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OVERRINGS OF PRIMITIVE FACTOR RINGS OF $U(\mathrm{sl}(2,\mathbb{C}))$

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This paper shows that certain primitive factor rings of $U(\mathrm{sl}(2,\mathbb{C}))$ embed in the rings of differential operators $\mathscr{D}$ on the curves $y^2=x^{2n+1}$. There is an action of $\mathrm{SL}(2,\mathbb{C})$ as automorphisms of $\mathscr{D}$ making $\mathscr{D}$ a $(\mathrm{sl}(2)\times \mathrm{sl}(2),\mathrm{SL}(2))$ Harish–Chandra bimodule in such a way that the invariant subring under the centre of SL(2), $\mathscr{D}^{\mathbb{Z}_2}$, is the primitive factor of $U(\mathrm{sl}(2))$. This result describes all the Dixmier algebras for $\mathrm{SL}(2,\mathbb{C})$.

Introduction

We consider a special case of a problem posed by Vogan [11]. Let G be a connected semi-simple algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. Let J be a completely prime primitive ideal in $U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$. Let $\mathscr{A}$ be the class of all completely prime associative $\mathbb{C}$-algebras A equipped with

(1) an algebra homomorphism $\varphi: U(\mathfrak{g})\to A$ with $\ker\varphi=J$, making A a finitely generated right (and left) $U(\mathfrak{g})$-module;

(2) a locally finite action of G on A as automorphisms, the differential of which agrees with the adjoint action of $\mathfrak{g}$ induced by φ.

The problem is to classify the objects in $\mathscr{A}$. Vogan proposes that the algebras A should be classified by certain coverings of the coadjoint orbits.

For $G=\mathrm{SL}(2)$ we describe the algebras A in $\mathscr{A}$. For $n\in\mathbb{N}$, let X_{2n+1} be the plane curve defined by the equation $y^2=x^{2n+1}$. Let $\mathscr{D}(X_{2n+1})$ denote the ring of differential operators on this curve (see [10]). Suppose that A is in $\mathscr{A}$. If $\dim_{\mathbb{C}} A=\infty$, and φ is not surjective (this eliminates the trivial cases), then $A\cong\mathscr{D}(X_{2n+1})$ for some n. This is our main result (see Proposition 1.3, Corollary 1.6 and Theorem 3.1).

The case $n=0$ will be familiar. When $n=0$, X_{2n+1} is just the affine line $\mathbb{A}^1$. Write $\mathscr{D}(\mathbb{A}^1)\cong\mathbb{C}(t,\partial]$ where t is an indeterminate and $\partial=\mathrm{d}/\mathrm{d}t$. There is an action of the group $\mathbb{Z}_2$ as automorphisms of $\mathscr{D}(\mathbb{A}^1)$ with the non-identity element of $\mathbb{Z}_2$ acting by $t\mapsto -t$, $\partial\mapsto -\partial$. The ring of invariants is $\mathscr{D}(\mathbb{A}^1)^{\mathbb{Z}_2}=\mathbb{C}[t^2,t\partial,\partial^2]$, which is isomorphic to a primitive factor ring of $U(\mathrm{sl}(2,\mathbb{C}))$. The action of SL(2) on $\mathbb{C}t\oplus\mathbb{C}\partial$ extends to an algebra automorphism of $\mathscr{D}(\mathbb{A}^1)$ in such a way that the $\mathbb{Z}_2$-action is as described.

The general problem of Vogan (and a more precise description of the problem) is considered by McGovern in [7]. He calls the algebras A satisfying the above conditions *Dixmier algebras*. Thus, we describe all the Dixmier algebras for $SL(2,\mathbb{C})$. The Dixmier algebras are a slightly wider class than the completely prime primitive factor rings, and one expects that from a ring-theoretic point of view they will have many properties in common with factors of enveloping algebras. Furthermore, since the primitive factor ring is a fixed ring of the Dixmier algebra under a finite group action, this is an interesting context in which to consider finite group actions on noncommutative algebras.

Fix a basis e,f,h for $sl(2,\mathbb{C})$ with relations $[e,f]=h$, $[h,e]=2e$ and $[h,f]=-2f$. As usual $M(\lambda)$ denotes the Verma module of highest weight $\lambda-\frac{1}{2}\alpha$ where α is the simple root for sl(2), $\alpha(h)=2$, $L(\lambda)$ denotes the simple quotient of $M(\lambda)$, and $J(\lambda)$ denotes the annihilator of $L(\lambda)$.

In Section 1 we describe the rings $\mathscr{D}(X_{2n+1})$, the subring which is isomorphic to $U(sl(2))/J(\frac{1}{4}(2n+1)\alpha)$, and the action of SL(2) as automorphisms of $\mathscr{D}(X_{2n+1})$. The action of $\mathscr{D}(X_{2n+1})$ on $\mathscr{O}(X_{2n+1})$, the coordinate ring of the curve, makes $\mathscr{O}(X_{2n+1})\cong M(\lambda)\oplus M(-\lambda)$ as an sl(2)-module where $\lambda=\frac{1}{4}(2n+1)\alpha$. It is proved that as a $U(sl(2))$-bimodule, $\mathscr{D}(X_{2n+1})\cong L(M(\lambda),M(\lambda))\oplus L(M(\lambda),M(-\lambda))$. We describe the associated graded rings of the $\mathscr{D}(X_{2n+1})$ (coming from the filtration by order of differential operators) and of $U(sl(2))/J(\frac{1}{4}(2n+1)\alpha)$. This inclusion of commutative rings corresponds to an SL(2)-equivariant covering of the cone of nilpotent elements in $sl(2,\mathbb{C})$. This is the geometric part of Vogan's problem for SL(2).

Section 2 gives some preliminary results on overrings of primitive quotients of $U(\mathfrak{g})$ which are also Harish-Chandra bimodules. These are to some extent either implicit or explicit in [1, Chapter 11] and [2, Section 4]. We include them here for the reader's convenience.

Section 3 proves the main theorem. It is shown that if S is a domain properly containing a primitive factor of $U(sl(2))$, and S is a Harish-Chandra bimodule, then S must be of the form described in Section 1.

This problem was brought to my attention by T. Levasseur who relayed a question of C. Moeglin. The associated graded ring of a (minimal) primitive factor of $U(sl(2))$ may be realised as the subring $\mathbb{C}[X^2,XY,Y^2]$ of the polynomial ring $\mathbb{C}[X,Y]$. Lying between $\mathbb{C}[X^2,XY,Y^2]$ and $\mathbb{C}[X,Y]$ are the SL(2)-stable subalgebras

$$\mathbb{C}[X^2,XY,Y^2][X^{2n+1},X^{2n}Y,\dots,XY^{2n},Y^{2n+1}]$$

one for each $n\in\mathbb{N}$. Moeglin, motivated by her work on Whittaker modules, asked whether there were overrings of primitive factor rings of $U(sl(2))$ whose associated graded rings where precisely these commutative algebras. Some time before, motivated by joint work with Stafford [10], I had computed the associated graded rings of the rings $\mathscr{D}(X_{2n+1})$: these were the above commutative algebras. Guessing that these might be the rings sought by Moeglin, it was fairly straightforward to verify that they were.

Preliminaries

Jantzen's book [1] is our basic reference for enveloping algebras of semisimple Lie algebras.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$. Fix a Borel subalgebra $\mathfrak{b}$ for $\mathfrak{g}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ determined by $\mathfrak{b}$. Denote the roots of $\mathfrak{g}$ relative to $\mathfrak{b}$ by R. Write $P(R)$ for the weight lattice and $Q(R)$ for the root lattice. The half-sum of the positive roots is denoted ϱ.

Fix $\lambda \in \mathfrak{h}^*$. Let $M(\lambda)$ be the Verma module of highest weight $\lambda - \varrho$, and let $L(\lambda)$ be the unique simple quotient of $M(\lambda)$. Write $J(\lambda) = \operatorname{Ann} L(\lambda)$.

If $\lambda \in \mathfrak{h}^*$ is dominant integral, write $E(\lambda)$ for the finite dimensional simple of highest weight λ. Thus $E(\lambda) = L(\lambda + \varrho)$. If E and V are $\mathfrak{g}$-modules, write $[V:E]$ for the multiplicity of E in V. If M is an arbitrary $\mathfrak{g}$-module, write M^λ for the λ-weight space.

Let $\mathcal{O}$ denote the category of $U(\mathfrak{g})$-modules M such that (i) $\dim_{\mathbb{C}} Z(\mathfrak{g})m < \infty$, for all $m \in M$, (ii) $\dim_{\mathbb{C}} U(\mathfrak{b})m < \infty$ for all $m \in M$, (iii) M is a direct sum of its $\mathfrak{h}$ weight spaces each of which is finite dimensional. Write $\mathcal{O}_\Lambda$ for the full subcategory of $\mathcal{O}$ whose weights lie in $\Lambda := \lambda + P(R)$.

Set $U := U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \times \mathfrak{g})$, and consider $U(\mathfrak{g})$-bimodules as U-modules. Set $\mathfrak{k} = \{(X, -X) \in \mathfrak{g} \times \mathfrak{g} \mid X \in \mathfrak{g}\}$, and consider $U(\mathfrak{k}) \subset U$. If V is a $U(\mathfrak{g})$-bimodule then the *adjoint action* of $\mathfrak{g}$ on V is given by $X \cdot v = Xv - vX$ for $X \in \mathfrak{g}$ and $v \in V$, so we may consider V as a left $U(\mathfrak{k})$-module. Write $[V:E]$ for the multiplicity of the $\mathfrak{k}$-module E in the $\mathfrak{k}$-module V.

Let $\mathcal{H}$ denote the category of U-modules V such that

(i) $\dim_{\mathbb{C}} Z(\mathfrak{g} \times \mathfrak{g})v < \infty$ for all $v \in V$,

(ii) $\dim_{\mathbb{C}} U(\mathfrak{k})v < \infty$ for all $v \in V$,

(iii) $[V:E] < \infty$ for each finite-dimensional simple $\mathfrak{k}$-module E.

The objects of $\mathcal{H}$ are called *Harish-Chandra modules* for $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{k})$, or simply Harish-Chandra bimodules. An object of $\mathcal{H}$ has finite length, and is finitely generated as a left (and as a right) $U(\mathfrak{g})$-module [1, 6.30]. The simple objects in $\mathcal{H}$ are described in [1, 6.29]. Take $\lambda \in \mathfrak{h}^*$ and write $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ for the corresponding central character. Let $\mathcal{H}_\lambda$ denote the full subcategory of $\mathcal{H}$ consisting of those V such that $V(z - \chi_\lambda(z)) = 0$ for all $z \in Z(\mathfrak{g})$.

If M and N are $\mathfrak{g}$-modules, then $\operatorname{Hom}_{\mathbb{C}}(M,N)$ is a $U(\mathfrak{g})$-bimodule. Write $L(M,N) = \{\psi \in \operatorname{Hom}_{\mathbb{C}}(M,N) \mid \dim_{\mathbb{C}} U(\mathfrak{k})\psi < \infty\}$.

When λ is dominant regular, there is an equivalence of categories $\mathcal{H}_\lambda \to \mathcal{O}_\Lambda$ given by $V \to V \otimes_{U(\mathfrak{g})} M(\lambda)$ [1, 6.27]. The inverse is given by $M \to L(M(\lambda), M)$. The proof depends on the fact that $M(\lambda)$ is projective in $\mathcal{O}_\Lambda$, and that $U(\mathfrak{g}) \to L(M(\lambda), M(\lambda))$ is surjective. Thus the simple objects in $\mathcal{H}_\lambda$ are the $L(M(\lambda), L(\mu))$ for μ in the Weyl group orbit of λ.

Fix a connected algebraic group K with $\operatorname{Lie} K = \mathfrak{g}$, maximal torus T, and write $\chi(T)$ for the character group of T. Identify $\chi(T)$ with a sublattice of $P(R)$ containing $Q(R)$. Thus $Q(R) \subset \chi(T) \subset P(R)$. Therefore $E(\lambda)$ lifts to a representation of

K if and only if $\lambda \in \chi(T)$. Write G^{ad} for the adjoint group of $\mathfrak{g}$, and G^{sc} for the simply connected algebraic group with Lie algebra $\mathfrak{g}$. The associated character groups for G^{ad}, and G^{sc} are $Q(R)$ and $P(R)$ respectively.

If the representation V for $\mathfrak{k}$ lifts to a representation of K, we call V a $(\mathfrak{g}\times\mathfrak{g}, K)$ *Harish-Chandra module*. If V is a $(\mathfrak{g}\times\mathfrak{g}, K)$ Harish-Chandra module, then V is in $\mathscr{H}$, but the converse is not true. If $V \in \mathscr{H}$, then V is a $(\mathfrak{g}\times\mathfrak{g}, K)$ Harish-Chandra module if and only if $V = \bigoplus \{n_\lambda E(\lambda) \mid \lambda \in \chi(T)\}$ where $n_\lambda = [V : E(\lambda)]$.

When V is a Harish-Chandra module, the Gelfand-Kirillov dimension of V, whether viewed as a left $U(\mathfrak{g})$-module or as a U-module is the same. We shall unambiguously write $d(V)$ for this number. If M is a left R-module, $d(M)$ will denote the Gelfand-Kirillov dimension of M.

1. Differential operators on the curve $y^2 = x^{2n+1}$

Fix $n \in \mathbb{N}$, and denote the plane curve $y^2 = x^{2n+1}$ by X_{2n+1}. This section describes the ring $\mathscr{D}(X_{2n+1})$ of differential operators on X_{2n+1}.

The ring of regular functions on X_{2n+1} is $\mathscr{O}(X_{2n+1}) \cong \mathbb{C}[t^2, t^{2n+1}] \subset \mathbb{C}[t]$. Define $\Lambda = 2\mathbb{N} + (2n+1)\mathbb{N}$. Thus $\mathscr{O}(X_{2n+1})$ has a $\mathbb{C}$-vector space basis $\{t^\lambda \mid \lambda \in \Lambda\}$. Write $\partial = \mathrm{d}/\mathrm{d}t$. Since $\mathbb{C}[t, t^{-1}]$ is a localisation of $\mathscr{O}(X_{2n+1})$, $\mathscr{D}(X_{2n+1})$ is a subalgebra of $\mathbb{C}[t, t^{-1}, \partial]$.

The inner derivation $\mathrm{ad}(t\partial)$ gives an eigenspace decomposition for $\mathbb{C}[t, t^{-1}, \partial]$ as

$$\mathbb{C}[t, t^{-1}, \partial] = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[t\partial] t^k.$$

This makes $\mathbb{C}[t, t^{-1}, \partial]$ a $\mathbb{Z}$-graded ring, and if $\mathbb{C}[t, t^{-1}]$ is given its usual grading by degree, then it becomes a graded $\mathbb{C}[t, t^{-1}, \partial]$-module.

Proposition 1.1. $\mathscr{D}(X_{2n+1}) = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}[t\partial] t^k f_k$ *where* $f_k = \prod \{t\partial - j \mid j \in \Lambda \setminus (\Lambda - k)\} \in \mathbb{C}[t\partial]$.

Proof. It is standard that

$$\mathscr{D}(X_{2n+1}) = \{D \in \mathbb{C}[t, t^{-1}, \partial] \mid D(f) \in \mathscr{O}(X_{2n+1}) \text{ for all } f \in \mathscr{O}(X_{2n+1})\}.$$

Note that $\mathbb{C}[t^2, t^{2n+1}]$ is a graded subring of $\mathbb{C}[t, t^{-1}]$. It follows that

$$\mathscr{D}(X_{2n+1}) = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[t\partial] t^k \cap \mathscr{D}(X_{2n+1}).$$

The rest is straightforward calculation. □

Because of the above grading, there is an action of $\mathbb{Z}_2$ as automorphisms of $\mathscr{D}(X_{2n+1})$: let the non-identity element of $\mathbb{Z}_2$ act as scalar multiplication by $(-1)^k$ on $\mathbb{C}[t\partial] t^k f_k$. The next lemma gives generators for the fixed ring, and immediately afterwards it is shown that this fixed ring is a primitive factor of $U(\mathrm{sl}(2))$.

Lemma 1.2. $\bigoplus_{k\in 2\mathbb{Z}} \mathbb{C}[t\partial]t^k f_k = \mathbb{C}[t^2, t\partial, t^{-2}(t\partial)(t\partial - 2n - 1)]$.

Proof. Write R for the ring on the right-hand side. Because the left-hand side is a $\mathbb{C}[t\partial]$-module, and $t, t\partial \in R$, it is enough to show that $t^{2k} f_{2k} \in R$ for all $k \in \mathbb{Z}$. First observe that if $k \geq 0$, then $t^{2k} f_{2k} = t^{2k}$. Furthermore a straightforward calculation shows that $t^{-2} f_{-2} = t^{-2}(t\partial)(t\partial - 2n - 1)$, and that for $k \geq 0$, $t^{-2k} f_{-2k} = (t^{-2} f_{-2})^k$. For the last calculation use the fact that $(t\partial)t^j = t^j(t\partial + j)$. □

Proposition 1.3. *There is an algebra homomorphism* $\Phi : U(\mathrm{sl}(2)) \to \mathscr{D}(X_{2n+1})$ *given by*

$$e \mapsto -\tfrac{1}{2} t^{-2}(t\partial)(t\partial - 2n - 1), \qquad f \mapsto \tfrac{1}{2} t^2, \qquad h \mapsto -t\partial + n - \tfrac{1}{2}.$$

The kernel is $J(\frac{1}{4}(2n+1)\alpha)$. *Furthermore the image of* Φ *is* $\bigoplus_{k\in 2\mathbb{Z}} \mathbb{C}[t\partial]t^k f_k$.

Proof. By Proposition 1.1 these elements belong to $\mathscr{D}(X_{2n+1})$. It is easy to check that $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Thus these elements span a copy of sl(2) contained in $\mathscr{D}(X_{2n+1})$. This gives the existence of Φ. The Casimir element gets mapped to $\Phi(2ef + 2fe + h^2) = n^2 + n - \frac{3}{4}$. Since the Casimir element also acts on $M(\frac{1}{4}(2n+1)\alpha)$ as scalar multiplication by $n^2 + n - \frac{3}{4}$, $\ker \Phi \supset J(\frac{1}{4}(2n+1)\alpha)$. However, $J(\frac{1}{4}(2n+1)\alpha)$ is a maximal ideal of $U(\mathrm{sl}(2))$ since $\frac{1}{4}(2n+1)\alpha$ is a non-integral weight. Thus $\ker \Phi = J(\frac{1}{4}(2n+1)\alpha)$.

The image of Φ is $\mathbb{C}[t^2, t\partial, t^{-2}(t\partial)(t\partial - 2n - 1)]$, and the proof is completed by Lemma 1.2. □

Proposition 1.4. *The* sl(2)*-module action on* $\mathbb{C}[t^2, t^{2n+1}]$ *obtained through* Φ *as in Proposition* 1.3 *makes*

$$\mathbb{C}[t^2, t^{2n+1}] = \mathbb{C}[t^2] \oplus t^{2n+1}\mathbb{C}[t^2] \cong M(\tfrac{1}{4}(2n+1)\alpha) \oplus M(-\tfrac{1}{4}(2n+1)\alpha).$$

The two highest weight vectors are 1 *and* t^{2n+1}, *which are of weight* $\frac{1}{4}(2n-1)\alpha$ *and* $-\frac{1}{4}(2n+3)\alpha$ *respectively.*

Proof. This is easy to verify by looking at the action of the elements e, f, h defined in Proposition 1.3. □

Lemma 1.5. *Both* $e = -\frac{1}{2} t^{-2}(t\partial)(t\partial - 2n - 1)$ *and* $f = \frac{1}{2} t^2$ *act ad-nilpotently on* $\mathscr{D}(X_{2n+1})$. *Hence* $\mathscr{D}(X_{2n+1})$ *becomes a sum of finite-dimensional* sl(2)*-modules under the adjoint action of* sl(2).

Proof. By definition of differential operators $f = \frac{1}{2} t^2 \in \mathscr{O}(X_{2n+1})$ acts ad-nilpotently on $\mathscr{D}(X_{2n+1})$.

Since $t\partial$ is in the image of Φ, e acts ad-nilpotently on $t\partial$. The elements of $\mathscr{D}(X_{2n+1})$ on which ad(e) is nilpotent form a subalgebra, so it is enough to show that ad(e) is nilpotent on the elements $t^k f_k$ defined in (1.1). Because ad(e) is locally nilpotent on $U(\mathrm{sl}(2))$, we only need to show that ad(e) is nilpotent on the elements $t^k f_k$ for k odd. This is a tedious but crucial calculation.

First verify that $[e, t^{-(2n+1)} f_{-(2n+1)}] = 0$. For odd $k < -(2n+1)$ one has $t^k f_k = t^{k+2} f_{k+2} e$, so by downwards induction from $-(2n+1)$, e commutes with $t^k f_k$ for all odd $k < -(2n+1)$. On the other hand, for odd $k > -(2n+1)$, one has $[e, t^k f_k] \in \mathbb{C}[t\partial] t^{k-2} f_{k-2}$, and upward induction from $-(2n+1)$ proves that ad(e) is nilpotent on $t^k f_k$. It follows that ad(e) is locally nilpotent on all of $\mathscr{D}(X_{2n+1})$.

Hence if $D \in \mathscr{D}(X_{2n+1})$, then $(\mathrm{ad}\, e)^r(D) = (\mathrm{ad}\, f)^r(D) = 0$ for r sufficiently large. It follows from [9, Lemma 3.3] that the ad-sl(2) module generated by D is finite dimensional. □

Corollary 1.6. *There is a locally finite action of* SL(2) *as automorphisms of* $\mathscr{D}(X_{2n+1})$, *the differential of which is the natural adjoint action of* sl(2). *Thus* $\mathscr{D}(X_{2n+1})$ *becomes a Harish-Chandra bimodule for* $(\mathrm{sl}(2) \times \mathrm{sl}(2), \mathrm{SL}(2))$.

Proof. If δ is a locally nilpotent derivation on a ring, then $\exp(\delta) = 1 + \delta + \delta^2/2! + \cdots$ is an automorphism of the ring. Since $\mathscr{D}(X_{2n+1})$ is a locally finite ad-sl(2) module by Lemma 1.5, it follows that every nilpotent $X \in \mathrm{sl}(2)$ acts ad-nilpotently on $\mathscr{D}(X_{2n+1})$. Therefore the group generated by $\{\exp(\mathrm{ad} X) \mid X \in \mathrm{sl}(2)$ is nilpotent$\}$ acts as automorphisms of $\mathscr{D}(X_{2n+1})$. But this group is a homomorphic image of SL(2), and by construction its differential is the sl(2) adjoint action. □

The ad-sl(2) submodule structure of $\mathscr{D}(X_{2n+1})$ can be described explicitly. The lowest weight vectors are those weight vectors that commute with f. A straightforward calculation shows that these are simply the powers of t, $\{t^j \mid j \in \Lambda\}$. Furthermore $(\mathrm{ad}\, e)^j(t^j) \neq 0$ and $(\mathrm{ad}\, e)^{j+1}(t^j) = 0$. Hence t^j generates a $(j+1)$-dimensional simple ad-sl(2) module. Thus $\mathscr{D}(X_{2n+1})$ is the sum of the simple modules $E(\frac{1}{2} j\alpha)$, $j \in \Lambda$.

Proposition 1.7. *Set* $M = M(\lambda) \oplus M(-\lambda)$ *with* $\lambda = \frac{1}{4}(2n+1)\alpha$. *Then the natural action of* $\mathscr{D}(X_{2n+1})$ *on* $M = \mathbb{C}[t^2, t^{2n+1}]$ *gives an injection* $\mathscr{D}(X_{2n+1}) \to L(M, M)$.

Proof. The natural action of $\mathscr{D}(X_{2n+1})$ on $M = \mathbb{C}[t^2, t^{2n+1}]$ gives a ring homomorphism $\mathscr{D}(X_{2n+1}) \to \mathrm{End}_{\mathbb{C}} M$. It must be injective because $\mathscr{D}(X_{2n+1})$ is a simple ring [10], and since $\mathscr{D}(X_{2n+1})$ is a locally finite ad-sl(2) module, the image is contained in $L(M, M)$. □

Proposition 1.8. *The restriction of the* $\mathscr{D}(X_{2n+1})$*-action on* $\mathbb{C}[t^2, t^{2n+1}]$ *to* $\mathbb{C}[t^2]$ *is a* $U(\mathrm{sl}(2)) - U(\mathrm{sl}(2))$ *bimodule isomorphism*

$$\mathscr{D}(X_{2n+1}) \to L(M(\lambda), M(\lambda)) \oplus L(M(\lambda), M(-\lambda)).$$

Proof. There is certainly such a bimodule homomorphism. If the kernel were non-zero, then the kernel would contain a lowest weight vector t^j. As t^j acts on $\mathbb{C}[t^2]$ by multiplication this cannot happen.

To see that the map is surjective compare the multiplicities of the finite-dimensional simple ad-sl(2))-modules in each bimodule. We have already seen that $\mathscr{D}(X_{2n+1}) \cong \bigoplus \{E(\frac{1}{2}j\alpha) \mid j \in \Lambda\}$. By [1, 6.9(7)] $[L(M(\mu), M(\nu)) : E] = \dim_{\mathbb{C}} E^{\nu-\mu}$, whence $L(M(\lambda), M(\lambda)) \oplus L(M(\lambda), M(-\lambda))$ has the same simple components. $\square$

We now consider the associated graded rings. Give $\mathscr{D}(X_{2n+1})$ and $\mathscr{D}(\mathbb{A}^1)$ the usual filtration by order of operator. Thus t is of degree zero, and ∂ of degree 1, whence $\operatorname{gr} \mathscr{D}(\mathbb{A}^1) = \mathbb{C}[t, \xi]$. By [10, 3.9–3.10], $\operatorname{gr} \mathscr{D}(X_{2n+1}) \subset \operatorname{gr} \mathscr{D}(\mathbb{A}^1)$.

Using Proposition 1.1, and the description of the f_k, one sees that

$$\operatorname{gr}(t^k f_k) = \begin{cases} t^k & \text{if } k \in 2\mathbb{N}, \\ \xi^k & \text{if } k \geq 2n+1, \text{ and } k = 1 \pmod 2). \end{cases}$$

Because $\operatorname{gr} \mathscr{D}(X_{2n+1})$ is a domain, $\operatorname{gr}\{(t\partial)^m t^k f_k\} = \operatorname{gr}(t\partial)^m \operatorname{gr}(t^k f_k) = (t\xi)^m \operatorname{gr}(t^k f_k)$. Thus $\operatorname{gr} \mathscr{D}(X_{2n+1}) = \mathbb{C}[t^2, t\xi, \xi^2][t^{2n+1}, t^{2n}\xi, \ldots, t\xi^{2n}, \xi^{2n+1}]$. Let Y_{2n+1} be the surface with $\mathscr{O}(Y_{2n+1}) \cong \operatorname{gr} \mathscr{D}(X_{2n+1})$.

Furthermore, $\operatorname{gr}(e) = -\frac{1}{2}\xi^2$, $\operatorname{gr}(f) = \frac{1}{2}t^2$, $\operatorname{gr}(h) = t\xi$. Hence, with the induced filtration, $\operatorname{gr} U(\mathrm{sl}(2))/J(\frac{1}{4}(2n+1)\alpha) = \mathbb{C}[t^2, t\xi, \xi^2]$. This subalgebra is the ring of regular functions on $\mathscr{N}$, the cone of nilpotent elements in sl(2). Thus the inclusion of graded rings gives a covering $\pi : Y_{2n+1} \to \mathscr{N}$. Furthermore, the natural action of SL(2) on $\mathbb{C}[t, \xi]$ leaves both $\operatorname{gr} \mathscr{D}(X_{2n+1})$ and $\operatorname{gr} U(\mathrm{sl}(2))/J(\frac{1}{4}(2n+1)\alpha)$ stable. This gives an action of SL(2) on Y_{2n+1}, and of course the action of SL(2) on $\mathscr{N}$ is the usual one. Hence π is an SL(2)-equivariant covering.

2. Harish-Chandra overrings

The results in this section about overrings of primitive factors of $U(\mathfrak{g})$ which are themselves Harish-Chandra modules are in [1, Chapter 11] and [2, Section 4]. We include them for the reader's convenience.

Let R be a primitive factor ring of $U(\mathfrak{g})$; that is, $R = U(\mathfrak{g})/P$ with P primitive. Let S be a prime ring containing R, such that S is a Harish-Chandra bimodule. Write $Q = \operatorname{Fract} R$. Write V for the *socle* of R as an object of $\mathscr{H}$. Because R is a prime ring, V is a simple object in $\mathscr{H}$. Thus V is the unique minimal non-zero ideal in R, and $d(V) = d(R)$. Since R/V is a torsion R-module, the inclusion $Q \otimes_R V \to Q \otimes_R R = Q$ is an isomorphism. More generally, if $M \supset N$ are left R-modules with $d(M/N) < d(R)$, then the inclusion $Q \otimes_R N \to Q \otimes_R N$ is an isomorphism. Recall [1, 11.12(a)] that if $X \in \mathscr{H}$, then $Q \otimes_R X$ has a Q-bimodule structure, and $Q \otimes_R X \cong X \otimes_R Q$ as Q-bimodules.

Let K be a complex semi-simple algebraic group with $\operatorname{Lie} K = \mathfrak{g}$. Suppose that there is a rational action of K as automorphisms of S, the differential of which agrees with the $\mathfrak{k}$-action on S. Furthermore, suppose that S is a $(\mathfrak{g}\times\mathfrak{g},K)$ Harish-Chandra module.

Lemma 2.1. *Let R be a primitive factor ring of $U(\mathfrak{g})$. Write $\mathscr{C}$ for the regular elements in R. Let $S\supset R$ be a prime overring which is a Harish-Chandra module. Then* $\operatorname{Fract} S=\operatorname{Fract} R$ *if and only if $d(S/R)<d(R)$.*

Proof. As in [4, Corollary 3.7] $\operatorname{Fract} S\supset\operatorname{Fract} R$, and $\operatorname{Fract} S=\mathscr{C}^{-1}R$. In particular, S is torsion free as an R-module on both sides, and if $0\neq I$ is an ideal of S, then $I\cap R\neq 0$. Thus $\operatorname{Fract} S=\operatorname{Fract} R$ if and only if $d(S/R)<d(R)$. □

Lemma 2.2. *The only K-invariant elements of* $\operatorname{Fract} S$ *are the scalars. That is,* $(\operatorname{Fract} S)^K=\mathbb{C}$.

Proof. $(\operatorname{Fract} R)^K$ is the center of R, which is $\mathbb{C}$ because R is primitive. Pick $x\in(\operatorname{Fract} S)^K$. Since $\operatorname{Fract} S$ is a finitely generated $\operatorname{Fract} R$ module, choose m minimal such that $x^m+\gamma_{m-1}x^{m-1}+\cdots+\gamma_0=0$, with $\gamma_i\in Q$. Apply $g\in K$ to this expression, and subtract. By choice of m, $g(\gamma_i)=\gamma_i$ for all i, and for all $g\in K$. Thus each $\gamma_i\in(\operatorname{Fract} R)^K$. Thus x is algebraic over $\mathbb{C}$, hence in $\mathbb{C}$. □

Lemma 2.3. *Let M be a U-submodule of S, such that $M\supset V$. Let N be a maximal U-submodule of M and suppose that $d(M/N)=d(V)$, and $d(N/V)<d(V)$. Then M/N is not isomorphic to V.*

Proof. The earlier comments, and the hypotheses ensure that there is a short exact sequence $0\to Q\otimes_R V\to Q\otimes_R M\to Q\otimes_R(M/N)\to 0$ of Q-bimodules. Suppose that $M/N\cong V$. Then there is an isomorphism of Q-bimodules $\varphi: Q\to Q\otimes_R(M/N)$. Let $x\in Q\otimes_R M$ be such that $\varphi(1)=\bar{x}$, the image of x. For all $q\in Q$, $q\bar{x}-\bar{x}q=0$, so $[q,x]\in Q\otimes_R V=Q$. Since $\operatorname{Fract} S$ is a finitely generated Q-module, choose m minimal such that

$$x^m+\gamma_{m-1}x^{m-1}+\cdots+\gamma_0=0,\quad \text{with } \gamma_i\in Q.$$

Apply $[q,-]$ to this expression to obtain one of lower degree, with leading term $(m[q,x]+[q,\gamma_{m-1}])x^{m-1}$. By minimality of m, $[q,mx+\gamma_{m-1}]=0$ for all $q\in Q$. In particular, if $X\in\mathfrak{g}$ $[X,mx+\gamma_{m-1}]=0$, so $mx+\gamma_{m-1}\in(\operatorname{Fract} S)^K=\mathbb{C}$ by Lemma 2.2. Thus $x\in Q$, and $\bar{x}=0$. This is absurd. □

Corollary 2.4. *Suppose* $\operatorname{Fract} R\neq\operatorname{Fract} S$. *Then there exists a simple subquotient M/N of S, with the properties that $d(M/N)=d(V)$ and $M/N\not\cong V$.*

Proof. S is a torsion free R-module, so the hypothesis ensures that $d(S/R)=d(R)$. Thus there exists a simple subquotient of S/R of GK-dimension $d(V)$. Choose

$M \supsetneqq V$ such that M is minimal with respect to having a simple quotient M/N with $d(M/N) = d(V)$. Apply Lemma 2.3 to obtain the result. □

Now apply this to $U(\mathrm{sl}(2))$.

Proposition 2.5. *Let* $R = U(\mathrm{sl}(2))/J(\lambda)$ *with* $\dim_{\mathbb{C}} R = \infty$. *Suppose that* $S \neq R$. *Then*
(a) Fract $S \neq$ Fract R.
(b) λ *is regular and non-integral. In particular* R *is a simple ring.*

Proof. (a) Suppose that Fract $S =$ Fract R. Then $R \subsetneqq S \subset Q =$ Fract R. But R is a maximal order in Q by [3, Corollary 2.10], whence (as S_R is finitely generated) $S = R$, a contradiction.

(b) If λ is either non-regular or integral, then there is only one simple object in $\mathcal{H}_\lambda$ of GK-dimension 2, and this must be V, the socle of R. Hence, by Corollary 2.4, it would follow that Fract $S =$ Fract R, contradicting (a). □

3. The SL(2) Problem

Theorem 3.1. *Fix* $\lambda \in \mathfrak{h}^*$, *such that* $J(\lambda)$ *is a minimal, primitive ideal. Let* $S \supset R = U(\mathrm{sl}(2))/J(\lambda)$ *be a ring such that*
(a) S *is completely prime,*
(b) *there is an action of* $K = \mathrm{SL}(2)$ *as automorphisms of* S, *such that the differential of the* K*-action coincides with the adjoint action on* $\mathrm{sl}(2)$ *on* S,
(c) S *is a* $(\mathrm{sl}(2) \times \mathrm{sl}(2), K)$ *Harish-Chandra bimodule,*
(d) $S \neq R$.
Then $\lambda = \pm\frac{1}{4}(2n+1)\alpha$ *for some* $n \in \mathbb{N}$, *and* $S = \mathcal{D}(X_{2n+1})$ *contains* $U(\mathrm{sl}(2))/J(\lambda)$, *as in Section* 1.

Remark. The theorem says that only a discrete set of the $U(\mathrm{sl}(2))/J(\lambda)$ ($\lambda \in \mathbb{C}$) admits such overrings. The well known example with $S = \mathcal{D}(\mathbb{A}^1)$ is the case $n = 0$. The other cases are obtained from this by the translation principle (although translation does not appear in our proof). Let us briefly explain.

Because the curves $y^2 = x^{2n+1}$ all have the same normalisation, namely the affine line $\mathbb{A}^1(y^2 = x)$, all the rings $\mathcal{D}(X_{2n+1})$ are Morita equivalent by [10]. So too are the fixed rings $\mathcal{D}(X_{2n+1})^{\mathbb{Z}_2} \cong U(\mathfrak{g})/J(\frac{1}{4}(2n+1)\alpha)$ where the Morita equivalence comes from the translation principle. It can be shown that the Morita equivalences between the various $\mathcal{D}(X_{2n+1})$ 'induce' the Morita equivalences between the fixed rings.

The proof of Theorem 3.1 will follow from a sequence of simple lemmas. Thus the hypotheses (a)–(d) apply throughout this section. Furthermore, because of Proposition 2.5 we may suppose that λ is dominant, regular, and non-integral.

Lemma 3.2. *Write* $\mathbb{Z}_2$ *for the center of* $\mathrm{SL}(2)$.

(a) *Decompose* S *as a* $\mathbb{Z}_2$*-module,* $S=S_+\oplus S_-$, *with the* $\mathbb{Z}_2$*-character on* $S_\pm$ *being* ± 1. *Then*

$$S_+ = \sum \textit{all } \text{ad-sl}(2) \textit{ submodules } E\cong E(\delta) \quad \textit{with } \delta\in Q(R)$$

$$S_- = \sum \textit{all } \text{ad-sl}(2) \textit{ submodules } E\cong E(\delta) \quad \textit{with } \delta\in P(R)\setminus Q(R)$$

(b) $S_+=S^{\mathbb{Z}_2}=R$.

Proof. (a) Trivial.

(b) Since $\mathbb{Z}_2$ acts trivially on R, R is contained in $S_+=S^{\mathbb{Z}_2}$. Thus S_+ is an overring of R satisfying the same hypotheses as S.

In $\mathscr{H}_\lambda$ there are two simple objects namely $L(M(\lambda),M(\lambda))$ and $L(M(\lambda),M(-\lambda))$. By [1, 6.9(7)], $[L(M(\mu),M(\nu)):E]=\dim_{\mathbb{C}} E^{\nu-\mu}$. Hence, if $\delta\in P(R)$, then $\delta\in Q(R) \Leftrightarrow [L(M(\lambda),M(\lambda)):E(\delta)]\neq 0 \Leftrightarrow [L(M(\lambda),M(-\lambda)):E(\delta)]=0$. So the only composition factor occurring in S_+ must be $L(M(\lambda),M(\lambda))$. Hence by Corollary 2.4, Fract $S_+=$ Fract R. Hence, by Proposition 2.5(a) applied to S_+, $S_+=R$. □

Lemma 3.3. *As an R–R bimodule,*

$$S\cong L(M(\lambda),M(\lambda))\oplus L(M(\lambda),M(-\lambda))$$

where $S_+=L(M(\lambda),M(\lambda))$ *and* $S_-=L(M(\lambda),M(-\lambda))$.

Proof. Since λ is dominant regular, and $S\in\mathscr{H}_\lambda$, the categories $\mathscr{H}_\lambda$ and $\mathscr{O}_\Lambda$ are equivalent via the functor $-\otimes_{U(\mathfrak{g})}M(\lambda)$. Hence the length of $S\otimes_R M(\lambda)$ as a left R-module equals the length of S as a U-module. This is just the R–R bimodule length of S.

Let $0\neq a\in S_-$. Then $aS_-\subset R$ and is a right R-submodule. Since S is a domain, $aS_-\cong S_-$ as right R-modules. Hence the rank of S_- as a right R-module is 1. Since R is a simple ring, this forces S_- to be a simple R–R bimodule. Hence S_- has length 1, and S has length 2.

As a left R-module, $S\otimes_R M(\lambda)\cong(S_+\otimes_R M(\lambda))\oplus(S_-\otimes_R M(\lambda))$. Hence $S\otimes_R M(\lambda)$ contains a copy of $M(\lambda)$ so is a faithful R-module. Since a non-zero ideal of S has non-zero intersection with R (see for example, [8, 4.3]), $S\otimes_R M(\lambda)$ is also a faithful S-module.

Write $M=S\otimes_R M(\lambda)$. Since M is a faithful left S-module, the map $S\to\operatorname{End}_{\mathbb{C}} M$ is injective. Since S is a Harish–Chandra module, the image is contained in $L(M,M)$. We now determine precisely what M is.

Since λ is dominant regular and non-integral, $\mathscr{O}_\Lambda$ is equivalent to the category Mod$(\mathbb{C}\oplus\mathbb{C})$. There are two simples in $\mathscr{O}_\Lambda$, namely $M(\lambda)$ and $M(-\lambda)$. Since $S\otimes_R M(\lambda)$ is an object in $\mathscr{O}_\Lambda$ of length 2, there are 3 possibilities for $S\otimes_R M(\lambda)$. These are $M(\lambda)\oplus M(\lambda)$, and $M(-\lambda)\oplus M(-\lambda)$, and $M(\lambda)\oplus M(-\lambda)$.

Recall that $[S:E(\delta)]\neq 0$ for some $\delta\in P(R)\setminus Q(R)$. However, $[L(M(\mu),M(\nu)):E]=\dim_{\mathbb{C}} E^{\nu-\mu}$ by [1, 6.9(7)]. Hence, if $\delta\in P(R)\setminus Q(R)$, then $[L(M(\lambda),M(\lambda)):E(\delta)]=$

$[L(M(-\lambda), M(-\lambda)) : E(\delta)] = 0$. Hence, if M is either $M(\lambda) \oplus M(\lambda)$ or $M(-\lambda) \oplus M(-\lambda)$, then $[L(M,M) : E(\delta)] = 0$. Therefore $S \otimes_R M(\lambda) \cong M(\lambda) \oplus M(-\lambda)$.

Applying the functor $L(M(\lambda), -)$ gives the result. □

Remark. The connection between S and the curves X_{2m+1} is now apparent. Choose $0 \neq y \in S_-$ a highest weight vector. Suppose $y \in E \cong E(\delta)$ with $\delta = \frac{1}{2}(2m+1)\alpha \in P(R) \setminus Q(R)$ for some $m \in \mathbb{N}$. Then $[e, y] = 0$, and $[H, y] = (2m+1)y$. Therefore $[e, y^2] = 0$, and y^2 is a highest weight vector in $S_+ = R$, of weight $(2m+1)\alpha$. So too is e^{2m+1} a highest weight vector in $S_+ = R$, of weight $(2m+1)\alpha$. But $[R : E((2m+1)\alpha)] = 1$, so (after replacing y by a suitable βy, $\beta \in \mathbb{C}$) $y^2 = e^{2m+1}$. Since S is a domain, $\mathbb{C}[e, y] \cong \mathcal{O}(X_{2m+1})$. Thus $\mathcal{O}(X_{2m+1}) \subset S$.

Lemma 3.4. $\lambda = \frac{1}{4}(2n+1)\alpha$ *for some* $n \in \mathbb{N}$.

Proof. Set $M = S \otimes_R M(\lambda) \cong M(\lambda) \oplus M(-\lambda)$, and set $\varrho = \frac{1}{2}\alpha$. By the previous remark, there exists $m \in \mathbb{N}$ and $x, y \in S \subset L(M,M)$ such that $[x, f] = 0$, $x^2 = f^{2m+1}$, $[y, e] = 0$, $y^2 = e^{2m+1}$. Furthermore, x is of weight $-\frac{1}{2}(2m+1)\alpha$. Let $v_{-\lambda-\varrho}$ and $v_{\lambda-\varrho}$ be the highest weight vectors in $M(-\lambda)$ and $M(\lambda)$ respectively.

Since $2m+1$ is odd, no element in $M(\lambda)$ is of weight $\lambda - \varrho - \frac{1}{2}(2m+1)\alpha$. Hence $xv_{\lambda-\varrho} \in M(-\lambda)$. Since $x^2 = f^{2m+1}$, certainly $xv_{\lambda-\varrho} \neq 0$. Therefore $xv_{\lambda-\varrho}$ is a non-zero weight vector in $M(-\lambda)$. So there exists $k \in \mathbb{N}$ with $\lambda - \varrho - \frac{1}{2}(2m+1)\alpha = -\lambda - \varrho - k\alpha$. Hence $\lambda = \frac{1}{2}(\frac{1}{2}(2m+1) - k)\alpha \in \frac{1}{4}\mathbb{Z}\alpha$. Since λ is also dominant, $\lambda \in \frac{1}{4}\mathbb{N}\alpha$, so write $\lambda = \frac{1}{4}r\alpha$ with $r \in \mathbb{N}$. Since λ is not integral, r must be odd. □

Let $\lambda = \frac{1}{4}(2n+1)\alpha$ where $n \in \mathbb{N}$. The ring structure on S gives a ring structure on $L(M(\lambda), M(\lambda)) \oplus L(M(\lambda), M(-\lambda))$. But there is also a ring structure on $L(M(\lambda), M(\lambda)) \oplus L(M(\lambda), M(-\lambda))$ coming from that on $\mathcal{D}(X_{2n+1})$, and the results of Section 1. The proof of Theorem 3.1 now follows from a very nice argument of McGovern [7, Theorem 1.2].

Proposition 3.5. *There is at most one way to extend the ring structure on* $R = L(M(\lambda), M(\lambda))$ *making* $T = L(M(\lambda), M(\lambda)) \oplus L(M(\lambda), M(-\lambda))$ *a ring such that*

(a) *T is a domain, and*

(b) *the multiplication on T gives T its natural R–R bimodule structure.*

Proof. This is easily seen to be part of the proof of [7, Theorem 1.2]. The key point in our situation is as follows. Suppose we have two multiplication maps μ_1, $\mu_2 : T \otimes_R T \to T$. On restriction we get R–R bimodule maps $\tau_1, \tau_2 : L(M(\lambda), M(-\lambda)) \otimes_R L(M(\lambda), M(-\lambda)) \to R$. These are non-zero because T is a domain. Hence they must be isomorphisms because all are simple R–R bimodules. But then $0 \neq \tau_1 \tau_2^{-1} : R \to R$ is in $\operatorname{End}_{\mathcal{H}} R$ which is $\mathbb{C}$ because R is simple. Using this scalar one may then construct an explicit isomorphism $(T, \mu_1) \to (T, \mu_2)$. □

The proof of Theorem 3.1 is now complete.

Remarks. (1) I know of no apriori reason why there should be any connection between these curves and the group SL(2); perhaps the fact that the Dixmier algebras for sl(2, $\mathbb{C}$) coincide with the rings $\mathscr{D}(X_{2n+1})$ should be seen as simply a coincidence, maybe a consequence of the fact that there are not very many algebras of Gelfand–Kirillov dimension 2 which have a commutative associated graded algebra.

(2) The results in this paper are also part of a program to understand primitive factor rings of $U(\mathfrak{g})$ in terms of rings of differential operators on varieties related to nilpotent orbits. As is made clear in [5] and [6], one should also consider singular varieties as well as the generalised flag variety and the Beilinson–Bernstein construction.

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