Differential Operators on Commutative Algebras.

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Abstract. This article discusses some of the similarities/differences between the theory of differential operators on (a) a non-singular variety in characteristic zero (b) a non-singular variety in positive characteristic (c) a singular variety in characteristic zero.

§1. Introduction.

The goal is to describe some of the ring theoretic structure of the rings of differential operators described in the abstract. Our main interest will be in singular varieties in characteristic zero, and non-singular varieties in characteristic $p > 0$. The theory for differential operators on non-singular varieties in characteristic zero is well developed. We begin by recalling some of this theory, which will provide the background against which the other cases will be viewed.

Let X be a non-singular irreducible affine algebraic variety over an algebraically closed field k of characteristic zero. The ring of differential operators on X, denoted $\mathfrak{D}(X)$, may be defined as follows: denote by A the co-ordinate ring of X (i.e. $A = \mathcal{O}(X)$, the ring of regular functions on X) and define $\mathfrak{D}(X)$ to be the k-subalgebra of $\mathrm{End}_k A$ generated by A (acting on A by multiplication) and $\mathrm{Der}_k A$, the module of k-linear derivations on A. For example, if $X \cong \mathbb{A}^n$, affine n-space, then $\mathfrak{D}(X) \cong k[t_1,..,t_n,\partial_1,..,\partial_n]$ where $\partial_j = \partial/\partial t_j$, the partial derivative with respect to t_j. The following properties hold for any such X (some details may be found in Bjork's book [2])

(a) $\mathfrak{D}(X)$ is a simple, noetherian, domain, finitely generated as a k-algebra;

(b) $\mathfrak{D}(X)$ is a filtered algebra, filtered by the order of the differential operators and the associated graded algebra is $\mathrm{gr}\,\mathfrak{D}(X) \cong S_A(\mathrm{Der}_k A)$, the symmetric algebra of the A-module $\mathrm{Der}_k A$;

(c) The global homological dimension of $\mathfrak{D}(X)$ is $\mathrm{gl.dim}\ \mathfrak{D}(X) = \dim X$.

As yet non-commutative algebraists do not have sufficient techniques to penetrate the mysteries of simple, noetherian domains. For example, one would conjecture that if X and Y are non-isomorphic curves then $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ are non-isomorphic k-algebras - but this question remains wide open (of course the real question is to allow X and Y to be of any dimension, but why add insult to injury!).

In this article we will neatly sidestep this difficulty by considering singular varieties over a field of characteristic zero, and non-singular varieties over a field of positive characteristic. As this conference is primarily for ring theorists, we hope to convince the audience/reader that the rings of differential operators on such varieties are worthy of their interest.

§2. Singular Varieties.

The results on singular varieties in this section are joint work with J.T. Stafford [8]. Many of these results have also been obtained independently by J. Muhasky and will appear in his Ph.D. Thesis.

The definition of $\mathfrak{D}(X)$ given in §1 for X non-singular, char $k = 0$ is not the appropriate definition when X is singular, or when char $k = p > 0$. We begin by giving the appropriate definition (for any commutative k-algebra A) of $\mathfrak{D}(A)$, the ring of k-linear differential operators on A. Not surprisingly if X is as in §1, and $A = \mathcal{O}(X)$ then $\mathfrak{D}(A) = \mathfrak{D}(X)$.

Let k be any commutative ring, A any commutative k-algebra. For M,N any A-modules, give $\mathrm{Hom}_k(M,N)$ an $A \otimes_k A$-module structure by $(a \otimes b)\theta(m) = a\theta(bm)$ for $a,b \in A$, $\theta \in \mathrm{Hom}_k(M,N)$, $m \in M$. Denote by μ

the multiplication map $\mu: A \otimes_k A \to A$, $\mu(a \otimes b) = ab$. This is a k-algebra homomorphism so the kernel, J say, is an ideal. It is easily shown that J is generated as an ideal by $\{1 \otimes a - a \otimes 1 \mid a \in A\}$.

Definition 1. For $n \geq -1$ define $\mathfrak{D}_A{}^n(M,N)$, the *space of k-linear differential operators from* M *to* N *of order* $\leq n$, by

$$\mathfrak{D}_A{}^n(M,N) = \{\theta \in \mathrm{Hom}_k(M,N) \mid J^{n+1}.\theta = 0\} .$$

Write $\mathfrak{D}_A(M,N) := \bigcup_{n \geq 0} \mathfrak{D}_A{}^n(M,N)$, for the space of differential operators from M to N.

We shall drop the subscript A from $\mathfrak{D}_A$ whenever convenient. It is clear that $\mathfrak{D}^n(M,N) \subset \mathfrak{D}^{n+1}(M,N)$, and $\mathfrak{D}^{-1}(M,N) = 0$. Observe that $\theta \in \mathfrak{D}^0(M,N)$, if and only if, $(1 \otimes a - a \otimes 1)\theta = 0$ for all $a \in A$ (as J is generated by such elements). This is equivalent to $(1 \otimes a - a \otimes 1)\theta(m) = 0$ for all $m \in M$, and from the definition of the $A \otimes_k A$ action this is saying that $a\theta(m) = \theta(am)$ for all $a \in A$, $m \in M$; that is, $\mathfrak{D}^0(M,N) = \mathrm{Hom}_A(M,N)$.

In the special case where $M = N$, write $\mathfrak{D}(M) := \mathfrak{D}(M,M)$. It is straightforward to check that this is a k-subalgebra of $\mathrm{End}_k M$, and that $\mathfrak{D}(M,N)$ becomes a $\mathfrak{D}(N)$ - $\mathfrak{D}(M)$ bimodule. The module action comes from the fact that $\mathrm{Hom}_k(M,N)$ is a $\mathrm{End}_k N$ - $\mathrm{End}_k M$ bimodule.

Some work is involved in proving the following:

THEOREM 2.1 ([5], [9]) *Let* k *be an algebraically closed field of characteristic zero, and let* X *be a non-singular irreducible affine variety over* k. *Let* A *be a localisation of* $\mathcal{O}(X)$. *Then* $\mathfrak{D}(A)$ *is generated by* A *and* $\mathrm{Der}_k A$.

It was shown above that $\mathfrak{D}^0(A,A) = A$, and it is an easy exercise to prove that $\mathfrak{D}^1(A,A) = A \oplus \mathrm{Der}_k A$; so one sees that the subalgebra of $\mathrm{End}_k A$ generated by A and $\mathrm{Der}_k A$ lies in $\mathfrak{D}(A)$ for any k, any A.

NOTATION. For the remainder of this section, k will be an algebraically closed field of characteristic zero, and X an irreducible affine variety over k.

Define $\mathcal{D}(X)$, the differential operators on X to be $\mathcal{D}(\mathcal{O}(X))$, where $\mathcal{O}(X)$ is the ring of regular functions on X, and $\mathcal{D}(\mathcal{O}(X))$ is obtained through Definition 1 in the case $M = N = \mathcal{O}(X) = A$. By Theorem 2.1 this agrees for X non-singular with the definition given in §1.

Recall that for X non-singular $\mathcal{D}(X)$ is a (right and left) noetherian finitely generated k-algebra, but this is not necessarily true for X singular. In [1] it is shown that if X is the zeroes over $\mathbb{C}$ of $X_1^3 + X_2^3 + X_3^3$ then $\mathcal{D}(X)$ is neither noetherian nor finitely generated. In this example $\dim X = 2$, however, we have

THEOREM 2.2 [8] *Let X be a curve. Then*

(a) $\mathcal{D}(X)$ *is (right and left) noetherian, and a finitely generated* k-*algebra;*

(b) $\mathcal{D}(X)$ *has a unique minimal non-zero two sided ideal,* $J(X)$ *say, and* $H(X) := \mathcal{D}(X)/J(X)$ *is a finite dimensional* k-*algebra.*

This theorem is proved by relating $\mathcal{D}(X)$ and $\mathcal{D}(\tilde{X})$ where $\tilde{X}$ denotes the normalisation of X. The morphism $\pi: \tilde{X} \to X$ corresponds to the k-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, where $\mathcal{O}(\tilde{X})$ is the integral closure of $\mathcal{O}(X)$ in its field of fractions. Viewing $\mathcal{O}(X)$ and $\mathcal{O}(\tilde{X})$ as $\mathcal{O}(X)$-modules, the definition above allows us to construct $\mathcal{D}(\tilde{X},X) :- \mathcal{D}_{\mathcal{O}(X)}(\mathcal{O}(\tilde{X}),\mathcal{O}(X))$. More concretely, one may show $\mathcal{D}(\tilde{X},X) = \{D \in \mathcal{D}(\tilde{X}) \mid D(f) \in \mathcal{O}(X)$ for all $f \in \mathcal{O}(\tilde{X})\}$. There is a natural $\mathcal{D}(X) - \mathcal{D}(\tilde{X})$ bimodule structure on $\mathcal{D}(\tilde{X},X)$, where the module action is just composition of maps. Thus, there is a functor $\mathcal{D}(\tilde{X},X) \otimes_{\mathcal{D}(\tilde{X})} - : \mathcal{D}(\tilde{X})\text{-Mod} \to \mathcal{D}(X)\text{-Mod}$. The following holds:

THEOREM 2.3 [8] *Let X be a curve. The functor* $\mathcal{D}(\tilde{X},X) \otimes_{\mathcal{D}(\tilde{X})} -$ *gives an equivalence between the categories* $\mathcal{D}(\tilde{X})$-Mod *and* $\mathcal{D}(X)$-Mod (i.e. $\mathcal{D}(\tilde{X})$ *and* $\mathcal{D}(X)$ *are Morita equivalent), if and only if,* $\pi: \tilde{X} \to X$ *is injective.*

Remark. In fact a little more is true, in that, if π is not injective then $\mathfrak{D}(X)$ is not a simple ring (so cannot be Morita equivalent to $\mathfrak{D}(\tilde{X})$ which is a simple ring).

One step in the proof of Theorem 2.3 is to show that the functor induces an equivalence of categories, if and only if, the natural map $\mathfrak{D}(\tilde{X},X) \otimes \mathcal{O}(\tilde{X}) \to \mathcal{O}(X)$ is surjective, or (what is the same thing) that there exist differential operators $D_\lambda \in \mathfrak{D}(\tilde{X},X)$ and regular functions $f_\lambda \in \mathcal{O}(\tilde{X})$ such that $\sum_\lambda D_\lambda(f_\lambda) = 1$.

Example. This gives an easy case where finding the D_λ and f_λ is child's play. Consider the curve $X \subset \mathbb{A}^2$ given by $y^2 = x^3$. Then $\tilde{X} \cong \mathbb{A}^1$ and $\pi:\mathbb{A}^1 \to X$ is given by $\pi:t \to (t^2,t^3)$. This is injective so such D_λ, f_λ exist. Consider $\mathcal{O}(\tilde{X}) = k[t] \supset \mathcal{O}(X) = k[t^2,t^3]$. Put $D = (t\partial/\partial t-1) \in \mathfrak{D}(\tilde{X},X)$ and observe that $D(-1) = 1$. In this case, $\mathfrak{D}(\tilde{X})$ and $\mathfrak{D}(X)$ are Morita equivalent.

One consequence of Theorem 2.3 is that one loses information about the existence of singularities when passing from X, or $\mathcal{O}(X)$-Mod to $\mathfrak{D}(X)$-Mod. A natural question is whether $\mathfrak{D}(X)$ still retains this information - one would at least like to know that if $\tilde{X} \neq X$ then $\mathfrak{D}(\tilde{X})$ and $\mathfrak{D}(X)$ are not isomorphic. Suppose this is the case; then what structural aspects of $\mathfrak{D}(X)$ reflect the existence of singularities on X? Of course, one can recognise the existence of singularities on X from $\mathcal{O}(X)$ just by determining the global dimension of $\mathcal{O}(X)$; that is, X is non-singular, if and only if, $\mathrm{gl.dim}(X) < \infty$ (this is not a sensible way to see if X has singularities but at least shows how the geometric information is reflected in the algebraic structure of $\mathcal{O}(X)$). Of course, the global dimension of $\mathfrak{D}(X)$ does not retain the necessary information since if $\pi:\tilde{X} \to X$ is injective then $\mathrm{gl.dim}\, \mathfrak{D}(X) = 1$ (by Theorem 2.3 and (c) of §1).

Returning to Theorem 2.2 an obvious question is to determine the structure of the finite dimensional algebra $H(X)$. First, we remark that $J(X) = \mathrm{Ann}_{\mathfrak{D}(X)}(\mathcal{O}(X)/\mathfrak{D}(\tilde{X},X)*\mathcal{O}(\tilde{X}))$ where $\mathfrak{D}(\tilde{X},X)*\mathcal{O}(\tilde{X})$ denotes the image of the natural map $\mathcal{D}(\tilde{X},X) \otimes_{\mathcal{D}(\tilde{X})} \mathcal{O}(\tilde{X}) \to \mathcal{O}(X)$, $D \otimes f \to D(f)$. In particular, if X is non-singular then $\mathcal{D}(X)$ is simple so $H(X) = 0$. But also after Theorem 2.3, if $\pi: \tilde{X} \to X$ is injective then $H(X) = 0$. In fact, as (implicitly) remarked earlier, $H(X) = 0$, if and only if, π is injective.

During the meeting A. Schofield asked whether $H(X)$ was a direct sum of algebras, one for each singular point. This is the case (as is proved in [8]), and thus we write $H(X) = \bigoplus_{x \in \mathrm{Sing} X} H_x$; it can be shown that if $\mathcal{O}_{X,x}$ is the local ring at x, and $\mathcal{D}_{X,x} = \mathcal{D}(\mathcal{O}_{X,x})$, then $\mathcal{D}_{X,x}$ has a unique minimal non-zero ideal J_x and $\mathcal{D}_{X,x}/J_x \cong H_x$. The point is that determining $H(X)$ is a local problem, and the question is to determine how the structure of H_x depends on the nature of the singularity at x. The results in [8] are a long way from answering this question completely and we just mention two examples.

Example 1. Let $\tilde{X} = \mathbb{A}^1$, and $\mathcal{O}(\tilde{X}) = k[t]$. Let X be the curve with $\mathcal{O}(X) = k[t^2, t(t^2-\lambda_1)\dots(t^2-\lambda_n)]$ where $\lambda_1,\dots,\lambda_n$ are distinct non-zero elements of k. In this case $H(X) = k \oplus \dots \oplus k$ a direct sum of n copies of k.

Example 2. Let $\tilde{X} = \mathbb{A}^1$, and $\mathcal{O}(\tilde{X}) = k[t]$. Let X be the curve with $\mathcal{O}(X) = k[t(t-1)(t-2), t^2(t-1)(t-2)] = k + kf + ktf + f^2k[t]$ where $f = t(t-1)(t-2)$. Then (after much computation) one has $H(X) \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$.

Another interesting aspect of Theorem 2.2 is that although $\mathcal{D}(X)$ is finitely generated $\mathrm{gr}\,\mathcal{D}(X)$ need not be. In [8] it is proved that $\mathrm{gr}\,\mathcal{D}(X)$ is finitely generated if and only if $\pi: \tilde{X} \to X$ is injective. The proof is somewhat tricky, but in the special case where $\pi: \tilde{X} \to X$ is unramified at all points it is easy to prove that $\mathrm{gr}\,\mathcal{D}(X)$ is not noetherian (and hence not finitely generated). To start, when π is unramified then $\mathcal{D}(X) \subseteq \mathcal{D}(\tilde{X})$ by [3]. Hence $\mathcal{D}(\tilde{X},X)$ becomes a two-sided ideal of $\mathcal{D}(X)$; however, the endomorphism ring of any $\mathcal{D}(\tilde{X})$-module of finite length is finite dimensional over k by Quillen's Lemma, and hence $\dim_k \mathcal{D}(X)/\mathcal{D}(\tilde{X},X) < \infty$. Consider $R = \operatorname{gr} \mathcal{D}(\tilde{X}) \supset S = \operatorname{gr} \mathcal{D}(X) \supset J = \operatorname{gr} \mathcal{D}(\tilde{X},X)$, inside the commutative k-algebra $\operatorname{gr} \mathcal{D}(\tilde{X})$. It is an easy exercise to show that because $\dim_k(R/J) = \infty$ and $\dim_k(S/J) < \infty$, then S cannot be noetherian. We do not have an explicit description of $\operatorname{gr}\mathcal{D}(X)$ in terms of $\mathcal{O}(X)$ - it would be interesting to have such a description.

Recall that when X is non-singular, then $\operatorname{gr}\mathcal{D}(X) \cong \mathcal{O}(T^*X)$ where T^*X is the cotangent bundle. As we have just said, $\operatorname{gr}\mathcal{D}(X)$ need not be the co-ordinate ring of any affine variety when X is singular, hence it is not possible to give a similar geometric definition of what it means for a module to be holonomic. Is there some "suitable" algebraic definition? If X is non-singular then for $0 \neq f \in \mathcal{O}(X)$, $\mathcal{O}(X)_f$ is a $\mathcal{D}(X)$-module of finite length. Is this true when X is singular?

To end we state a result for higher dimensional varieties.

THEOREM 2.4 *Let X be a singular variety of dimension ≥ 2; suppose that the normalisation $\tilde{X}$, is non-singular, and that* Sing X *is finite. Then $\mathcal{D}(X)$ is a finitely generated* k-*algebra which is right but not left noetherian.*

§3. Positive Characteristic.

The differences between the characteristic zero and positive characteristic theories are striking. Yet so are the similarities. Let us explain by giving two theorems. In this section k denotes an algebraically closed field of characteristic $p > 0$, and X denotes a non-singular, irreducible affine variety over k. Write $A = \mathcal{O}(X)$ and for each $r \geq 0$ define $A_r = \{a^{p^r} \mid a \in A\}$. This is a k-subalgebra of A isomorphic to A.

THEOREM 3.1 ([4], [7]) $\mathcal{D}(X) = \bigcup_{n=1}^{\infty} \operatorname{End}_{A_n} A$.

Notation Write $D_n := \operatorname{End}_{A_n} A$.

THEOREM 3.2 [7] gl.dim $\mathcal{D}(X) = \dim X$.

Certainly Theorem 3.1 has no analogue in characteristic zero and illustrates a substantial difference. A good example to keep in mind is $X = \mathbb{A}^1$ in which case $A = k[t]$ and $A_r = k[t^{p^r}]$. Thus $\mathrm{End}_{A_r} A \cong M_{p^r}(k[t^{p^r}])$ as $k[t]$ is a free $k[t^{p^r}]$-module of rank p^r . More explicitly, if $p = 2$, then $D_0 = k[t]$, $D_1 \cong M_2(k[t^2])$, $D_2 \cong M_4(k[t^4])$ etc., and the inclusions $D_0 \subset D_1 \subset D_2 \subset \ldots$ are easy to describe in terms of basis elements viz.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & t^4 & 0 \\ 0 & 0 & 0 & t^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{etc.}$$

Some of the differences from the characteristic zero theory (which are immediate from Theorem 3.1) are that $\mathcal{D}(X)$ is no longer a domain, $\mathcal{D}(X)$ is not finitely generated, $\mathcal{D}(X)$ is not noetherian (all for $\dim X \geq 1$) .

For example, $\mathcal{D}(X)$ is not finitely generated because any finite set of elements can at best generated some D_n , but it is an easy matter to see that $D_n \neq D_{n+1}$. If $K = \mathrm{Fract}\, A$, then $\mathcal{D}(K)$ is a localisation of $\mathcal{D}(X)$ and similarly $\mathcal{D}(K) = \bigcup_{n=1}^{\infty} \mathrm{End}_{K_n} K$ where $K_n = \{\alpha^{p^n} \mid \alpha \in K\}$. As K is a free K_n-module of rank p^n , $\mathrm{End}_{K_n} K \cong M_{p^n}(K_n)$ so $\mathcal{D}(K)$ is not a domain, hence neither is $\mathcal{D}(X)$. Also the argument of [6 , Corollary 2.2 (4)] shows that $\mathcal{D}(K)$ is not noetherian, hence neither is $\mathcal{D}(X)$.

Theorem 3.2 illustrates one of the similarities with the characteristic zero theory. The characteristic zero proof.makes use of $\mathrm{gr}\,\mathcal{D}(X)$. In characteristic p , $\mathrm{gr}\,\mathcal{D}(X)$ appears to be of little help in understanding $\mathcal{D}(X)$ (for example gl.dim $\mathcal{D}(\mathbb{A}^1) = \infty$) . Instead one makes use of the description of $\mathcal{D}(X)$ given in Theorem 3.1. The following summarises some of the good properties of the D_n .

PROPOSITION 3.3 [7] *For all* $n \in \mathbb{N}$, *the following hold.*

(a) D_n *is Morita equivalent to* A_n, *the progenerator being the* D_n-A_n *bimodule* A [4];

(b) D_{n+1} *is a finitely generated projective (right or left)* D_n*-module and* D_n *is a direct summand of* D_{n+1} *as a* D_n*-module;*

(c) $\mathfrak{D}(X)$ *is a projective (right or left)* D_n*-module;*

(d) *If* $m \geq n$, *then* $\mathrm{Hom}_{D_n}(D_m, D_n)$ *is a rank 1 projective right* D_m*-module*;

(e) *If* M *is a simple left* D_n*-module then* $\mathfrak{D}(X) \otimes_{D_n} M$ *is a simple left* $\mathfrak{D}(X)$*-module.*

One further similarity with the characteristic zero theory is that $\mathfrak{D}(X)$ is a simple ring, and $\mathcal{O}(X)$ is a simple $\mathfrak{D}(X)$-module. Before proving this note that if $0 \neq I$ is an ideal of $\mathfrak{D}(A)$ then $A \cap I \neq 0$. To see this choose $0 \neq D \in I$ of lowest order; if $a \in A$, then $[a,D] \in I$ is of lower order, hence zero by choice of D; but $[a,D] = 0$ for all $a \in A$ implies that D is a multiplication operator.

Proposition 3.4 *If* X *is non-singular then* $\mathfrak{D}(X)$ *is a simple ring.*

Proof. In [6] this is proved for $X \cong \mathbb{A}^1$. Since $\mathfrak{D}(\mathbb{A}^n) \cong \mathfrak{D}(\mathbb{A}^1)^{\otimes n}$ it is easy to see that $\mathfrak{D}(\mathbb{A}^n)$ is also simple.

As $\mathfrak{D}_X$ is a quasi-coherent $\mathcal{O}_X$-module it is enough to show that each stalk $\mathfrak{D}_{X,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}(X)} \mathfrak{D}(X)$ is a simple ring for $x \in X$. As $\mathcal{O}_{X,x}$ is regular local, there is a local system of parameters $t_1,..,t_n$ $(n = \dim X)$ (which we may choose to be elements of $\mathcal{O}(X)$) such that $\Omega_{\mathcal{O}}$, the module of Kahler differentials is free on $dt_1,..,dt_n$. Hence by [5 , §16] $\mathfrak{D}_{X,x}$ is generated by $\mathcal{O}_{X,x}$ and a set of differential operators $\{D_I \mid I = (i_1,...,i_n)\ 0 \leq i < \infty\}$ which satisfy $D_I(t^J) = \binom{J}{I} t^{J-I}$ where $J = (j_1,...,j_n)$ and we are using standard multi-index notation. The point is that $\mathfrak{D}_{X,x}$ contains a copy of $\mathfrak{D}(\mathbb{A}^n)$, namely the subalgebra generated by $k[t_1,...,t_n]$

and all the D_I .

Let $0 \neq I$ be an ideal of $\mathcal{D}_{X,x}$. Then (as argued above) $I \cap \mathcal{O}_{X,x} \neq 0$. Hence $I \cap \mathcal{O}(X) \neq 0$, and consequently $I \cap k[t_1,,...,t_n] \neq 0$. In particular $I \cap \mathcal{D}(\mathbb{A}^n) \neq 0$, so by the simplicity of $\mathcal{D}(\mathbb{A}^n)$ it follows that $1 \in I$. Hence $\mathcal{D}_{X,x}$ is simple as required. $\square$

COROLLARY 3.5 $\mathcal{O}(X)$ *is a simple* $\mathcal{D}(X)$*-module.*

Proof. If not then any proper submodule would be an ideal of $\mathcal{O}(X)$ as $\mathcal{O}(X) \subseteq \mathcal{D}(X)$; if $I \subset \mathcal{O}(X)$ were the proper submodule then $0 \neq I\, \mathcal{D}(X) \subseteq \mathrm{Ann}_{\mathcal{D}(X)}(\mathcal{O}(X)/I)$ would be a proper ideal of the simple ring $\mathcal{D}(X)$. Contradiction. $\square$

Remark. Of course the above proof works for any commutative k-algebra A ; viz. $\mathcal{D}(A)$ simple, implies A is a simple $\mathcal{D}(A)$-module. Is the converse true?

Questions.

Some problems/questions have already been mentioned above. Let us give a few more which relate to the characteristic p theory - so in what follows X,k are as in §3.

1. What is the appropriate definition of a holonomic module? It is tempting to hope that an algebraic rather than a geometric definition is possible viz. M is *holonomic* if $\mathrm{Ext}^i_{\mathcal{D}(X)}(M,\mathcal{D}(X)) = 0$ for all $0 \le i < \dim X$. In characteristic zero this is equivalent to the geometric definition in terms of the dimension of the associated variety [2]. As Bjork pointed out during the meeting, this definition would lead to a "good" theory if question (2) has a positive answer.
2. Is $\mathcal{D}(X)$ a Gorenstein ring? That is, if N is a (right) $\mathcal{D}(X)$-submodule of $\mathrm{Ext}^j_{\mathcal{D}(X)}(M,\mathcal{D}(X))$ for some left $\mathcal{D}(X)$-module M , is $\mathrm{Ext}^i_{\mathcal{D}(X)}(N,\mathcal{D}(X)) = 0$ for all $0 \le i < j$?

3. Does a version of Quillen's Lemma hold? That is, if M is a simple $\mathfrak{D}(X)$-module, is $\mathrm{End}_{\mathfrak{D}(X)}M = k$?

4. If $0 \neq f \in \mathcal{O}(X)$, is $\mathcal{O}(X)_f$ of finite length as a $\mathfrak{D}(X)$-module? This may well require an answer to question (3) as a preliminary.

A curiosity. Take $A = k[t]/(t^n)$. Then $\mathfrak{D}(A) \cong M_n(k)$, the ring of $n \times n$ matrices over k. To see this observe that $\mathrm{End}_k A \cong M_n(k)$ so we only need show for $\theta \in \mathrm{End}_k A$, that $J^r\theta = 0$ for $r >> 0$. As J is generated by $1 \otimes t - t \otimes 1$, and $1 \otimes t$, $t \otimes 1$ are both nilpotent elements in $A \otimes_k A$ it follows that J is a nilpotent ideal. Hence $J^r = 0$ for $r >> 0$ and the result follows.

Another curiosity. The Steenrod Algebra acts as differential operators. This observation was made with J.D.S. Jones, and I am grateful for his allowing me to include it here.

Let F_2 denote the field of two elements, and write $S = F_2[Sq^1, Sq^2, \ldots]$ for the Steenrod algebra - for brevity write $s_j = Sq^j$. The co-product $\Delta: S : \to S \otimes_k S$ is given by $\Delta(s_n) = \sum_{i=0}^{n} s_{n-i} \otimes s_i$ where $s_0 = 1$. For $x \in S$, adopt Sweedler's notation and write $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$. Suppose that A is a commutative F_2-algebra which is an S-module, such that for all $x \in S$, $a \in A$, $b \in A$ one has

$x(ab) = \sum_{(x)} x_{(1)}(a)\ x_{(2)}(b)$. Then we claim that the algebra homomorphism $S \to \mathrm{End}_F\ A$, actually has its image inside $\mathfrak{D}(A)$.

To prove this notice first that $\Delta(s_1) = s_1 \otimes 1 + 1 \otimes s_1$, and thus $s_1(ab) = s_1(a)b + as_1(b)$, hence s_1 is a derivation on A. By induction on n, show $s_n \in \mathfrak{D}^n(A)$. Notice that $s_n(ab) = \sum_{i=1}^{n-1} s_{n-i}(a)s_i(b) + as_n(b)$, whence $[a,s_n](b) = -\sum_{i=0}^{n-1} s_{n-i}(a)s_i(b)$. That is, for all $a \in A$, $[a,s_n] = -\sum_{i=0}^{n-1} s_{n-i}(a)s_i$; by induction this element is in $\mathfrak{D}^{n-1}(A)$, and hence $s_n \in \mathfrak{D}^n(A)$.

As concrete examples take the action of S on $H^*(\mathbb{R}P^\infty, F_2) = F_2[u]$, the polynomial ring in u. The action of S is given by $s_n = Sq^n : u^m \to \binom{m}{n} u^{m+n}$, and it is immediate that this action coincides with that of $u^{2n}(1/n!)\partial^n/\partial u^n$. This explicitly describes the algebra homomorphism, $S \to \mathfrak{D}(F_2[u])$.

The action of S on $H^*(X,F_2)$ where X is a product of countably many copies of $\mathbb{R}P^\infty$, is a faithful representation of S. In this case $H^*(X,F_2) = F_2[u_1,u_2,\dots]$, the polynomial ring on countably many indeterminates. Here the algebra homomorphism is given by $Sq^n \to \sum_{|I|=n} u^{2I}\partial_I$, where $I = (i_1,i_2,\dots)$, and $u^{2I} = u_1^{2i_1} u_2^{2i_2}\dots$, and $|I| = i_1 + i_2 + \dots$, and $\partial_I = (1/i_1!i_2!\dots)\,\partial^{i_1+i_2+\dots}/\partial u_1^{i_1}\partial u_2^{i_2}\dots$.

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