be the map described in [D, §2.8.12]. The restriction $\theta|\otimes^p \underset{\sim}{n}$ induces a map $\underset{\sim}{n}^p \longrightarrow \underset{\sim}{n}$ defined by $\theta(x_1,\dots,x_p) = \theta(x_1 \otimes \dots \otimes x_p)$. The map θ has the important properties that, if $x_1,\dots,x_p \varepsilon \underset{\sim}{n}$ then $\theta(x_1,\dots,x_p) \varepsilon \langle \underset{\sim}{n} \rangle^p$; and if $y \varepsilon \underset{\sim}{n} \cap \langle \underset{\sim}{n} \rangle^s$ then $\theta(y,x_1,\dots,x_p) \varepsilon \langle n \rangle^{p+s}$.

For $x_1,\dots,x_p \varepsilon \underset{\sim}{n}$, denote by $(x_1,\dots,x_p)$ the sum $\frac{1}{p!} \Sigma_{\sigma \varepsilon S_p} x_{\sigma 1} \dots x_{\sigma p}$. So in particular, $(x_1 \dots, x_p) = \omega(\omega^{-1}(x_1)\dots\omega^{-1}(x_p))$ and if each of $x_1,\dots,x_p$ is an element of $\{X_1,\dots,X_n\}$ then $(x_1,\dots,x_p)$ is a scalar multiple of some $\omega(Y^J)$.

The result from [S] which we require is that for $x_1,\dots,x_p \varepsilon \underset{\sim}{n}$, $x_1 x_2 \dots x_p = \sum_{s=1}^{p} \Sigma_{1 \leq i_1 < \dots < i_s \leq p} (\Theta(x_{i_p},\dots,x_{i_s}), x_1,\dots,\hat{x}_{i_1},\dots,\hat{x}_{i_s},\dots,x_p)$ where $\hat{x}_{i_j}$ means that the term x_{i_j} is omitted.

Consider one of the terms appearing in this sum. Write each of $x_1,\dots,x_p$ (except $x_{i_1},\dots,x_{i_s}$) as a linear combination of the basis elements $\{X_1,\dots,X_n\}$. Because $\Theta(x_{i_1},\dots,x_{i_s}) \varepsilon \langle \underset{\sim}{n} \rangle^s \cap \underset{\sim}{n}$ the choice of the basis $X_1,\dots,X_n$ ensures that $\Theta(x_{i_1},\dots,x_{i_s})$ is a linear combination of basis elements X_j which belong to $\langle \underset{\sim}{n} \rangle^s \cap \underset{\sim}{n}$. By the multi-linearity of $(x_1,\dots,x_m)$ each term $(\Theta(x_{i_1},\dots,x_{i_s}), x_1,\dots,\hat{x}_{i_1},\dots,\hat{x}_{i_s},\dots,x_p)$ may be written as a linear combination of terms of the form $(y_o, y_1, \ldots, y_{p-s})$ where each $y_j \in \{X_1, \ldots, X_n\}$ and $y_o \in \langle \underset{\sim}{n} \rangle^s$; hence $(y_o, y_1, \ldots, y_{p-s})$ is an element of $\langle \underset{\sim}{n} \rangle^p$ and is a scalar multiple of $\omega(Y^J)$ for some Y^J. Thus $x_1 x_2 \ldots x_p$ is a linear combination of terms $\omega(Y^J)$ which belong to $\langle \underset{\sim}{n} \rangle^p$.

As $\langle \underset{\sim}{n} \rangle^m$ is spanned by elements of the form $x_1 x_2 \ldots x_p$ with $p \geq m$ and each $x_j \in \underset{\sim}{n}$, the result follows.

COROLLARY. _Each_ of $q_1, \ldots, q_n$ _belongs to_ $U(\underset{\sim}{n})^\circ$.

Proof. As $\underset{\sim}{n}$ is nilpotent $\bigcap \langle \underset{\sim}{n} \rangle^m = 0$, so given any X_i, there exists m with $X_i \notin \langle \underset{\sim}{n} \rangle^m$. Now by the Lemma q_i vanishes on $\langle \underset{\sim}{n} \rangle^m$, so belongs to $U(\underset{\sim}{n})^\circ$.

3.5 The following notation is standard. For a multi-index $J = (j_1, \ldots, j_n)$ put $|J| = j_1 + \ldots + j_n$, put $J! = j_1! \ldots j_n!$. If $I = (i_1, \ldots, i_n)$ write $I \leq J$ if $i_1 \leq j_1, \ldots, i_n \leq j_n$ and put $J - I = (j_1 - i_1, \ldots, j_n - i_n)$. If $J \leq I$ put $\binom{J}{I} = \binom{j_1}{i_1} \ldots \binom{j_n}{i_n}$. Write e_p for $(0, \ldots, 0, 1, 0, \ldots, 0)$, the index with 1 in the p-th position and zeroes elsewhere. Define $\delta_{IJ} = 1$ if $I = J$ and 0 otherwise. Write $q^J = q_1^{j_1} \ldots q_n^{j_n}$.

LEMMA $q^J(\omega(Y^K)) = \delta_{JK} J!$

Proof. If $|J| = 1$ this is true by the definition of the q_i. The Lemma is proved by induction on $|J|$. Suppose the result is true for all q^L with $1 \leq |L| < |J|$. We write $q^J = q_p q^L$ with $p \in \{1, 2, \ldots, n\}$ and $L = J - e_p$. Now consider $\omega(Y^K)$.

If $|K| = 1$ then $Y^K = Y_r$ for some $r \varepsilon \{1,2,\ldots,n\}$, and $q^J(\omega(Y^K)) = q_p q^L(X_r) = q_p q^L(X_r) = (q_p \otimes q^L)\Delta(X_r)$ $= q_p(X_r)q^L(1) + q_p(1)q^L(X_r) = 0$. So assume $|K|>1$. Then Y^K $= Y_r Y^M$ for some $r \varepsilon \{1,2,\ldots,n\}$ where $M = K - e_r$.

Now $q^J(\omega(Y^K)) = q_p q^L(\omega(Y_r Y^M)) = (q_p \otimes q^L)(\Delta\omega(Y_r Y^M))$. Denote the comultiplication in $S(\underset{\sim}{n})$ by Δ'; that is $\Delta'(Y_j)$ $= Y_j \otimes 1 + 1 \otimes Y_j$ for all $j = 1,2,\ldots,n$. As remarked in [D, §2.8.13], $(\omega \otimes \omega)\Delta' = \Delta\omega$. Using this and the fact that $\Delta'(Y^M) = \sum_{I \leq M} \binom{M}{I} Y^I \otimes Y^{M-I}$ we have

$$q^J(\omega(Y^K)) = \sum_{I \leq M} \binom{M}{I} \{q_p(\omega(Y_r Y^I))q^L(\omega(Y^{M-I})) + q_p(\omega(Y_r Y^{M-I}))\}$$

Applying the induction hypothesis to q^L, the first term will only make a non-zero contribution when $I = 0$, $r = p$ and $M = L$. So the first part of the sum equals $\delta_{pr}\delta_{ML}M!$

The second term will only make a non-zero contribution when $I = e_p$ (so we would require $e_p \leq M$) and $L = M - e_p + e_r$. So, if $m_p = 0$ the second term makes no contribution to the sum; and if $m_p \geq 1$ the second term contributes $\binom{M}{e_p} \delta_{L, M-e_p+e_r} L!$ Notice that $\delta_{L,M-e_p+e_r} = \delta_{JK}$. Hence

$$q^J(\omega(Y^K)) = \begin{cases} \delta_{pr}\delta_{ML}M! & \text{if } m_p = 0 \\ \\ \delta_{pr}\delta_{ML}M! + \binom{M}{e_p}\delta_{JK}L! & \text{if } m_p \neq 0 \end{cases}$$

It is rather tedious to give the deails but both these expressions are identical to $\delta_{JK}\, J!$; we leave the details to the reader.

3.6 THEOREM. $U(\underset{\sim}{n})^\circ = k[q_1,\ldots,q_n]$.

Proof. We have already seen that $k[q_1,\ldots,q_n] \subset U(\underset{\sim}{n})^\circ$. The Lemma just proved may be interpreted as saying that the set $\{q^J \mid \omega(Y^J) \notin \langle\underset{\sim}{n}\rangle^m\}$ forms a basis (on restriction) for $(U(\underset{\sim}{n})/\langle\underset{\sim}{n}\rangle^m)^*$, because the Lemma implies that the images of $\{\omega(Y^J) \mid \omega(Y^J) \notin \langle\underset{\sim}{n}\rangle^m\}$ form a basis for $U(\underset{\sim}{n})/\langle\underset{\sim}{n}\rangle^m$. As $U(\underset{\sim}{n})^\circ$ may be realized as the direct limit of the $(U(\underset{\sim}{n})/\langle\underset{\sim}{n}\rangle^m)^*$, one sees that the q^J form a basis for $U(\underset{\sim}{n})^\circ$.

Although there are easier ways to prove it we have shown that $U(\underset{\sim}{n})^\circ$ is a polynomial ring (the q_j are algebraically independent by Lemma 3.5).

3.7 The ring of differential operators on $k[N]$ will be denoted by $A = k[q_1,\ldots,q_n,p_1,\ldots,p_n]$ where $p_j = \partial/\partial q_j$, so that $[q_i,q_j] = \delta_{ij}$. The map $i_2 \colon U(\underset{\sim}{n}) \to A$ is given by $i_2(X) = \sum_{j=1}^{n} X(q_j)p_j$ for $X \in \underset{\sim}{n}$.

The ring of differential operators on $S(\underset{\sim}{n})$ will be denoted by $B = k[Y_1,\ldots,Y_n,\partial_1,\ldots,\partial_n]$ where $\partial_j = \partial/\partial Y_j$ so that $[\partial_i,Y_j] = \delta_{ij}$. For each X_j put $A_j = \mathrm{Ad}X_j$ considered as acting on $S(\underset{\sim}{n})$. Define $s(A_{i_1},\ldots,A_{i_m}) = \frac{1}{m!} \sum_{\sigma\in S_m} A_{i_{\sigma 1}}\cdots A_{i_{\sigma m}}$. We denote by b_i the Bernoulli numbers, defined by $\sum_{i\geq 0} b_i x^i = x(e^x-1)^{-1}$. For each multi-index $I = (i_1,\ldots,i_n)$ we define $b_I = b_{|I|}|I|!(I!)^{-1}$.

For $X \in \underset{\sim}{n}, a \in S(\underset{\sim}{n})$ one has $X\omega(a) = \sum_I b_I \omega(s(A^I)(\omega^{-1}(X))\partial^I(a))$. Hence if the action of X on $S(\underset{\sim}{n})$ is given by $X.a = \omega^{-1}(X\omega(a))$, we have $i_1 : U(\underset{\sim}{n}) \to B$ given by $i_1(X) = \sum_I b_I\ s(A^I)(\omega^{-1}(X))\partial^I$. Details concerning the above appear in [G], [C],[B].

Let $\psi : A \to B$ be the ring isomorphism given by $\psi(q_j) = \partial_j$ and $\psi(p_j) = -Y_j$.

THEOREM. $\psi i_2 = i_1$.

<u>Proof.</u> Fix $X \in \underset{\sim}{n}$ and put $Y = \omega^{-1}(X)$. We begin by giving a more explicit description for $i_2(X) = \sum_{j=1}^{n} X(q_j)p_j$.

For any $f \in k[N]$, $X(f)$ is the functional on $U(\underset{\sim}{n})$ given by $X(f)(u) = -f(Xu)$. If $X(f) = \Sigma\alpha_I q^I$ for some scalars α_I, then $X(f)(\omega(Y^J)) = \alpha_J J!$ after Lemma 3.5. Hence $X(f) = -\Sigma_I (I!)^{-1} f(X\omega(Y^I))q^I$. We want to consider the case when $f = q_j$.

Consider $q_j(\omega(s(A^K)(Y)\partial^K(Y^I)))$; as $s(A^K)(Y)$ is a linear combination of $Y_1,..,Y_n$ and q_j vanishes on homogeneous terms of degree greater than 1, a necessary condition for this expression to be non-zero is that $\partial^K(Y^I)$ be a non-zero scalar. This can only happen if $K = I$, and then $\partial^I(Y^I) = I!$. Hence $q_j(X\omega(Y^I)) = q_j(\sum_K b_K\omega(s(A^K)(Y)\partial^K(Y^I)))$ $= b_I I! q_j(\omega(s(A^I)(Y)))$. Hence

$$i_2(X) = -\sum_I b_I \sum_{j=1}^{n} q_j(\omega(s(A^I)(Y)))q^I p_j, \text{ and}$$

$$\psi i_2(X) = \sum_I b_I \partial^I \sum_{j=1}^{n} q_j(\omega(s(A^I)(Y)))Y_j.$$

But, when the q_j are viewed as functionals on $kY_1 + \ldots + kY_n$ then $q_1,\ldots,q_n$ is the dual basis to $Y_1,\ldots,Y_n$. Consequently $\psi i_2(X) = \sum_I b_I \partial^I s(A^I)(Y)$.

Comparing this with the expression for $i_1(X)$ given prior to the Theorem, the proof will be complete once we have shown that ∂^I and $s(A^I)(Y)$ commute. This is a consequence of the way the basis for $\underset{\sim}{n}$ was chosen. Consider the A_i as acting on $\underset{\sim}{n}$. If A_i appears in A^I then $A^I(X) \in \underset{\sim}{n}_{i+1}$ since $[X_i,\underset{\sim}{n}] \subset [\underset{\sim}{n}_i, \underset{\sim}{n}] \subset \underset{\sim}{n}_{i+1}$. In particular, if A_i appears in A^I, then writing $s(A^I)(Y)$ as a linear combination of $Y_1,\ldots,Y_n$, the coefficient of Y_i is zero. In other words, if ∂_i appears in ∂^I, then Y_i does not appear in $s(A^I)(Y)$. Hence ∂^I and $s(A^I)(Y)$ commute.

4. The Conze Embedding

4.1 We return to the themes of Section 2. Denote by $j_\lambda : D_\lambda \to A$ the restriction map $\Gamma(X,\mathcal{D}_\lambda) \to \Gamma(V,\mathcal{D}_\lambda)$. Denote by B the ring of differential operators on $S(\mathfrak{n})$. We adopt the notation in §3 for the generators of A and B, and define $\psi: A \to B$ as in §3. First observe that the restriction of j_λ to $U(\mathfrak{n})$ is just the map $i_2 : U(\mathfrak{n}) \to A$ given in §3. So ψj_λ restricted to $U(\mathfrak{n})$ coincides with $i_1 : U(\mathfrak{n}) \to B$.

Let L be the left ideal $Aq_1 + \ldots + Aq_n$. Then $B/\psi(L)$ may be identified with $S(\mathfrak{n})$ and after the next Lemma we are able to describe the action of $\mathfrak{g}$ on $S(\mathfrak{n})$ that is induced by the map $\psi j_\lambda : D_\lambda \to B$.

4.2 LEMMA. *Let* $X \in \mathfrak{b}^-$. *Suppose* $1 \otimes X + \Sigma f_\gamma \otimes X_\gamma \in \mathfrak{b}_V^\circ$ *where the sum is over* $\gamma \in R^+$.

(i) *If* $X \in \mathfrak{n}^-$, *then for each* $\gamma \in R^+$, $X_\gamma(f_\gamma) \in \underline{m}$.

(ii) *If* $X \in \mathfrak{h}$, *then for each* $\gamma \in R^+$, $X_\gamma(f_\gamma) = \gamma(X)$.

Proof. We already know each $f_\gamma \in \underline{m}$ by Lemma 2.9.

For any $\beta \in R^+$ the following is an element of the ideal $\mathfrak{b}_V^\circ$ of $\mathcal{O}_V \otimes \mathfrak{g}$:

$$[1 \otimes X + \Sigma f_\gamma \otimes X_\gamma, 1 \otimes X_\beta] = 1 \otimes [X, X_\beta] + \Sigma\{f_\gamma \otimes [X_\gamma, X_\beta] - X_\beta(f_\gamma) \otimes X_\gamma\} \quad (*)$$

To prove (i) it is enough to prove it for $X = X_{-\alpha}, \alpha \in R^+$. If $[X_{-\alpha}, X_\beta] \in \mathfrak{b}^-$ then $1 \otimes [X_{-\alpha}, X_\beta] \in \mathfrak{b}_V^\circ + \underline{m} \otimes \mathfrak{n}$ by Lemma 2.9. So from (*) we have $\Sigma X_\beta(f_\gamma) \otimes X_\gamma \in \underline{m} \otimes \mathfrak{n}$. In particular each $X_\beta(f_\gamma) \in \underline{m}$. So $X_\beta(f_\beta) \in \underline{m}$.

Consider now the possibility that $[X_{-\alpha}, X_\beta] \in \underset{\sim}{n}$. If it is zero it is already in $\underset{\sim}{b}^-$, so assume $[X_{-\alpha}, X_\beta]$ is a non-zero scalar multiple of $X_{\beta-\alpha} \in \underset{\sim}{n}$. In this case the element in (*) is in $(\mathcal{O}_V \otimes \underset{\sim}{n}) \cap \underset{\sim}{b}_V^\circ$ so must equal zero. Now $1 \otimes [X, X_\beta]$ is not in $\underline{m} \otimes \underset{\sim}{n}$ so for it to cancel in (*) we must have $X_\beta(f_{\beta-\alpha}) \otimes X_{\beta-\alpha}$ equal to $1 \otimes [X_{-\alpha}, X_\beta]$ i.e. $X_\beta(f_{\beta-\alpha})$ is a non-zero scalar. And for the other terms to cancel we must have $X_\beta(f_\gamma) \in \underline{m}$ (for $\gamma \neq \beta-\alpha$). In particular $X_\beta(f_\beta) \in \underline{m}$. As $\beta \in R^+$ was arbitrary we have $X_\beta(f_\beta) \in \underline{m}$ for all $\beta \in R^+$.

Turning to (ii), one has $[X, X_\beta] = \beta(X) X_\beta \in \underset{\sim}{n}$. As remarked above, for $1 \otimes [X, X_\beta]$ to cancel it is necessary that $X_\beta(f_\beta) \otimes X_\beta = 1 \otimes [X, X_\beta]$, hence $\beta(X) = X_\beta(f_\beta)$.

4.3 PROPOSITION. <u>Consider</u> $S(\underset{\sim}{n})$ <u>as a</u> $\underset{\sim}{g}$<u>-module through the embedding</u> $\psi j_\lambda : D_\lambda \to B$. <u>Then</u> $S(\underset{\sim}{n})$ <u>has the following properties:</u>

(i) <u>The action of</u> $U(\underset{\sim}{n})$ <u>on</u> $S(\underset{\sim}{n})$ <u>is that obtained through the symmetrisation map.</u>

(ii) $\underset{\sim}{n}^- . 1 = 0$

(iii) <u>If</u> $H \in \underset{\sim}{h}$ <u>then</u> $H.1 = (\lambda+\rho)(H)$

<u>Proof.</u> (i) follows from Theorem 3.7 because $j_\lambda | U(\underset{\sim}{n})$ equals i_2, hence $\psi j_\lambda | U(\underset{\sim}{n})$ equals i_1, in the terminology of §3.

Let $X \in \underset{\sim}{b}^-$ and choose $f_\gamma \in \underline{m}$ such that $1 \otimes X + \sum_\gamma f_\gamma \otimes X_\gamma \in \underset{\sim}{b}_V^\circ$ the sum being over $\gamma \in R^+$. The image of $1 \otimes X + \Sigma f_\gamma \otimes X_\gamma$ in A is $(\lambda-\rho)^\circ p(1 \otimes X + \Sigma f_\gamma \otimes X_\gamma) = (\lambda-\rho)^\circ p(1 \otimes X)$. So the image of $1 \otimes X$ in A equals $(\lambda-\rho)^\circ p(1 \otimes X) - \Sigma f_\gamma X_\gamma = (\lambda-\rho)^\circ p(1 \otimes X)$

$- \Sigma(X_\gamma f_\gamma - X_\gamma(f_\gamma))$. Now the action of $\underset{\sim}{g}$ on $S(\underset{\sim}{n}) = B/\psi(L)$ coincides with the action of $\underset{\sim}{g}$ on A/L. As each $f_\gamma \varepsilon L$, the action of X on $\bar{1} \varepsilon A/L$ is $X.\bar{1} = ((\lambda - \rho)^\circ p(1 \otimes X) + X_\gamma(f_\gamma)).\bar{1}$.

If $X \varepsilon \underset{\sim}{n}^-$ then by Corollary 2.9, $(\lambda - \rho)^\circ p(1 \otimes X).\bar{1} = 0$ and by Lemma 4.2, $X_\gamma(f_\gamma).\ \bar{1} = 0$ also. Thus $X.\bar{1} = 0$, and this gives (ii).

If $X \varepsilon \underset{\sim}{h}$ then $X.\bar{1} = ((\lambda - \rho)(X) + \Sigma\gamma(X)).\bar{1}$ by Lemma 4.2. But $\Sigma\gamma(X) = 2\rho(X)$, so $X.\bar{1} = (\lambda + \rho)(X)\bar{1}$. This gives (iii).

4.4 Denote by τ an automorphism of $\underset{\sim}{g}$ which is induced by $w_o \varepsilon W$. That is $\tau(kX_\alpha) = kX_{w_o\alpha}$ for all $\alpha \varepsilon R$ and $\tau(H_\alpha) = H_{w_o\alpha}$.

THEOREM. $\psi j_\lambda \tau = i_{w_o\lambda}$

Proof. After the proposition it is clear that if we view B as the ring of differential operators on $S(\underset{\sim}{n}^-)$ (through the isomorphism $\tau : S(\underset{\sim}{n}) \to S(\underset{\sim}{n}^-)$) then the action of $\underset{\sim}{g}$ on $S(\underset{\sim}{n}^-)$ (obtained through $\psi j_\lambda \tau : D_\lambda \to B$) satisfies the following:

(i) the $U(\underset{\sim}{n}^-)$ action on $S(\underset{\sim}{n}^-)$ is that obtained by the symmetrisation map,

(ii) $\underset{\sim}{n}.1 = 0$,

(iii) if $H \varepsilon \underset{\sim}{h}$ then $H.1 = (\lambda + \rho)(w_o H) = (w_o\lambda - \rho)(H)$. So the action on $S(\underset{\sim}{n}^-)$ makes it isomorphic to $M(w_o\lambda)$, and the action of $U(\underset{\sim}{n}^-)$ is that obtained from the symmetrisation map. This is all that is required to show that $\psi j_\lambda \tau = i_{w_o\lambda}$.

Remark. The reader will have realised that τ is not well-defined; τ is only determined up to a "scalar multiple" (that is, an automorphism of $\mathfrak{g}$ which is the identity on $\mathfrak{h}$ and sends each X_α to a scalar multiple of itself). The subsequent ambiguity in the statement of Theorem 4.4 is, however, resolved by realising that a similar problem occurs in the definition of the Conze embedding; viz. the Conze embedding obtained through the action of $\mathfrak{g}$ on $M(w_0\lambda)$ depends on the choice of a highest weight vector (which is only defined up to multiplication by a scalar) when defining the isomorphism $M(w_0\lambda) \cong U(\mathfrak{n}^-)$.

4.5 For each $w \in W$, put $V_w = wBw_oB$, and put $A_w = \Gamma(V_w, \mathcal{D}_\lambda)$. Let $j_w : D_\lambda \to A_w$ be the restriction map.

PROPOSITION. Put $S = \oplus A_w$. The diagonal map $D_\lambda \to S$ obtained from the restriction maps makes S faithfully flat as a right D_λ-module.

Proof. Suppose we can show for each open affine $U \subset G/B$, that $\Gamma(U, \mathcal{D}_\lambda \otimes M) = \Gamma(U, \mathcal{D}_\lambda) \otimes_{D_\lambda} M$.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of D_λ-modules. Put $\mathcal{M}_i = \mathcal{D}_\lambda \otimes M_i$. The Beilinson-Bernstein equivalence of categories shows that, for each w,

$$0 \to \Gamma(V_w,\mathcal{M}_1) \to \Gamma(V_w,\mathcal{M}_2) \to \Gamma(V_w,\mathcal{M}_3) \to 0$$

is exact. But, by the first paragraph, $\Gamma(V_w,\mathcal{M}_i) = A_w \otimes_{D_\lambda} M_i$, and so A_w is flat as a right D_λ-module. To prove the "faithfulness", suppose M is a left D_λ-module with $S \otimes_{D_\lambda} M = 0$; by the first paragraph, $\Gamma(V_w, \mathcal{D}_\lambda \otimes M) = 0$, for all w. Hence $\mathcal{D}_\lambda \otimes M = 0$, and so by the equivalence of categories M = 0.

We now prove the statement at the beginning of the proof. Consider the presheaf $\mathcal{D}_\lambda|_U \otimes M$; we claim this is already a sheaf. Put $D_U = \Gamma(U, \mathcal{D}_\lambda)$ and $\mathcal{O}_U = \Gamma(U, \mathcal{O})$. Consider $D_U \otimes_{D_\lambda} M$ as a left $\mathcal{O}_U$-module and let $D_U \otimes M$ be the associated sheaf on the (open affine) set U. By the definition of $D_U \otimes M$, for any open affine $W \subset U$, $\Gamma(W, D_U \otimes M) = \Gamma(W, \mathcal{O}) \otimes_{\mathcal{O}_U} D_U \otimes_D M$. However, as $\mathcal{D}_\lambda$ is quasi-coherent, this equals $\Gamma(W, \mathcal{D}_\lambda) \otimes M$; and hence the presheaf $\mathcal{D}_\lambda|_U \otimes M$ equals the sheaf $D_U \otimes M$.

If $\mathcal{F}$ is a presheaf on G/B and $\mathcal{F}^+$ denotes the associated sheaf then for any open affine U, $\mathcal{F}^+|_U = (\mathcal{F}|_U)^+$. Thus $\mathcal{D}_\lambda \otimes M|_U = \mathcal{D}_\lambda|_U \otimes M$, and consequently $\Gamma(U, \mathcal{D}_\lambda \otimes M) = \Gamma(U, \mathcal{D}_\lambda) \otimes M$ as required. □

4.6 COROLLARY. <u>The Conze embedding</u> $i_{w_o\lambda} : D_\lambda \to A_n$ (<u>where</u> A_n <u>is realised as the differential operators on</u> $S(\underset{\sim}{n}^-) \simeq M(w_o\lambda)$) <u>makes</u> A_n <u>flat as a right</u> D_λ-<u>module</u>.

<u>Remark</u>. A direct proof of this is given [JS].

<u>Proof</u>. This follows at once from Theorem 4.4 and Proposition 4.5 as $i_{w_o\lambda} = \psi j_\lambda \tau$ with ψ and τ both isomorphisms. □

References

[BB] A. BEILINSON and J.N. BERNSTEIN, Localisation de $\mathfrak{g}$-modules, C.R. Acad. Sci 292 (1981) 15-18

[B] F.A. BEREZIN, Some remarks on the associative envelope of a Lie algebra, Funk. Anal. i Priloz. 1 (1967) 1-14

[C] N. CONZE, Algebres d'opérateurs differentiels et quotients des algèbres enveloppantes, Bull. Soc. Math. France 102 (1974) 379-415

[D] J. DIXMIER, Enveloping Algebras, North-Holland Math. Library, Amsterdam (1977)

[G] R. GOODMAN, Differential operators of finite order on a Lie Group II, Indiana Math. J. 21 (1971) 383-409

[H] G.P. HOCHSCHILD, Basic Theory of Algebraic Groups and Lie Algebras, Springer-Verlag, New York (1981)

[JS] A. JOSEPH and J. T. STAFFORD, Modules of $\mathfrak{k}$-finite vectors over semi-simple Lie algebras, Preprint.

[S] L. SOLOMON, On the Poincare-Birkhoff-Witt Theorem, J. Combin. Theory, 4 (1968) 363- 375

[W] W.C. WATERHOUSE, Introduction to Affine Group Schemes, Springer-Verlag, New York (1979)

T.J. Hodges
Department of Mathematics,
University of Utah,
Salt Lake City,
Utah 84112,
USA

S.P. Smith
Mathematics Institute
University of Warwick
Coventry CV4 7AL
ENGLAND